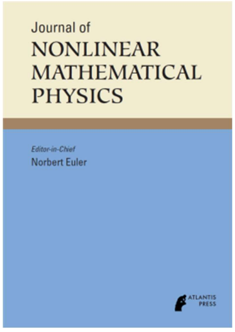


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# Hyperelliptic function solutions with finite genus $\mathscr{G}$ of coupled nonlinear differential equations *

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In this paper, using the properties of hyperelliptic $\sigma$- and $\wp$- functions, $\wp_{\mu\nu} := \partial_\mu \partial_\nu \log \sigma$, we propose an algorithm to obtain particular solutions of the coupled nonlinear differential equations, such as a general (2+1)-dimensional breaking soliton equation and static Veselov-Novikov(SVN) equation, the solutions of which can be expressed in terms of the hyperelliptic Kleinian functions for a given curve $y^2 = f(x)$ of (2g+1)- and (2g+2)-degree with genus $\mathscr{G}$. In particular, owing to the idea of CK direct method, the algorithm can generate a series of new forms of hyperelliptic function solutions with the same genus $\mathscr{G}$.

*Keywords*: Nonlinear differential equations; Hyperelliptic function solutions; Analytic solutions.

2000 Mathematics Subject Classification: 35Q51, 35Q53, 35C99, 68W30, 74J35.

## 1. Introduction

In nonlinear mathematical physics, analytic solutions of nonlinear differential equations (NLDE) play the more and more important role. Different approaches, particularly in soliton theory, provide many tools for searching exact solutions. Various kinds of exact solutions have been presented for NLDE. Successful methods include inverse scattering transform [1], Lie group [2], Darboux transformation [3], Hirota direct method [4], algebro-geometrical approach [5], et al. The algebro-geometrical approach presents quasi-periodic or algebro-geometric solutions to many nonlinear differential equations, which were originally obtained on the Korteweg-de Vries (KdV) equation based inverse spectral theory and algebro-geometric method developed by pioneers such as Novikov, Dubrovin, Mckean, Lax, Its, Matveev, and co-workers [5-10] in the late 1970s. Recently, this theory has been extended to a large class of nonlinear integrable equations[11-17]. By virtue of Riemann theta function, we obtain some quasi-periodic wave solutions of nonlinear equations, discrete equations and supersymmetric equations [43-48].

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The goal of this paper is to present an algorithm for obtain particular solutions of nonlinear differential equations by virtue of CK direct method [18]. Taking a general (2+1)-dimensional breaking soliton equation as an example, the solutions of which can be expressed in terms of the hyperelliptic Kleinian functions with a given curve $y^2 = f(x)$ of (2g+1)- and (2g+2)- degrees whose genus is $\mathscr{G}$, along the lines of the study of Baker's sigma function [19,20]. This construction means re-evaluation of Baker's studies on hyperelliptic functions which were conducted 100 years ago. According to [20], in around 1898 he discovered series of partial differential equations which led to the hyperelliptic $\sigma$-function , and $\wp$-functions, $\wp_{\mu\nu} := \partial_\mu\partial_\nu \log\sigma$. If one saw the partial differential equations, one would know that they are related to soliton equations such as the KdV equation and KP equation. Further as the paper [20] requires knowledge of hyperelliptic $\sigma$- and $\wp$- functions which might not be familiar nowadays [19], it is not easy to understand its contents and to confirm the derivation. Recently, many authors, such as Buchstaber, Eilbeck, Ônishi and others [21-31], have extended it from the point of view of soliton theory. Their methods are consistent with the zero curvature condition in modern soliton theory. Using the hyperelliptic sigma function and defining natural sigma functions of more general algebraic curves, the authors in [21-26] have been constructing deeper theories of Abelian functions and soliton equations[35-42].

Firstly, let us give the conventions which express the hyperelliptic functions throughout this paper. We denote the set of complex numbers by $\mathbb{C}$ and the set of integers by $\mathbb{Z}$.

**Convention 1.1.** A hyperelliptic curve-Riemann surface-$C_g$ with genus $g$ ($g > 0$): Hyperelliptic curve of (2$g$+1)- and (2$g$+2)- degrees, respectively, given by the affine equations

$$y^2 = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_{2g+1} x^{2g+1} = (x-a_1)\cdots(x-a_g)(x-a_{g+1})(x-b_1)\cdots(x-b_g),$$
$$y^2 = \bar{\lambda}_0 + \bar{\lambda}_1 x + \bar{\lambda}_2 x^2 + \cdots + \bar{\lambda}_{2g+2} x^{2g+2} = (x-c_1)\cdots(x-c_g)(x-c_{g+1})(x-d_1)\cdots(x-d_{g+1}),$$

where $\lambda_{2g+1}{\equiv}1$, $\bar{\lambda}_{2g+1}{\equiv}1$ and the $\lambda_j$, $\bar{\lambda}_j$ are complex numbers, the $a_i$, $b_j$, $c_i$, $d_i$ ($i$=$1,\ldots,g+1$,$j$=$1,\ldots,g$) are complex values.

**Proposition 1.2.** *There exist several symmetries which express the same curve $C_g$.*
*(i) Translational symmetry: for $\forall\alpha_0 \in \mathbb{C}, (x,y) \to (x+\alpha_0, y)$, with $b_j \to b_j + \alpha_0$.*
*(ii) Dilatation symmetry: for $\forall\alpha_1 \in \mathbb{C}, (x,y) \to (\alpha_0 x, \alpha_0^{2g+1} y)$, with $b_j \to \alpha_0 b_j$.*
*(iii) Inversion symmetry: for fixing $b_{j_1}, (x,y) \to (1/(x-b_{j_1}), y\prod_{j_1\neq j_2}\sqrt{b_{j_1}-b_{j_2}}/(x-b_{j_1})^{(2g+1)/2})$, with $b_{j_1} \to 1/(b_{j_2}-b_{j_1})$.*

Let the infinite point be located on this curve, we should embed it in a projective space. However as this is not difficult, we assume that the curve $y^2 = f(x)$ includes the infinite point. Further, for simplicity, we also assume that $f(x) = 0$ is not degenerate. We sometimes express a point $P$ in the curve by the affine coordinate $(x,y)$.

**Definition 1.3.** ([19.p.195],[20,p.137],[27,p.385])

(i) Let us denote the homology of a hyperelliptic curve $C_g$ by

$$H_1(C_g,\mathbb{Z}) = \bigoplus_{j=1}^{g}\mathbb{Z}\alpha_j \oplus \bigoplus_{j=1}^{g}\mathbb{Z}\beta_j, \tag{1.1}$$

where these intersections are given as $[\alpha_i,\alpha_j] = 0$, $[\beta_i,\beta_j] = 0$ and $[\alpha_i,\beta_j] = \delta_{ij}$.

(ii) The unnormalized differentials of the first kind are defined as

$$\omega_1 := \frac{dx}{2y},\ \ \omega_2 := \frac{xdx}{2y}, \cdots, \omega_g := \frac{x^{g-1}dx}{2y}. \tag{1.2}$$

(iii) The unnormalized differentials of the second kind are defined as

$$\eta_j := \frac{1}{2y}\sum_{k=j}^{2g-j}(k-j+1)\lambda_{k+j+1}x^k dx, \quad (j=1,2,\dots,g). \tag{1.3}$$

(iv) The complete hyperelliptic integral matrices of the second kind are defined by

$$\zeta' := \left[\int_{\alpha_j}\eta_i\right], \ \zeta'' := \left[\int_{\beta_j}\eta_i\right], \ \zeta := \begin{pmatrix}\zeta' \\ \zeta''\end{pmatrix}. \tag{1.4}$$

(v) The unnormalized period matrices are defined as

$$\Omega' := \left[\int_{\alpha_j}\omega_i\right], \ \Omega'' := \left[\int_{\beta_j}\omega_i\right], \ \Omega := \begin{pmatrix}\Omega' \\ \Omega''\end{pmatrix}. \tag{1.5}$$

(vi) The normalized period matrices are given by

$$^t[\hat{\omega}_1\cdots\hat{\omega}_g] := \Omega'^{-1}\,{}^t[\omega_1\cdots\omega_g], \ \mathbb{T} = \Omega'^{-1}\Omega'', \ \hat{\Omega} := \begin{pmatrix}1_g \\ \mathbb{T}\end{pmatrix}. \tag{1.6}$$

(vii) By defining the Abel map for the $g$th symmetric product of the curve $X_g$ and for points $\{Q_i\}_{i=1,2,\dots,g}$ in the curve:

$$\begin{aligned}\omega: \ Sym^g(X_g) \to \mathbb{C}^g \ \left(\omega_k(Q_i) := \sum_{i=1}^{g}\int_\infty^{Q_1}\omega_k\right),\\ \hat{\omega}: \ Sym^g(X_g) \to \mathbb{C}^g \ \left(\hat{\omega}_k(Q_i) := \sum_{i=1}^{g}\int_\infty^{Q_1}\hat{\omega}_k\right)\end{aligned} \tag{1.7}$$

the Jacobi varieties $\mathscr{J}_g$ and $\hat{\mathscr{J}}_g$ are defined as complex torus,

$$\mathscr{J}_g := \mathbb{C}^g/\Lambda, \ \hat{\mathscr{J}}_g := \mathbb{C}^g/\hat{\Lambda}. \tag{1.8}$$

Here $\Lambda$ ($\hat{\Lambda}$) is a lattice generated by $\Omega$ ($\hat{\Omega}$).
(viii) We define the theta function over $\mathbb{C}^g$, characterized by $\hat{\Lambda}$, as [43, 44]

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z) := \theta\begin{bmatrix}a\\b\end{bmatrix}(z;\mathbb{T}) := \sum_{n\in\mathbb{Z}^g}\exp\left[2i\pi\left\{\frac{1}{2}{}^t(n+a)\mathbb{T}(n+a) + {}^t(n+a)(z+b)\right\}\right] \tag{1.9}$$

for $g$-dimensional vectors $a$ and $b$.
**Remark 1.4.** The first and second period matrices satisfy the generalized Legendre relations to the genus $g$,

$$\omega'\omega^T - \omega\omega'^T = 0, \ \zeta'\omega^T - \zeta\omega'^T = -\frac{i\pi}{2}I_g, \ \zeta'\zeta^T - \zeta\zeta'^T = 0, \tag{1.10}$$

where $I_g$ is the $g\times g$ unit matrix.

We should note that these contours in the integrals are, for example, given in p 3.83 in Ref. [30]. It is also noted that in Eq.(1.2), we have employed the convention of Refs. [27], which differs from Baker's original one by a factor of 1/2. Due to the difference, the results and definitions in Refs. [19, 20] will be slightly modified but the factor set us free from extra constant factors in various situations [27,29]. And based on references, we give the following definition.

**Definition 1.5.** ($\wp$*-function, Baker*). The coordinate in $\mathbb{C}^g$ for points $(x_i, y_i)_{i=1,2,\ldots,g}$ of the cure $y^2 = f(x)$ is given by

$$u_j := \sum_{i=1}^{g} \int_{g}^{(x_i,y_i)} du_j^{(i)}. \tag{1.11}$$

(i) The hyperelliptic function $\sigma$, which is a holomorphic function over $u \in \mathbb{C}^g$, is defined by

$$\sigma(u) = \sigma(u; C_g) := \gamma \exp(-\frac{1}{2}{}^t u\zeta'\omega'^{-1}u)\vartheta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}(\omega'^{-1}u;\tau), \tag{1.12}$$

where

$$\delta' :={}^t\begin{bmatrix}\frac{g}{2} & \frac{g-1}{2} & \cdots & \frac{1}{2}\end{bmatrix},\ \delta'' :={}^t\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2}\end{bmatrix}. \tag{1.13}$$

(ii) In terms of the $\sigma$-function, the $\wp$-function and Kleinian $\mathfrak{S}$-function over the hyperelliptic curve are defined as logarithmic derivatives of the fundamental $\sigma$

$$\wp_{\mu\nu} = -\frac{\partial^2}{\partial u_\mu \partial u_\nu}\log\sigma(u),\ \ \mu,\nu = 1,2,\ldots,g. \tag{1.14}$$

The multi-index symbols $\wp_{i_1,i_2,\cdots,i_n}$ $(n \geq 2)$ are defined in similar way,

$$\wp_{i_1,i_2,\cdots,i_n}(u) = -\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\cdots\frac{\partial}{\partial u_{i_n}}\log\sigma(u),\ \ i_1 \leq i_2 \leq \cdots \leq i_n, \tag{1.15}$$

with the vector $u$ belongs to the Jacobian variety $Jav(X_g)$ of hyperelliptic curves.

One also consider these functions as hyperelliptic Abelian functions under the construction of Kleinian, called *hyperelliptic function* below, which is a natural generalization of the Weierstrass approach in elliptic functions theory to the case of a hyperelliptic cure of genus $g > 1$.

The rest of paper is organized as follows. In Sect. 2, we briefly introduce the hyperelliptic functions of finite genus $\mathscr{G}$. In sect. 3, an algorithm is proposed to obtain particular solutions of a general (2+1)-dimensional breaking soliton equation (3.1) and static Veselov-Novikov(SVN) equation (3.18) , the solutions of which can be expressed in terms of the hyperelliptic Kleinian functions for a given curve $y^2 = f(x)$ of (2g+1)- and (2g+2)- degree with genus $\mathscr{G}$. Finally, some conclusions and remarks are presented.

## 2. The hyperelliptic functions of finite genus $\mathscr{G}$

The hyperelliptic functions $\wp_{ij\ldots k}$ with finite genus $\mathscr{G}$ are the generalization of the Weierstrass function $\wp$. As the $\wp$ is generated by Weierstrass elliptic function $\sigma$, the hyperelliptic functions $\wp_{ij\ldots k}$ are generated by the fundamental Kleinian $\mathfrak{S}$-function which is the natural generalization of the $\sigma$.

It turns out that the $\sigma$-function is a well tuned theta function. Equation (1.13) is related to the so-called *Riemannian constant* $\mathscr{R}$ as mentioned on p 3.80-82 in Ref. [30]; $\delta' + \mathbb{T}\delta''$ agrees with $\mathscr{R}$. As the $\sigma$-function consists of the shifting Riemann theta function (1.9), the Riemann constant

$\mathscr{R}$ outwardly disappears (Thus the $\sigma$-function vanishes just over the theta divisor). Using the $\sigma$-function, Baker derived the multiple relations of $\wp$-functions and so on. Hereafter we assume that the genus $\mathscr{G}=2$, and $\mathscr{G}=3$ of the curve of (2g+1)- and (2g+2)- degree are given in the following propositions 2.1, 2.3 and 2.4, respectively.

**Proposition 2.1.** *(Genus two) Let us express $\wp_{\mu\nu\rho} := \partial\wp_{\mu\nu}(u)/\partial u_\rho$ and $\wp_{\mu\nu\rho\lambda} := \partial^2\wp_{\mu\nu}(u)/\partial u_\rho\partial u_\lambda$. Then the hyperelliptic functions $\wp_{\mu\nu\rho}$ are expressed (extended cubic relation) as follows in terms of $\wp_{11}$, $\wp_{12}$, $\wp_{22}$ and the constants $(\lambda_0,\ldots,\lambda_4)$ of defining hyperelliptic curve $C_2$*

$$
\begin{aligned}
(\mathrm{I}-1):\quad & \wp_{122}^2 = \lambda_0 - 4\wp_{11}\wp_{12} + \lambda_4\wp_{12}^2 + \lambda_4\wp_{22}\wp_{12}^2,\\
(\mathrm{I}-2):\quad & \wp_{211}^2 = \lambda_0\wp_{22}^2 - \lambda_1\wp_{22}\wp_{12} + \lambda_2\wp_{12}^2 + 4\wp_{11}\wp_{12}^2,\\
(\mathrm{I}-3):\quad & \wp_{222}^2 = \lambda_2 + 4\wp_{11} + \lambda_3\wp_{22} + 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + \lambda_4\wp_{22}^2,\\
(\mathrm{I}-4):\quad & \wp_{111}^2 = \lambda_0\wp_{12}^2 + \lambda_1\wp_{11}\wp_{12} + \lambda_2\wp_{11}^2 + 4\lambda_0\wp_{11}\wp_{22} + 4\wp_{11}^3 + \frac{1}{2}\lambda_0\lambda_3\wp_{12}\\
& + \frac{1}{16}(\lambda_1^2\lambda_4 + \lambda_0\lambda_3^2 - 4\lambda_0\lambda_3\lambda_4) + \left(\frac{1}{4}\lambda_1\lambda_3 - \lambda_0\lambda_4\right)\wp_{11} + \left(\frac{1}{4}\lambda_1^2 - \lambda_0\lambda_2\right)\wp_{22}.
\end{aligned}
\tag{2.1}
$$

*The $\wp_{\mu\nu\rho\lambda}$ functions are expressed as follows:*

$$
\begin{aligned}
(\mathrm{I}-5):\quad & \wp_{1222} = 6\wp_{12}\wp_{22} + \lambda_4\wp_{12} - 2\wp_{11},\\
(\mathrm{I}-6):\quad & \wp_{1122} = 2\wp_{11}\wp_{22} + \frac{1}{2}\lambda_3\wp_{12} + 4\wp_{12}^2,\\
(\mathrm{I}-7):\quad & \wp_{2222} = \frac{1}{2}\lambda_3 + 6\wp_{22}^2 + \lambda_4\wp_{22} + 4\wp_{12},\\
(\mathrm{I}-8):\quad & \wp_{1112} = -\lambda_0 + 6\wp_{11}\wp_{12} - \frac{1}{2}\lambda_1\wp_{22} + \lambda_2\wp_{12},\\
(\mathrm{I}-9):\quad & \wp_{1111} = -\frac{1}{2}\lambda_0\lambda_4 + \frac{1}{8}\lambda_1\lambda_3 + 6\wp_{11}^2 + \lambda_2\wp_{11} + \lambda_1\wp_{12} - 3\lambda_0\wp_{22}.
\end{aligned}
\tag{2.2}
$$

*The $\wp_{\mu_1\nu_1\rho_1}\wp_{\mu_2\nu_2\rho_2}$ functions are expressed as follows:*

$$
\begin{aligned}
(\mathrm{I}-10):\quad & \wp_{112}\wp_{122} = 2\wp_{12}^3 + \frac{1}{2}\lambda_3\wp_{12}^2 + \frac{1}{2}\lambda_1\wp_{12} + 2\wp_{11}\wp_{12}\wp_{22} - \lambda_0\wp_{22},\\
(\mathrm{I}-11):\quad & \wp_{122}\wp_{222} = \frac{1}{2}\lambda_1 + 2\wp_{12}^2 - 2\wp_{11}\wp_{22} + \frac{1}{2}\lambda_3\wp_{12} + 4\wp_{12}\wp_{22}^2 + \lambda_4\wp_{12}\wp_{22},\\
(\mathrm{I}-12):\quad & \wp_{112}\wp_{222} = -\frac{1}{2}\lambda_1\wp_{22} + 2\wp_{11}\wp_{22}^2 + 2\wp_{22}\wp_{12}^2 + \lambda_2\wp_{12} + 4\wp_{11}\wp_{12} + \frac{1}{2}\lambda_3\wp_{12}\wp_{22},\\
(\mathrm{I}-13):\quad & \wp_{111}\wp_{122} = -\frac{1}{4}\lambda_0\lambda_3 - \lambda_0\wp_{12} - \frac{1}{2}\lambda_1\wp_{11} - 2\lambda_0\wp_{22}^2 + \frac{1}{4}\lambda_1\lambda_4\wp_{12} - \frac{1}{2}\lambda_0\lambda_4\wp_{22}\\
& + \lambda_1\wp_{12}\wp_{22} + \frac{1}{2}\lambda_3\wp_{11}\wp_{12} + 2\wp_{11}\wp_{12}^2 + 2\wp_{11}^2\wp_{22},\\
(\mathrm{I}-14):\quad & \wp_{111}\wp_{222} = -18\frac{\lambda_3}{\lambda_1} - 4\wp_{11}^2 - 2\wp_{12}^3 - \frac{1}{2}\lambda_1\wp_{12} - \frac{1}{4}\lambda_1\lambda_4\wp_{22} - \lambda_1\wp_{22}^2 - \lambda_2\wp_{11} - \frac{1}{2}\lambda_3\wp_{11}\wp_{22}\\
& + 6\wp_{11}\wp_{12}\wp_{22} - \lambda_3\wp_{12}^2 + \frac{1}{2}\lambda_2\lambda_4\wp_{12} + 2\lambda_2\wp_{12}\wp_{22} + 2\lambda_4\wp_{11}\wp_{12} - \frac{1}{8}\lambda_3^2\wp_{12}.
\end{aligned}
\tag{2.3}
$$

**Remark 2.2.** These relations in proposition 2.1 are generalizations of the elliptic PDE:

$$
\wp''(u) - 6\wp(u)^2 - \frac{1}{2}g_2 = 0,\quad [\wp'(u)]^2 - 4\wp(u)^3 - g_2\wp(u) - g_3 = 0. \tag{2.4}
$$

**Proposition 2.3.** *(Genus three: (2g+1)-degree) As in the case of the $\wp$ Weierstrass elliptic function, and let $\wp_{\mu\nu\rho\lambda} := \wp_{\mu\nu\rho\lambda}(u)$ and $\wp_{\mu\nu} := \wp_{\mu\nu}(u)$ for simplicity, the hyperelliptic functions $\wp_{\mu\nu\rho\lambda}$ are expressed (extended cubic relation) as follows in terms of $\wp_{11}$, $\wp_{12}$, $\wp_{13}$, $\wp_{22}$, $\wp_{23}$, $\wp_{33}$ and the constants $(\lambda_0,\ldots,\lambda_7)$ of defining hyperelliptic curve $C_3$*

$$\begin{aligned}
&(\mathrm{II}-1): && \wp_{1133} = 2\Delta + 4\wp_{13}^2 + 2\wp_{11}\wp_{33},\\
&(\mathrm{II}-2): && \wp_{1333} = 6\wp_{13}\wp_{33} + 4\lambda_6\wp_{13} - 2\lambda_7\wp_{12},\\
&(\mathrm{II}-3): && \wp_{1233} = 2\wp_{12}\wp_{33} + 4\wp_{13}\wp_{23} + 2\lambda_5\wp_{13},\\
&(\mathrm{II}-4): && \wp_{3333} = 2\lambda_5\lambda_7 + 6\wp_{33}^2 + 4\lambda_6\wp_{33} + 4\lambda_7\wp_{23},\\
&(\mathrm{II}-5): && \wp_{2333} = 6\wp_{23}\wp_{33} + 4\lambda_6\wp_{23} + 2\lambda_7(3\wp_{13} - \wp_{22}),\\
&(\mathrm{II}-6): && \wp_{1113} = 6\wp_{11}\wp_{13} + 4\lambda_0\wp_{33} - 2\lambda_1\wp_{23} + 4\lambda_2\wp_{13},\\
&(\mathrm{II}-7): && \wp_{1123} = 4\wp_{12}\wp_{13} + 2\wp_{11}\wp_{23} - 4\lambda_0\lambda_7 + 2\lambda_3\wp_{13},\\
&(\mathrm{II}-8): && \wp_{2233} = 4\wp_{23}^2 + 2\wp_{22}\wp_{33} + 2\lambda_5\wp_{23} + 4\lambda_6\wp_{13} - 2\lambda_7\wp_{12},\\
&(\mathrm{II}-9): && \wp_{1223} = -2\Delta + 4\wp_{12}\wp_{23} + 2\wp_{13}\wp_{22} - 2\lambda_1\lambda_7 + 4\lambda_4\wp_{13},\\
&(\mathrm{II}-10): && \wp_{1112} = -2\lambda_0\lambda_5 + 6\wp_{11}\wp_{12} - 8\lambda_0\wp_{23} + 4\lambda_2\wp_{12} + 2\lambda_1(3\wp_{13} - \wp_{22}),\\
&(\mathrm{II}-11): && \wp_{2223} = -4\lambda_2\lambda_7 + 6\wp_{22}\wp_{23} - 2\lambda_3\wp_{33} + 4\lambda_4\wp_{23} - 6\lambda_7\wp_{11} + 4\lambda_5\wp_{13},\\
&(\mathrm{II}-12): && \wp_{1111} = -4\lambda_0\lambda_4 + \lambda_1\lambda_3 + 6\wp_{11}^2 + 4\lambda_1\wp_{12} + 4\lambda_2\wp_{11} + 4\lambda_0(4\wp_{13} - 3\wp_{22}),\\
&(\mathrm{II}-13): && \wp_{1122} = -8\lambda_0\lambda_6 + 4\wp_{12}^2 + 2\wp_{11}\wp_{22} - 8\lambda_0\wp_{33} - 2\lambda_1\wp_{23} + 4\lambda_2\wp_{13} + 2\lambda_3\wp_{12},\\
&(\mathrm{II}-14): && \wp_{1222} = -4\lambda_1\lambda_6 - 8\lambda_0\lambda_7 + 6\wp_{12}\wp_{22} - 6\lambda_1\wp_{33} + 4\lambda_3\wp_{13} + 4\lambda_4\wp_{12} - 2\lambda_5\wp_{11},\\
&(\mathrm{II}-15): && \wp_{2222} = 12\Delta + 2\lambda_3\lambda_5 - 6\lambda_1\lambda_7 - 8\lambda_2\lambda_6 + 6\wp_{22}^2 - 12\lambda_2\wp_{33} + 4\lambda_3\wp_{23} + 4\lambda_4\wp_{22}\\
& && \qquad + 4\lambda_5\wp_{12} - 12\lambda_6\wp_{11},
\end{aligned} \tag{2.5}$$

*where*

$$\Delta = \wp_{12}\wp_{23} - \wp_{13}\wp_{22} + \wp_{13}^2 - \wp_{11}\wp_{33}. \tag{2.6}$$

*These equations are presented under the convention that if $g = 1$ or $2$ then $\lambda_\mu$ with $\mu > 2g+1$ and $\wp$-functions whose suffix contain $\mu$ bigger than $\nu$ are all zero.*

Note that when $g$=1 the equation (II-12) above is the well-known equation derived from $\wp'(u)^2 - 4f(\wp(u)) = 0$.

**Proposition 2.4.** *(Genus three: (2g+2)-degree) The hyperelliptic $\wp$-functions of a curve $y^2 = \bar{f}(x)$ (g=3) obey the following relations:*

$$\begin{aligned}
&(\mathrm{III}-1): && \wp_{1133} = 4\wp_{13}^2 + 2\wp_{11}\wp_{33} + 2\Delta,\\
&(\mathrm{III}-2): && \wp_{1333} = 6\wp_{13}\wp_{33} + 4\bar{\lambda}_8\wp_{11} - 2\bar{\lambda}_7\wp_{12} + 4\bar{\lambda}_6\wp_{13},\\
&(\mathrm{III}-3): && \wp_{1233} = -4\bar{\lambda}_1\bar{\lambda}_8 + 2\wp_{12}\wp_{33} + 4\wp_{13}\wp_{23} + 2\bar{\lambda}_5\wp_{13},\\
&(\mathrm{III}-4): && \wp_{1223} = -8\bar{\lambda}_0\bar{\lambda}_8 - 2\bar{\lambda}_1\bar{\lambda}_7 + 4\wp_{12}\wp_{23} + 2\wp_{13}\wp_{22} + 4\bar{\lambda}_4\wp_{13} - 2\Delta,\\
&(\mathrm{III}-5): && \wp_{2333} = -4\bar{\lambda}_3\bar{\lambda}_8 + 6\wp_{23}\wp_{33} + 4\bar{\lambda}_6\wp_{23} + 8\bar{\lambda}_8\wp_{12} + 6\bar{\lambda}_7\wp_{13} - 2\bar{\lambda}_7\wp_{22},\\
&(\mathrm{III}-6): && \wp_{2233} = -8\bar{\lambda}_2\bar{\lambda}_8 + 4\wp_{23}^2 + 2\wp_{22}\wp_{33} + 2\bar{\lambda}_5\wp_{23} - 8\bar{\lambda}_8\wp_{11} - 2\bar{\lambda}_7\wp_{12} + 4\bar{\lambda}_6\wp_{13},\\
&(\mathrm{III}-7): && \wp_{2223} = -8\bar{\lambda}_1\bar{\lambda}_8 - 4\bar{\lambda}_2\bar{\lambda}_7 - 6\bar{\lambda}_7\wp_{11} + 4\bar{\lambda}_5\wp_{13} + 4\bar{\lambda}_4\wp_{23} + 6\wp_{23}\wp_{22} - 2\bar{\lambda}_3\wp_{33},\\
&(\mathrm{III}-8): && \wp_{3333} = -8\bar{\lambda}_4\bar{\lambda}_8 + 2\bar{\lambda}_5\bar{\lambda}_7 + 6\wp_{33}^2 + 16\bar{\lambda}_8\wp_{13} - 12\bar{\lambda}_8\wp_{22} + 4\bar{\lambda}_7\wp_{23} + 4\bar{\lambda}_6\wp_{33},
\end{aligned} \tag{2.7}$$

*together with relations (*II-*6), (*II-*7), (*II-*10) and (*II-*12)-(*II-*15), and* $\Delta$ *which have the same form as those in proposition 2.3 by replacing the* $\lambda$ *with the* $\bar{\lambda}$.

## 3. Applications of the hyperelliptic function

In this section, we will consider hyperelliptic solutions of some nonlinear differential equations such as a general (2+1)-dimensional breaking soliton equation (3.1) and static Veselov-Novikov(SVN) equation (3.18).

### 3.1. *A general (2+1)-dimensional breaking soliton equation*

A general (2+1)-dimensional breaking soliton equation reads [32–34, 43]

$$u_t + u_{xxx} + u_{xxy} + 6u_x + 4uu_y + 4u_x\partial_x^{-1}u_y = 0, \tag{3.1}$$

which describe the interaction of a Riemann wave propagating along the $y$-axis with a long wave along the $x$-axis, where $\alpha$ and $\beta$ are the real parameters. We make the transformation $v = \partial_x^{-1}u_y$, then the Eq.(3.1) becames

$$\begin{cases} u_t + u_{xxx} + u_{xxy} + 6u_x + 4uu_y + 4u_xv = 0, \\ u_y = v_x. \end{cases} \tag{3.2}$$

In the following, we will give some Theorems to obtain its hyperelliptic function solutions with genus $\mathscr{G} = 2$, and $\mathscr{G} = 3$ of hyperelliptic curve of $(2g+1)$-degree and $(2g+2)$-degree, respectively.

**Theorem 3.1.** *The following* $\{u, v\}$ *are two families of hyperelliptic function solutions of the (2+1)-dimensional breaking soliton equation*(3.2) *with genus* $\mathscr{G} = 2$*, and* $\mathscr{G} = 3$ *of hyperelliptic curve of* $(2g+1)$*-degree and* $(2g+2)$*-degree, respectively.*
*(i) When* $\mathscr{G} = 2$*, u, v are given of the form*

$$\begin{aligned} u(t,x,y) =& U_0 - \frac{3b_2^2(b_2+c_2)}{2c_2} \\ &\times \wp_{22}\Big(-4b_2^2(b_2+c_2)t, \Big(-6b_2 - 4c_2U_0 - 4b_2V_0 - \lambda_4 b_2^2(b_2+c_2)\Big)t + b_2x + c_2y\Big), \end{aligned} \tag{3.3a}$$

$$\begin{aligned} v(t,x,y) =& V_0 - \frac{3b_2(b_2+c_2)}{2} \\ &\times \wp_{22}\Big(-4b_2^2(b_2+c_2)t, \Big(-6b_2 - 4c_2U_0 - 4b_2V_0 - \lambda_4 b_2^2(b_2+c_2)\Big)t + b_2x + c_2y\Big), \end{aligned} \tag{3.3b}$$

*where* $\lambda_4$, $b_2$, $c_2(\neq 0)$, $U_0$ *and* $V_0$ *are arbitrary constants in* $\mathbb{C}$.
*(ii) When* $(2g+1)$*-degree, u, v are given of the form*

$$u(t,x,y) = U_0 + U_{33}\wp_{33}(a_1t + b_1x - b_1y, a_2t + b_2x - b_2y, a_3t + b_3x - b_3y), \tag{3.4a}$$

$$v(t,x,y) = V_0 + V_{33}\wp_{33}(a_1t + b_1x - b_1y, a_2t + b_2x - b_2y, a_3t + b_3x - b_3y), \tag{3.4b}$$

*where* $a_i = -6b_i + 4b_iU_0 - 4b_iV_0$, $b_i$, $U_0$, $V_0$, $U_{33}$ *and* $V_{33}$ *are arbitrary constants in* $\mathbb{C}$ $(i=1,2,3)$.
*(iii) When* $(2g+2)$*-degree, u, v are given of the form*

$$u(t,x,y) = U_0 - \frac{3b_3^2(b_3+c_3)}{2c_3}\wp_{33}\Big(-16\bar{\lambda}_8 b_3^2(b_3+c_3)t, -4\bar{\lambda}_7 b_3^2(b_3+c_3)t, a_3t + b_3x + c_3y\Big), \tag{3.5a}$$

$$v(t,x,y) = V_0 - \frac{3b_3(b_3+c_3)}{2}\wp_{33}\Big(-16\bar{\lambda}_8 b_3^2(b_3+c_3)t, -4\bar{\lambda}_7 b_3^2(b_3+c_3)t, a_3t + b_3x + c_3y\Big), \tag{3.5b}$$

*where* $a_3 = -6b_3 - 4c_3U_0 - 4b_3V_0 - 4\bar{\lambda}_6 b_3^2(b_3 + c_3)$, $\bar{\lambda}_6$, $\bar{\lambda}_7$, $\bar{\lambda}_8$, $b_3$, $c_3(\neq 0)$, $U_0$, $V_0$ *are arbitrary constants in* $\mathbb{C}$.

**Proof.** According to the arithmetic given in Sec. 3, we prove them in (i),(ii) and (iii) by using propositions 2.1,2.3 and 2.4, respectively.
(i) We make a transformation $u(t,x,y) = U(\omega_1, \omega_2)$, $v(t,x,y) = V(\omega_1, \omega_2)$, $\omega_i = a_it + b_ix + c_iy$ $(i = 1,2)$, then Eq.(3.2) becomes

$$\begin{aligned}&(a_1 + 6\alpha b_1)U_{\omega_1} + (a_2 + 6\alpha b_2)U_{\omega_2} + (\alpha b_1^3 + \beta b_1^2 c_1)U_{\omega_1\omega_1\omega_1} + (\alpha b_2^3 + \beta b_2^2 c_2)U_{\omega_2\omega_2\omega_2}\\&\quad + (3\alpha b_1^2 b_2 + \beta b_1^2 c_2 + 2\beta b_1 b_2 c_1)U_{\omega_1\omega_1\omega_2} + (3\alpha b_1 b_2^2 + 2\beta b_1 b_2 c_2 + \beta b_2^2 c_1)U_{\omega_1\omega_2\omega_2}\\&\quad + 4\beta c_1 U U_{\omega_1} + 4\beta c_2 U U_{\omega_2} + 4\beta b_1 U_{\omega_1} V + 4\beta b_2 U_{\omega_2} V = 0,\\&\quad c_1 U_{\omega_1} + c_2 U_{\omega_2} - b_1 V_{\omega_1} - b_2 V_{\omega_2} = 0.\end{aligned} \tag{3.6}$$

Substituting $U = U_0 + \sum_{j,k=1}^{2} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{2} V_{jk}\wp_{jk}$ and the relations **(I-1)**-**(I-14)** into Eq.(3.6), it becomes overdetermined equations in $\wp_{jk}$ $i,j = 1,2$. Making each coefficient of the overdetermined equations to zero yields equations about $a_i$, $b_i$, $c_i$, $U_0$, $V_0$, $U_{jk}$ and $V_{jk}(i,j,k = 1,2)$. Solving the equations, we obtain

$$\begin{aligned}&b_1 = c_1 = U_{jk} = V_{jk} = 0,\ \ a_1 = -4b_2^2(b_2 + c_2),\ \ a_2 = -6b_2 - 4c_2U_0 - 4b_2V_0\\&\quad - \lambda_4 b_2^2(b_2 + c_2),\ \ U_{33} = -\frac{3b_2^2(b_2 + c_2)}{2c_2},\ \ V_{33} = -\frac{3b_2(b_2 + c_2)}{2},\end{aligned} \tag{3.7}$$

where $j,k = 1,2$ and $(j,k) \neq (2,2)$, $\lambda_4$, $b_2$, $c_2(\neq 0)$, $U_0$ and $V_0$ are arbitrary constants in $\mathbb{C}$. Now substituting the above values in $U = U_0 + \sum_{j,k=1}^{2} U_{jk}\wp_{jk}$ and $V = V_0 + \sum_{j,k=1}^{2} V_{jk}\wp_{jk}$, we achieve a hyperelliptic function solution (3.3a) and (3.3b) with genus $\mathscr{G} = 2$ of the (2+1)-dimensional breaking soliton equation(3.2).
(ii) We make a transformation $u(t,x,y) = U(\omega_1, \omega_2, \omega_3)$, $v(t,x,y) = V(\omega_1, \omega_2, \omega_3)$, $\omega_i = a_it + b_ix + c_iy$ $(i = 1,2,3)$, then Eq.(3.2) becomes

$$\begin{aligned}&a_1 U_{\omega_1} + a_2 U_{\omega_2} + a_3 U_{\omega_3} + (b_1^3 + b_1^2 c_1)U_{\omega_1\omega_1\omega_1} + (b_2^3 + b_2^2 c_2)U_{\omega_2\omega_2\omega_2} + (b_3^3 + b_3^2 c_3)U_{\omega_3\omega_3\omega_3}\\&\quad + (3b_1^2 b_2 + b_1^2 c_2 + 2b_1 b_2 c_1)U_{\omega_1\omega_1\omega_2} + (3b_1^2 b_3 + b_1^2 c_3 + 2b_1 b_2 c_1)U_{\omega_1\omega_1\omega_3} + (3b_1 b_2^2\\&\quad + 2b_1 b_2 c_2 + b_2^2 c_1)U_{\omega_1\omega_2\omega_2} + (3b_2^2 b_3 + b_2^2 c_3 + 2b_2 b_3 c_2)U_{\omega_2\omega_2\omega_3} + (3b_1 b_3^2 + 2b_1 b_3 c_3\\&\quad + b_3^2 c_1)U_{\omega_1\omega_3\omega_3} + (3b_2 b_3^2 + 2b_2 b_3 c_3 + b_3^2 c_2)U_{\omega_2\omega_3\omega_3} + (6b_1 b_2 b_3 + 2b_1 b_2 c_3 + 2b_1 b_3 c_2\\&\quad + 2b_2 b_3 c_1)U_{\omega_1\omega_2\omega_3} + 6b_1 U_{\omega_1} + 6b_2 U_{\omega_2} + 6b_3 U_{\omega_3} + 4c_1 U U_{\omega_1} + 4c_2 U U_{\omega_2} + 4c_3 U U_{\omega_3}\\&\quad + 4b_1 U_{\omega_1} V + 4b_2 U_{\omega_2} V + 4b_3 U_{\omega_3} V = 0,\ \ c_1 U_{\omega_1} + c_2 U_{\omega_2} + c_3 U_{\omega_3} - b_1 V_{\omega_1}\\&\quad - b_2 V_{\omega_2} - b_3 V_{\omega_3} = 0.\end{aligned} \tag{3.8}$$

Substituting $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$ and the relations **(II-1)**-**(II-15)** into Eq.(3.8), it becomes overdetermined equations in $\wp_{jk}$ $i,j = 1,2,3$. Making each coefficient of the overdetermined equations to zero yields equations about $a_i$, $b_i$, $c_i$, $U_0$, $V_0$, $U_{jk}$ and $V_{jk}(i,j,k = 1,2,3)$. Solving the equations, we obtain

$$U_{jk} = V_{jk} = 0,\ \ a_i = -6b_i + 4b_iU_0 - 4b_iV_0,\ \ c_i = -b_i, \tag{3.9}$$

where $j,k = 1,2,3$ and $(j,k) \neq (3,3)$, $b_i(i{=}1,2,3)$, $U_0$, $V_0$, $U_{33}$ and $V_{33}$ are arbitrary constants in $\mathbb{C}$. Now substituting the above values in $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$ and $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$, we

achieve a hyperelliptic function solution (3.4a) and (3.4b) with genus $\mathscr{G}$=3 of the (2+1)-dimensional breaking soliton equation(3.2).
(iii) Substituting $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$ and the relations **(III-1)-(III-15)** into Eq.(3.8), it becomes overdetermined equations in $\wp_{jk}$ $i, j = 1,2,3$. Making each coefficient of the overdetermined equations to zero yields equations about $a_i$, $b_i$, $c_i$, $U_0$, $V_0$, $U_{jk}$ and $V_{jk}(i, j,k = 1,2,3)$. Solving the equations, we obtain

$$\begin{aligned} &b_i = c_i = U_{jk} = V_{jk} = 0,\ \ a_1 = -16\bar{\lambda}_8 b_3^2(b_3+c_3),\ \ a_2 = -4\bar{\lambda}_7 b_3^2(b_3+c_3),\ \ a_3 = -6b_3 \\ &-4c_3U_0 - 4b_3V_0 - 4\bar{\lambda}_6 b_3^2(b_3+c_3),\ \ U_{33} = -\frac{3b_3^2(b_3+c_3)}{2c_3},\ \ V_{33} = -\frac{3b_3(b_3+c_3)}{2}, \end{aligned} \tag{3.10}$$

where $i$=1,2, $j,k$=1,2,3 with $(j,k) \neq (3,3)$, and $\bar{\lambda}_6$, $\bar{\lambda}_7$, $\bar{\lambda}_8$, $b_3$, $c_3(\neq 0)$, $U_0$, $V_0$ are arbitrary constants in $\mathbb{C}$. Now substituting the above values in $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$ and $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$, we achieve a hyperelliptic function solution (3.5a) and (3.5b) with genus $\mathscr{G}$=3 of the (2+1)-dimensional breaking soliton equation(3.2). □

**Theorem 3.2.** *If* $\{U = U(t,x,y), V = V(t,x,y)\}$ *is a solution of the (2+1)-dimensional breaking soliton equation*(3.2) *then so is*

$$u(t,x,y) = \frac{\tau_{tt}y}{12\alpha\beta\tau_t} - \frac{\delta g'(t)}{4\beta(\tau_t)^{\frac{1}{3}}} + \delta(\tau_t)^{\frac{2}{3}}U(\tau,\xi,\eta), \tag{3.11a}$$

$$v(t,x,y) = \frac{3\alpha\tau_t - 3f'(t) - 3\delta\alpha(\tau_t)^{\frac{1}{3}} - \delta(\tau_t)^{-\frac{2}{3}}\tau_{tt}x}{12\delta\alpha\beta(\tau_t)^{\frac{1}{3}}} + \frac{1}{\delta}(\tau_t)^{\frac{2}{3}}V(\tau,\xi,\eta), \tag{3.11b}$$

*where* $\tau = \tau(t)$, $\xi = \delta(\tau_t)^{\frac{1}{3}}x + f(t)$, $\eta = \frac{1}{\delta}(\tau_t)^{\frac{1}{3}}y + g(t)$, $f(t)$, $g(t)$ *are arbitrary function of t, and the discrete value of the constant* $\delta$ *with* $\delta^3 = 1$.

**Proof.** We show here that it is sufficient to seek a similarity reduction of the (2+1)-dimensional breaking soliton equation(3.2) in the form

$$u(t,x,y) = A + BU(\tau,\xi,\eta),\ \ v(t,x,y) = C + DV(\tau,\xi,\eta), \tag{3.12}$$

where $A$, $B$, $C$, $D$, $\xi$, $\eta$, $\tau$ are functions of $x$, $y$, $t$, and $U(\tau,\xi,\eta)$, $V(\tau,\xi,\eta)$ satisfy the same equation as (3.2) of the form

$$\begin{cases} U_\tau + \alpha U_{\xi\xi\xi} + \beta u_{\xi\xi\eta} + 6\alpha U_\xi + 4\beta U U_\eta + 4\beta U_\xi V = 0, \\ U_\eta = V_\xi. \end{cases} \tag{3.13}$$

Substituting (3.12) into (3.2) and requiring $\{U(\tau,\xi,\eta), V(\tau,\xi,\eta)\}$ also satisfies Eq.(3.13) but with independent variables, we have two differential equations about $U(\tau,\xi,\eta)$ and $V(\tau,\xi,\eta)$. Making each coefficient of $U(\tau,\xi,\eta)$,$V(\tau,\xi,\eta)$ and its derivatives to zero [eliminating $U_\tau$,$V_\xi$ and its higher order derivatives by means of the (2+1)-dimensional breaking soliton equation], we obtain overdetermined equations for $A$, $B$, $C$, $D$, $\xi$, $\eta$ and $\tau$. It is straightforward to find out the overdetermined equations. The result reads

$$\tau=\tau(t),\ \ \xi=\delta(\tau_t)^{\frac{1}{3}}x+f(t),\ \ \eta=\frac{1}{\delta}(\tau_t)^{\frac{1}{3}}y+g(t),\ \ A=\frac{\tau_{tt}y}{12\alpha\beta\tau_t}-\frac{\delta g'(t)}{4\beta(\tau_t)^{\frac{1}{3}}},$$
$$B=\delta(\tau_t)^{\frac{2}{3}},\ \ C=\frac{3\alpha\tau_t-3f'(t)-3\delta\alpha(\tau_t)^{\frac{1}{3}}-\delta(\tau_t)^{-\frac{2}{3}}\tau_{tt}x}{12\delta\alpha\beta(\tau_t)^{\frac{1}{3}}},\ \ D=\frac{1}{\delta}(\tau_t)^{\frac{2}{3}}, \tag{3.14}$$

where $f(t)$, $g(t)$ are arbitrary function of $t$, and the discrete value of the constant $\delta$ with $\delta^3=1$. Now substituting the above values in transformation(3.12), we achieve a new solution (3.11a) and (3.11b) of the (2+1)-dimensional breaking soliton equation(3.2). □

**Theorem 3.3.** *Both the following* $\{u,v\}$ *are two new families of hyperelliptic function solutions of the (2+1)-dimensional breaking soliton equation*(3.2) *with genus* $\mathscr{G}=3$ *of hyperelliptic curve of (*$2g+1$*)-degree and (*$2g+2$*)-degree, respectively, given by*
*(i) When* $\mathscr{G}=2$, *the new solution u, v are given of the form*

$$u(t,x,y)=\overline{U}_0-\frac{3b_2^2B(b_2+c_2)}{2\beta c_2}\wp_{22}\Big(-4b_2^2(b_2+c_2)\tau,\Big(-6b_2-4c_2U_0-4b_2V_0-\lambda_4b_2^2(b_2+c_2)\Big)\tau+b_2\xi+c_2\eta\Big), \tag{3.15a}$$

$$v(t,x,y)=\overline{V}_0-\frac{3b_2D(b_2+c_2)}{2}\wp_{22}\Big(-4b_2^2(b_2+c_2)\tau,\Big(-6b_2-4c_2U_0-4b_2V_0-\lambda_4b_2^2(b_2+c_2)\Big)\tau+b_2\xi+c_2\eta\Big), \tag{3.15b}$$

*where* $\tau$, $\xi$, $\eta$, *A, B, C and D are accurately given in Eq.*(3.14) *with* $f(t)$, $g(t)$ *are arbitrary function of t,* $\overline{U}_0=A+\delta(\tau_t)^{\frac{2}{3}}U_0$, $\overline{V}_0=C+\frac{1}{\delta}(\tau_t)^{\frac{2}{3}}V_0$, $\lambda_4$, $b_1$, $b_2$, $c_1$, $c_2(\neq 0)$, $U_0$, $V_0$ *and* $\delta(\delta^3=1)$ *are arbitrary constants in* $\mathbb{C}$.
*(ii) When* $\mathscr{G}=3$*: hyperelliptic curve of (*$2g+1$*)-degree, the new solution u, v are given of the form*

$$u(t,x,y)=\overline{U}_0+\delta(\tau_t)^{\frac{2}{3}}U_{33}\wp_{33}\left(a_1\tau+b_1\xi-b_1\eta,a_2\tau+b_2\xi-b_2\eta,a_3\tau+b_3\xi-b_3\eta\right), \tag{3.16a}$$
$$v(t,x,y)=\overline{V}_0+\frac{1}{\delta}(\tau_t)^{\frac{2}{3}}V_{33}\wp_{33}\left(a_1\tau+b_1\xi-b_1\eta,a_2\tau+b_2\xi-b_2\eta,a_3\tau+b_3\xi-b_3\eta\right), \tag{3.16b}$$

*where* $\tau$, $\xi$, $\eta$, *A, B, C and D are accurately given in Eq.*(3.14) *with* $f(t)$, $g(t)$ *are arbitrary function of t, and* $a_i=-6b_i+4b_iU_0-4b_iV_0$, $\overline{U}_0=A+BU_0$, $\overline{V}_0=C+DV_0$, $b_i$, $U_0$, $V_0$, $U_{33}$, $V_{33}$ *and* $\delta(\delta^3=1)$ *are arbitrary constants in* $\mathbb{C}$ *(*$i=1,2,3$*).*
*(iii) When* $\mathscr{G}=3$*: hyperelliptic curve of (*$2g+2$*)-degree, the new solution u, v are given of the form*

$$u(t,x,y)=\overline{U}_0-\frac{3\delta b_3^2(\tau_t)^{\frac{2}{3}}(b_3+c_3)}{2c_3}\wp_{33}\Big(-16\bar{\lambda}_8b_3^2(b_3+c_3)\tau,-4\bar{\lambda}_7b_3^2(b_3+c_3)\tau,a_3\tau+b_3\xi+c_3\eta\Big), \tag{3.17a}$$

$$v(t,x,y)=\overline{V}_0-\frac{3b_3(\tau_t)^{\frac{2}{3}}(b_3+c_3)}{2\delta}\wp_{33}\Big(-16\bar{\lambda}_8b_3^2(b_3+c_3)\tau,-4\bar{\lambda}_7b_3^2(b_3+c_3)\tau,a_3\tau+b_3\xi+c_3\eta\Big), \tag{3.17b}$$

*where* $\tau$, $\xi$, $\eta$, *A, B, C and D are accurately given in Eq.*(3.14) *with* $f(t)$, $g(t)$ *are arbitrary function of t,* $a_i=-6b_i+4b_iU_0-4b_iV_0$, $\overline{U}_0=A+BU_0a_1$, $\overline{V}_0=C+DV_0a_1$, $\bar{\lambda}_6$, $\bar{\lambda}_7$, $\bar{\lambda}_8$, $b_3$, $c_3(\neq 0)$, $U_0$, $V_0$ *are arbitrary constants in* $\mathbb{C}$.

**Proof.** It is straightforward to prove this Theorem by using the Theorems 3.1 and 3.2. □

**Remark 3.4.** (i) Theorems 3.1-3.3 and the definition of $\wp$ mean that solutions of the (2+1)-dimensional breaking soliton equation are explicitly constructed. The quantities in definitions 1.3 and 1.5 can be, in principle, evaluated in terms of numerical computations because there is no ambiguous parameter.
(ii) The dispersion relations: $\omega_j$ behaves like $(1/\bar{x})^{2(g-j)-1}$ and $(1/\bar{x})^{g-j}$ around the infinity point if we use the local coordinate $\bar{x}^2 := x$, respectively. By comparing the order of $\bar{x}$, denoted by ord$\bar{x}$, we have the relations

$$\mathrm{ord}\bar{x}(\omega_2) = 3\mathrm{ord}\bar{x}(\omega_1), \quad \mathrm{ord}\bar{x}(\omega_2) = \mathrm{ord}\bar{x}(\omega_3),$$

which is the dispersion relations of the (2+1)-dimensional breaking soliton equation(3.2).

### 3.2. *The static Veselov-Novikov equation*

The static Veselov-Novikov(SVN) equation is given as

$$\begin{cases} u_{xxx} + u_{yyy} - 2(uv)_x - 3(uw)_y = 0, \\ u_x = v_y, \\ u_y = w_x, \end{cases} \tag{3.18}$$

The families of hyperelliptic function solutions of the SVN equation(3.18) with genus $\mathscr{G} = 2$ and 3 of the static Veselov-Novikov(SVN) equation are investigated in Theorem 4.18.

**Theorem 3.5.** *The following $\{u, v, w\}$ are three families of hyperelliptic function solutions of the SVN equation*(3.18) *with genus $\mathscr{G} = 2$, and $\mathscr{G} = 3$ of hyperelliptic curve of (2g+1)-degree and (2g+2)-degree, respectively, given by*
*(i) When $\mathscr{G} = 2$, u, v, w are given of the form*

$$u(t,x,y) = 2\wp_{12}(y,x), \tag{3.19a}$$

$$v(t,x,y) = \frac{1}{3}\left(\lambda_4 - \frac{1}{2}\lambda_1\right) + 2\wp_{22}(y,x), \tag{3.19b}$$

$$w(t,x,y) = \frac{1}{3}(\lambda_2 - 2\lambda_1) + 2\wp_{11}(y,x), \tag{3.19c}$$

*where $\lambda_1$, $\lambda_2$ and $\lambda_4$ are arbitrary constants in $\mathbb{C}$.*
*(ii) When $\mathscr{G} = 3$ of hyperelliptic curve of (2g+1)-degree, u, v, w are given of the form*

$$u(x,y) = U_0 + U_{33}\wp_{33}(a_1x - a_1y, a_2x - a_2y, a_3x - a_3y), \tag{3.20a}$$

$$v(x,y) = V_0 - U_{33}\wp_{33}(a_1x - a_1y, a_2x - a_2y, a_3x - a_3y), \tag{3.20b}$$

$$w(x,y) = -\frac{5}{3}U_0 + \frac{2}{3}V_0 - U_{33}\wp_{33}(a_1x - a_1y, a_2x - a_2y, a_3x - a_3y), \tag{3.20c}$$

*where $a_1$, $a_2$, $a_3$, $U_0$, $V_0$ and $U_{33}$ are arbitrary constants in $\mathbb{C}$.*
*(iii) When $\mathscr{G} = 3$ of hyperelliptic curve of (2g+2)-degree, u, v, w are given of the form*

$$u(x,y) = U_0 + 9\wp_{33}\left((\bar{\lambda}_6 - 2V_0 - 2W_0)x, (\bar{\lambda}_7 - 3U_0 - 3W_0)y, x+y\right), \tag{3.21a}$$

$$v(x,y) = V_0 - (9\bar{\lambda}_6 - 18V_0 - 18W_0)\wp_{13}\left((\bar{\lambda}_6 - 2V_0 - 2W_0)x, (\bar{\lambda}_7 - 3U_0 - 3W_0)y, x+y\right), \tag{3.21b}$$

$$w(x,y) = W_0 + (9\bar{\lambda}_7 - 27U_0 - 27W_0)\wp_{23}\left((\bar{\lambda}_6 - 2V_0 - 2W_0)x, (\bar{\lambda}_7 - 3U_0 - 3W_0)y, x+y\right), \tag{3.21c}$$

*where* $\bar{\lambda}_6$, $\bar{\lambda}_7$, $U_0$, $V_0$ *and* $W_0$ *are arbitrary constants in* $\mathbb{C}$.

The proof of Theorem 3.5 is given in Appendix. By virtue of the above theorem, one can further present the analogues, such as Theorems 3.2 and 3.3, of static Veselov-Novikov equation. We here do not give more information for that.

**Remark 3.6.** In Ref. [22], the authors gave the KdV equation by the $\wp$-functions, while the method only got the solutions of single equation. In our method, we can solve the coupled equations and systems of equations.

## 4. Conclusions and remarks

In Ref. [22], the authors gave the KdV equation by the $\wp$-functions, while the method only got the solutions of single equation. In our method, we can solve the coupled equations and systems of equations. Furthermore, we observe that the proposed method generalizes the auxiliary method [35] through which only travelling waves solutions are obtained. In Refs. [43, 44, 46], we give some results to explicitly construct multiperiodic Riemann theta function periodic wave solutions of some discrete soliton equations, nonlinear equations and supersymmetric equations, respectively, which is a lucid and straightforward generalization of the Hirota-Riemann method. In this paper, we present the algorithm for obtain particular solutions of a general (2+1)-dimensional breaking soliton equation (3.1) and static Veselov-Novikov(SVN) equation (3.18), whose solutions can be expressed in terms of the Kleinian hyperelliptic functions of a given curve $y^2 = f(x)$ of $(2g+1)$- and $(2g+2)$- degrees with genus $\mathscr{G}$, respectively. We hope the method could help to better understand the diversity and integrability of nonlinear differential equations in mathematical physics.

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## Appendix A. Proof of Theorem 3.5.

**Proof.** According to the arithmetic given in Sec. 3.1, we prove them in (i), (ii) and (iii), respectively. (i) We make a transformation $u(x,y) = U(\omega_1,\omega_2)$, $v(x,y) = V(\omega_1,\omega_2)$, $w(x,y) = W(\omega_1,\omega_2)$, $\omega_i = a_i x + b_i y$ $(i = 1,2)$, then Eq.(3.18) becomes

$$\begin{aligned}
&a_1^3 U_{\omega_1\omega_1\omega_1} + a_2^3 U_{\omega_2\omega_2\omega_2} + 3a_1^2 a_2 U_{\omega_1\omega_1\omega_2} + 3a_1 a_2^2 U_{\omega_1\omega_2\omega_2} + b_1^3 U_{\omega_1\omega_1\omega_1} + b_2^3 U_{\omega_2\omega_2\omega_2}\\
&+ 3b_1^2 b_2 U_{\omega_1\omega_1\omega_2} + 3b_1 b_2^2 U_{\omega_1\omega_2\omega_2} - 2a_1 U_{\omega_1} V - 2a_2 U_{\omega_2} V - 2a_1 U V_{\omega_1} - 2a_2 U V_{\omega_2}\\
&- 3b_1 U_{\omega_1} W - 3b_2 U_{\omega_2} W - 3U b_1 W_{\omega_1} - 3b_2 U W_{\omega_2} = 0, \;\; a_1 U_{\omega_1} + a_2 U_{\omega_2} - b_1 V_{\omega_1}\\
&- b_2 V_{\omega_2} = 0, \;\; b_1 U_{\omega_1} + b_2 U_{\omega_2} - a_1 W_{\omega_1} - a_2 W_{\omega_2} = 0.
\end{aligned} \tag{A.1}$$

Substituting $U = U_0 + \sum_{j,k=1}^{2} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{2} V_{jk}\wp_{jk}$, $W = W_0 + \sum_{j,k=1}^{2} W_{jk}\wp_{jk}$, and the relations **(I-1)**-**(I-14)** into Eq.(A.1), it becomes overdetermined equations in $\wp_{11}$, $\wp_{12}$ and $\wp_{22}$. Making each coefficient of the overdetermined equations to zero yields equations about $a_i$, $b_i$, $c_i$, $U_0$, $V_0$,$W_0$,

$U_{jk}$, $V_{jk}$ and $W_{jk}$ $(i,j,k$=1,2). Solving the equations, we have

$$a_1 = b_2 = 0,\ \ a_2 = b_1 = 1,\ \ U_0 = U_{11} = U_{22} = V_{11} = V_{12} = W_{12} = W_{22} = 0,$$
$$U_{12} = V_{22} = W_{11} = 2,\ \ V_0 = \frac{1}{3}\left(\lambda_4 - \frac{1}{2}\lambda_1\right),\ \ W_0 = \frac{1}{3}(\lambda_2 - 2\lambda_1), \tag{A.2}$$

where $\lambda_1$, $\lambda_2$ and $\lambda_4$ are arbitrary constants in $\mathbb{C}$. Now substituting the above values in $U = U_0 + \sum_{j,k=1}^{2} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{2} V_{jk}\wp_{jk}$ and $W = W_0 + \sum_{j,k=1}^{2} W_{jk}\wp_{jk}$ we achieve a hyperelliptic function solution (3.19a),(3.19b) and (3.19c) with genus $\mathscr{G} = 2$ of the SVN equation(3.18).

(ii) We make a transformation $u(x,y) = U(\omega_1,\omega_2,\omega_3)$, $v(x,y) = V(\omega_1,\omega_2,\omega_3)$, $w(x,y) = W(\omega_1,\omega_2,\omega_3)$, $\omega_i = a_i x + b_i y$ $(i = 1,2,3)$, then Eq.(3.18) becomes

$$\begin{aligned}&(a_1^3U_{\omega_1\omega_1\omega_1} + a_2^3U_{\omega_2\omega_2\omega_2} + a_3^3U_{\omega_3\omega_3\omega_3} + 3a_1^2a_2U_{\omega_1\omega_1\omega_2} + 3a_1^2a_3U_{\omega_1\omega_1\omega_3} + 3a_1a_2^2U_{\omega_1\omega_2\omega_2}\\ &+ 3a_2^2a_3U_{\omega_2\omega_2\omega_3} + 3a_1a_3^2U_{\omega_1\omega_3\omega_3} + 3a_2a_3^2U_{\omega_2\omega_3\omega_3} + 6a_1a_2a_3U_{\omega_1\omega_2\omega_3}) + (b_1^3U_{\omega_1\omega_1\omega_1} + b_2^3U_{\omega_2\omega_2\omega_2}\\ &+ b_3^3U_{\omega_3\omega_3\omega_3} + 3b_1^2b_2U_{\omega_1\omega_1\omega_2} + 3b_1^2b_3U_{\omega_1\omega_1\omega_3} + 3b_1b_2^2U_{\omega_1\omega_2\omega_2} + 3b_2^2b_3U_{\omega_2\omega_2\omega_3} + 3b_1b_3^2U_{\omega_1\omega_3\omega_3}\\ &+ 3b_2b_3^2U_{\omega_2\omega_3\omega_3} + 6b_1b_2b_3U_{\omega_1\omega_2\omega_3}) - 2(a_1U_{\omega_1} + a_2U_{\omega_2} + a_3U_{\omega_3})V - 2U(a_1V_{\omega_1} + a_2V_{\omega_2} + a_3V_{\omega_3})\\ &- 3(b_1U_{\omega_1} + b_2U_{\omega_2} + b_3U_{\omega_3})W - 3U(b_1W_{\omega_1} + b_2W_{\omega_2} + b_3W_{\omega_3}) = 0,\ \ a_1U_{\omega_1} + a_2U_{\omega_2} + a_3U_{\omega_3}\\ &- b_1V_{\omega_1} - b_2V_{\omega_2} - b_3V_{\omega_3} = 0,\ \ b_1U_{\omega_1} + b_2U_{\omega_2} + b_3U_{\omega_3} - a_1W_{\omega_1} - a_2W_{\omega_2} - a_3W_{\omega_3} = 0.\end{aligned} \tag{A.3}$$

Substituting $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$, $W = W_0 + \sum_{j,k=1}^{3} W_{jk}\wp_{jk}$, and the relations **(II-1)**-**(II-15)** into Eq.(A.3), it becomes overdetermined equations in $\wp_{jk}$ $i,j = 1,2,3$. Making each coefficient of the overdetermined equations to zero yields equations about $a_i$, $b_i$, $c_i$, $U_0$, $V_0$,$W_0$, $U_{jk}$, $V_{jk}$ and $W_{jk}(i,j,k = 1,2,3)$. Solving the equations, we obtain

$$U_{jk} = V_{jk} = W_{jk} = 0,\ \ b_i = -a_i,\ \ V_{33} = -U_{33},\ \ W_{33} = -U_{33},\ \ W_0 = -\frac{5}{3}U_0 + \frac{2}{3}V_0, \tag{A.4}$$

where $((j,k) \neq (3,3))$, $a_1$ ,$a_2$, $a_3$, $U_0$, $V_0$ and $U_{33}$ are arbitrary constants in $\mathbb{C}$. Now substituting the above values in $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$ and $W = W_0 + \sum_{j,k=1}^{3} W_{jk}\wp_{jk}$, we achieve a hyperelliptic function solution (3.20a),(3.20b) and (3.20c) with genus $\mathscr{G} = 3$ of the SVN equation(3.18).

(iii) Similarly, substituting $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$, $W = W_0 + \sum_{j,k=1}^{3} W_{jk}\wp_{jk}$ and the relations **(III-1)**-**(III-15)** into Eq.(A.3), we can achieve overdetermined equations in $\wp_{jk}$ $i,j = 1,2,3$. Making each coefficient of the overdetermined equations to zero yields equations about $a_i$, $b_i$, $c_i$, $U_0$, $V_0$,$W_0$, $U_{jk}$, $V_{jk}$ and $W_{jk}(i,j,k = 1,2,3)$. Solving the equations, we obtain

$$U_{j_1k_1} = V_{j_2k_2} = W_{j_3k_3} = 0,\ \ a_1 = \bar{\lambda}_6 - 2V_0 - 2W_0,\ \ b_2 = \bar{\lambda}_7 - 2U_0 - 2W_0,\ \ U_{33} = 9,$$
$$V_{13} = 9\bar{\lambda}_6 - 18V_0 - 18W_0,\ \ W_{23} = 9\bar{\lambda}_7 - 27U_0 - 27W_0,\ \ b_1 = a_2 = 0,\ \ a_3 = b_3 = 1, \tag{A.5}$$

where $i_1,i_2,i_3,j_1,j_2,j_3,k_1,k_2,k_3 = 1,2,3((j_1,k_1) \neq (3,3), (j_2,k_2) \neq (1,3), (j_3,k_3) \neq (2,3))$, $\bar{\lambda}_6$, $\bar{\lambda}_7$, $U_0$, $V_0$ and $W_0$ are arbitrary constants in $\mathbb{C}$. Now substituting the above values in $U = U_0 + \sum_{j,k=1}^{3} U_{jk}\wp_{jk}$, $V = V_0 + \sum_{j,k=1}^{3} V_{jk}\wp_{jk}$ and $W = W_0 + \sum_{j,k=1}^{3} W_{jk}\wp_{jk}$, we achieve a hyperelliptic function solution (3.21a),(3.21b) and (3.21c) with genus $\mathscr{G} = 3$ of the SVN equation(3.18). □

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