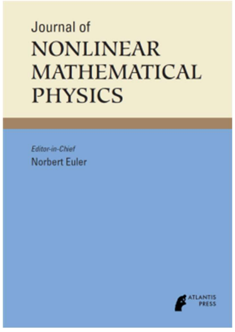

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Periodic Solutions, Stability and Non-Integrability in a Generalized Hénon-Heiles Hamiltonian System

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We consider the Hamiltonian function defined by the cubic polynomial $H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{A}{3}x^3 + Bxy^2 + Dx^2y$, where $A, B, D \in \mathbb{R}$ are parameters and so H is an extension of the well known Hénon-Heiles problem. Our main contribution for $D \neq 0$, $A + B \neq 0$ and other technical restrictions are in three aspects: existence of periodic solutions, stability and instability of these periodic solutions and the problem of non-integrability of the system associated to H. Initially we give sufficient conditions on the three parameters of these generalized Hénon-Heiles systems, which guarantees that at any positive energy level, the Hamiltonian system has periodic orbits. After that, we prove that its stability changes with the values of the parameters. Finally, we show that the generalized Hénon–Heiles systems cannot have any second first integral of class $\mathscr{C}^1$ in the sense of Liouville–Arnol'd. In fact, the parameters where our problem is not integrable in the sense of Liouville–Arnol'd are the same where the periodic orbits were analytically found through averaging theory.

Keywords: Generalized Hénon-Heiles Hamiltonian, periodic orbits, integrability, averaging theory.

2000 *Mathematics Subject Classification*: Primary 58F05; Secondary 70H99.

1. Introduction

The Hénon-Heiles system was originated in Celestial Mechanics as a model in an axially symmetric gravitational potential due to a galaxy. The system can also be regarded as an extension of the harmonic oscillator to the anharmonic case. The Hamiltonian function of this model is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{1}{3}x^3 - xy^2, \tag{1.1}$$

and it was introduced in 1964 in [16].

In this work we consider the cubic polynomial Hamiltonian function

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{A}{3}x^3 + Bxy^2 + Dx^2y, \tag{1.2}$$

where $A, B, D \in \mathbb{R}$ are parameters. By simplicity we will call this problem the generalized Hénon-Heiles Hamiltonian system, since if $A = 1$, $B = -1$ and $D = 0$, we have the classical Hénon-Heiles problem. It was constructed by adding three terms of third degree to the potential of a planar oscillator. Such a model also appears by expanding the potential corresponding to an integrable system, resulting via some canonical transformations applied to the motion of three particles on a circle

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under exponentially decreasing forces, to third degree terms (for more explanation we suggest to review [2] and [5]).

In this work we prove under an analytical point of view the existence of periodic orbits of the Hamiltonian system (1.2) as a function of the parameters A, B and D, but with $A+B \neq 0$ and $D \neq 0$. Our main result on the periodic orbits of the generalized Hénon-Heiles Hamiltonian system (1.2) is summarized in Theorem 3.1. In Theorem 4.1 we summarize our results concerning with the type of stability of each family of periodic solutions obtained previously. Also, in a third direction our study concerns with the problem of non-integrability in the sense of Liouville-Arnol'd (a Hamiltonian system with Hamiltonian H of two degrees of freedom is integrable in the sense of Liouville–Arnol'd if it has a first integral C independent with H) of the generalized Hénon-Heiles Hamiltonian system (1.2). See details in [1, 4] and the subsection 7.1.2 of [4] in a certain region determined by A, B and D, but with $A+B \neq 0$ and $D \neq 0$. The main result on non-integrability of the generalized Hénon-Heiles Hamiltonian system (1.2) can be found in Theorem 5.1. In particular, we have proved that for any second first integral of class C^1 many generalizations of the Hénon-Heiles system are not integrable in the sense of Liouville–Arnol'd.

It is known that the periodic orbits are the most simple non-trivial solutions of an ordinary differential system, and that their study is of particular interest because the motion in a neighborhood can be determined by their type of stability. On the other hand, if the system is non-integrable in the sense of Liouville-Arnol'd, the existence of isolated periodic orbits as in Theorem 3.1 found by the averaging theory in the energy level of the Hamiltonian with multipliers different from 1 is related to the non-existence of any second first integral of class C^1, so the study of these kind of periodic orbits becomes relevant.

Since 1964 lots of researchers have devoted his attention to the Hénon-Heiles problem (1.1). In [16] they were the first in computing numerically the Poincaré map in order to get qualitative information on the Hamiltonian flow. The study of the equilibrium point using important and standard theorems like the ones discussed in [14], [24], [29] and [30] permits us to construct periodic solutions close to the equilibrium point with some pure imaginary eigenvalues under appropriate conditions, but only for energy level sufficiently close to the energy level associated to the equilibrium solution. Other important aspects of this dynamical system like non-trivial periodic orbits and their stability have been systematically studied for several authors (see for example, [10] et al, Rod [27], Davies *et al.* [11] and others [6], [12], [25]. Maciejewski *et al.* in [21] consider an analytic study of more general Hénon-Heiles Hamiltonians including a third cubic term of the form Dx^2y, and two more parameters associated to the quadratic part of the potential. They proved the existence of connected branches of non-stationary periodic orbits in the neighborhood of the equilibrium point $(0,0,0,0)$.

The integrability and non-integrability related to the Hénon-Heiles problem with some parameters and with $D=0$ has been considered by many authors, as for example, [8], [9], [13], [15], [17], [18], [22], [23], [28].

The present article is a self-contained paper and in order to get our main results this work is organized as follows. In Section 2 we deduce the equation of motion, it is introduced a convenient small parameter ε and it is considered a convenient change of coordinates (like polar coordinates) in order to apply the average method of second order established by Buica and Llibre in [7]. Section 3 is devoted to prove the existence of different families of periodic solutions for our model in the case when $A+B \neq 0$ and $D \neq 0$ for any value of the energy level $h > 0$. We obtain one, two or three families according to the variation of the parameters. In Section 4 we characterize the stability or instability of each family of periodic orbits on each Hamiltonian level $h > 0$, again using the averaging method. In Section 5 we consider the problem of non-integrability in the sense of Liouville–Arnold.

Hamiltonian systems are usually classified as integrable and nonintegrable. If the system is integrable then its dynamics is essentially almost periodic and we can say that it is "well-behaved", but it does not mean that its dynamics is simple. In the nonintegrable case the dynamics is much more complicated, and a usual approach is to consider the nonintegrable system as some perturbation of an integrable one. Thus by practical importance and applications of the Hamiltonian systems is crucial to decide if a given system is or is not integrable. Related to integrability theory, it is known that Ziglin theory was extended to the so-called Morales–Ramis theory ([22]), which replaces the study of the monodromy group of the variational equations by the study of their Galois differential group, which is easier to analyze. See [22] for more details and the references therein, but as Ziglin's theory, Morales-Ramiss theory can only study the non-existence of meromorphic first integrals. As was described in ([19]) Poincaré's method allows us to prove under convenient assumptions that the non- Liouville-Arnold integrable systems do not have any second first integral of class C^1. We will apply the Poincaré criterion to the motion of the generalized Hénon–Heiles system (1.2) and show that its motion is integrable and the two constants of motion have dependent gradients along the periodic orbits found in Theorem 3.1, or that it is not Liouville–Arnold integrable with any second first integral of class C^1. In order to apply the Poincaré non-integrability theory to the generalized Hénon-Heiles system (1.2), we need to study some of the periodic orbits of these systems and compute their multipliers. The averaging theory is used to get these periodic orbits. As far as we known, the results of the previous theorems are new in the case $D \neq 0$. In [19] the case $D = 0$ was studied under the same point of view, but there the expressions are simpler than our case.

2. Statement of the problem and equations of motion

The equations of motion associated to the system (1.2) are

$$\begin{aligned} \dot{x} &= p_x, \\ \dot{p}_x &= -x - (Ax^2 + By^2 + 2Dxy), \\ \dot{y} &= p_y, \\ \dot{p}_y &= -y - (2Bxy + Dx^2), \end{aligned} \tag{2.1}$$

where $A, B, D \in \mathbb{R}$.

Firstly, we introduce a small parameter ε in the Hamiltonian system (2.1) by the change of variables (x, y, p_x, p_y) to (X, Y, p_X, p_Y) where $x = \varepsilon X$, $y = \varepsilon Y$, $p_x = \varepsilon p_X$ and $p_y = \varepsilon p_Y$ which is ε^{-2}-symplectic. So the system (2.1) becomes

$$\begin{aligned} \dot{X} &= p_X, \\ \dot{p}_X &= -X - \varepsilon(AX^2 + BY^2 + 2DXY), \\ \dot{Y} &= p_Y, \\ \dot{p}_Y &= -Y - \varepsilon(2BXY + DX^2). \end{aligned} \tag{2.2}$$

The Hamiltonian function associated to the previous system is

$$K = \frac{1}{2}(p_X^2 + p_Y^2) + \frac{1}{2}(X^2 + Y^2) + \varepsilon[\frac{A}{3}X^3 + BXY^2 + DX^2Y]. \tag{2.3}$$

By the standard theory of Hamiltonian dynamical systems, for all ε different from zero, the original and the transformed systems (2.1) and (2.2) have essentially the same phase portrait, and the additionally system (2.2) for ε sufficiently small is close to an integrable one

Now, we introduce the convenient polar coordinates by

$$X = r\cos\theta, \quad p_X = r\sin\theta, \quad Y = \rho\cos(\theta+\alpha), \quad p_Y = \rho\sin(\theta+\alpha).$$

Recall that this is a change of variables where $r > 0$ and $\rho > 0$. Now, we observe that the fixed value of the energy in polar coordinates is

$$h = \frac{1}{2}(r^2+\rho^2) + \varepsilon\left[\frac{A}{3}r^3\cos^3\theta + Br\rho^2\cos\theta\cos^2(\theta+\alpha) + Dr^2\rho\cos^2\theta\cos(\theta+\alpha)\right], \tag{2.4}$$

and the equations of motion assume the form

$$\begin{aligned}
\dot{r} &= -\varepsilon\sin\theta\left[A\,r^2\cos^2\theta + B\,\rho^2\cos^2(\theta+\alpha)2Dr\rho\cos\theta\cos(\theta+\alpha)\right],\\
\dot{\theta} &= -1-\varepsilon\cos\theta\left[A\,r\cos^2\theta + B\tfrac{\rho^2}{r}\cos^2(\theta+\alpha) + 2D\rho\cos\theta\cos(\theta+\alpha)\right],\\
\dot{\rho} &= -\varepsilon\, r\cos\theta\sin(\theta+\alpha)\left[2B\rho\cos(\theta+\alpha) + Dr\cos\theta\right],\\
\dot{\alpha} &= \varepsilon\cos\theta\left[A\,r\cos^2\theta + B\left(\tfrac{\rho^2}{r}-2r\right)\cos^2(\theta+\alpha) - D\left(\tfrac{r^2}{\rho}-2\rho\right)\cos\theta\cos(\theta+\alpha)\right].
\end{aligned} \tag{2.5}$$

In order to put our system as a periodic ordinary differential equations, we change to the θ variable as the independent one, and we use the prime notation for the derivative with respect to θ. The angular variable α can not be used as the independent variable since the new differential system would not have the appropriate form as in [7] to apply Theorem 3.1.

Of course this system has now only three equations because we do not need the θ equation. If we write the previous system as a Taylor series in powers of ε, we arrive to

$$\begin{aligned}
r' &= \varepsilon\sin\theta\left[A\,r^2\cos^2\theta + B\rho^2\cos^2(\theta+\alpha) + 2Dr\rho\cos(\theta)\cos(\theta+\alpha)\right] -\\
&\quad \varepsilon^2\,\tfrac{\sin(2\theta)}{2}\left[A\,r\cos^2(\theta) + 2D\rho\cos(\theta)\cos(\theta+\alpha) + B\tfrac{\rho^2}{r}\cos^2(\theta+\alpha)\right]^2 + O(\varepsilon^3),\\
\rho' &= \varepsilon\cos\theta\sin(\theta+\alpha)\left[2B\rho\cos(\theta+\alpha) + Dr\cos\theta\right]\{r-\\
&\quad \varepsilon\cos\theta\left[Ar^2\cos^2\theta + B\rho^2\cos^2(\theta+\alpha) + 2Dr\rho\cos\theta\cos(\theta+\alpha)\right]\} + O(\varepsilon^3),\\
\alpha' &= -\varepsilon\cos\theta\left[A\,r\cos^2\theta + B(\tfrac{\rho^2}{r}-2r)\cos^2(\theta+\alpha) - D(\tfrac{r^2}{\rho}-2\rho)\cos\theta\cos(\theta+\alpha)\right]\\
&\quad +\varepsilon^2\cos\theta\left[\tfrac{A}{\rho}\cos^2\theta + \tfrac{B}{r^2}\cos^2(\theta+\alpha) + 2\tfrac{D}{r}\cos\theta\cos(\theta+\alpha)\right]\cdot\\
&\quad \left[A\cos^2\theta + B(\rho^2-2r^2)\cos^2(\theta+\alpha) - Dr(r^2-2\rho^2)\cos\theta\cos(\theta+\alpha)\right] + O(\varepsilon^3).
\end{aligned} \tag{2.6}$$

Therefore, the system (2.6) is 2π-periodic in the variable θ. In order to apply Theorem 3.1. in [7] we will fix the value of the first integral at $h > 0$, and by solving equation (2.4) for ρ we obtain

$$\begin{aligned}
\rho = {} & \left[-3\,\varepsilon Dr^2\cos^2\theta\cos(\theta+\alpha) + \tfrac{1}{2}\{36\varepsilon^2D^2r^4\,\cos^4\theta\cos^2(\theta+\alpha) +\right.\\
& \left. 12(6h-3r^2-2\,\varepsilon Ar^3\cos^3\theta)(1+2\,\varepsilon Br\cos\theta\cos^2(\theta+\alpha)\}^{1/2}\right]\\
& \left[3+6Br\cos\theta\cos^2(\theta+\alpha)\right]^{-1}.
\end{aligned} \tag{2.7}$$

Thus, expanding in Taylor series we obtain

$$
\begin{aligned}
\rho = {} & \sqrt{2h-r^2} - \varepsilon \tfrac{r\cos\theta}{3\sqrt{2h-r^2}}\big[Ar^2\cos^2\theta + 3Dr\sqrt{2h-r^2}\cos\theta\cos(\theta+\alpha) - \\
& 3B(-2h+r^2)\cos^2(\theta+\alpha)\big] + \varepsilon^2 \tfrac{r^2\cos^2\theta}{18(2h-r^2)^{3/2}}\big[-A^2r^4\cos^4\theta - \\
& 3(2AB+3D^2)r^2(-2h+r^2)\cos^2\theta\cos^2(\theta+\alpha) + 36BDr\cdot \\
& (2h-r^2)^{3/2}\cos\theta\cos^3(\theta+\alpha) + 27B^2(-2h+r^2)^2\cos^4(\theta+\alpha)\big].
\end{aligned}
\tag{2.8}
$$

Using this value of ρ in equations (2.6), we obtain the following system of differential equations

$$
\begin{aligned}
r' = {} & \varepsilon\sin\theta\big[Ar^2\cos^2(\theta) - 2Dr\sqrt{2h-r^2}\cos\theta\cos(\theta+\alpha) + \\
& B(2h-r^2)\cos^2(\theta+\alpha)\big] + \varepsilon^2\tfrac{\cos(\theta)\sin(\theta)}{3r\sqrt{2h-r^2}}\big[-3A^2r^4\sqrt{2h-r^2}\cos(\theta)^4 + \\
& 2ADr^3(12h-5r^2)\cos(\theta)^3\cos(\theta+\alpha) + 2r^2\sqrt{2h-r^2}(3D^2(-4h+r^2) + \\
& 2AB(-3h+r^2))\cos^2(\theta)\cos^2(\theta+\alpha) + 24BDhr(2h-r^2)\cos(\theta)\cos^3(\theta+\alpha) + \\
& 3B^2\sqrt{2h-r^2}(-4h^2+r^4)\cos^4(\theta+\alpha) + O(\varepsilon^3) \\
\alpha' = {} & \varepsilon\cos(\theta)\big[-Ar\cos^2(\theta) + \tfrac{D(4h-3r^2)\cos(\theta)\cos(\theta+\alpha)}{\sqrt{2h-r^2}} + \tfrac{B(-2h+3r^2)\cos^2(\theta+\alpha)}{r}\big] - \\
& \varepsilon^2\tfrac{\cos^2(\theta)}{3r^2(2h-r^2)^{3/2}}\big[-3A^2r^4(2h-r^2)^{3/2}\cos^4(\theta) + 2ADr^3(24h^2-25hr^2+7r^4) \\
& \cos^3(\theta)\cos(\theta+\alpha) - r^2\sqrt{2h-r^2}\big\{(24(AB+2D^2)h^2 - 16(2AB+3D^2)hr^2 + \\
& 5(2AB+3D^2)r^4)\cos^2(\theta)\cos^2(\theta+\alpha)\big\} - 6BDr(-2h+r^2)(8h^2-9hr^2+ \\
& 3r^4)\cos(\theta)\cos^3(\theta+\alpha) - 3B^2(2h-r^2)^{7/2}\cos^4(\theta+\alpha)\big] + O(\varepsilon^3).
\end{aligned}
\tag{2.9}
$$

3. Existence of periodic solutions

For fixed $h > 0$, it is not difficult to check that the system (2.9) satisfies all the assumptions of the averaging theory of second order to find periodic orbits. In fact, we will apply Theorem 3.1 developed in [7] (see this paper for additional details and for the proofs of the results stated in this section related to averaging theory).

In this work we only need this theory up to second order. This is summarized as follows. Using the notation $x = (r,\alpha) \in D = (0,\sqrt{2h}) \times \mathbb{R}$ and $t = \theta$, and it has the convenient form as in Theorem 3.1 in [7], with $F_1 = (F_{11}, F_{12})$ and $F_2 = (F_{21}, F_{22})$, defined by

$$
\begin{aligned}
F_{11} = {} & [Ar^2\cos^2\theta + 2Dr\sqrt{2h-r^2}\cos\theta\cos(\theta+\alpha) + \\
& B(2h-r^2)\cos^2(\theta+\alpha)]\sin\theta, \\
F_{12} = {} & \cos\theta\left[-Ar\cos^2\theta + \tfrac{D(-4h+3r^2)\cos\theta\cos(\theta+\alpha)}{\sqrt{2h-r^2}} + \tfrac{B(-2h+3r^2)\cos^2(\theta+\alpha)}{r}\right]
\end{aligned}
\tag{3.1}
$$

and

$$\begin{aligned}
F_{21} = & \tfrac{\sin(2\theta)}{6r\sqrt{2h-r^2}}\big[-3A^2r^4\sqrt{2h-r^2}\cos^4\theta+2ADr^3(-12h+5r^2)\\
& \cos^3\theta\cos(\theta+\alpha)+2r^2\sqrt{2h-r^2}\big(3D^2(-4h+r^2)+2AB(-3h+r^2)\big)\\
& \cos^2\theta\cos^2(\theta+\alpha)+24BDhr(-2h+r^2)\cos\theta\cos^3(\theta+\alpha)+\\
& 3B^2\sqrt{2h-r^2}(-4h^2+r^4)\cos^4(\theta+\alpha)\big],\\
F_{22} = & \tfrac{\cos^2\theta}{3(-2hr+r^3)^2}\big[3A^2r^4(-2h+r^2)^2\cos^4\theta+2ADr^3\sqrt{2h-r^2}(24h^2-25hr^2+\\
& 7r^4)\cos^3\theta\cos(\theta+\alpha)-r^2(-2h+r^2)\big(24(AB+2D^2)h^2-\\
& 16(2AB+3D^2)hr^2+5(2AB+3D^2)r^4\big)\cos^2\theta\cos^2(\theta+\alpha)+\\
& 6BDr(2h-r^2)^{3/2}(8h^2-9hr^2+3r^4)\cos\theta\cos^3(\theta+\alpha)+\\
& 3B^2(-2h+r^2)^4\cos^4(\theta+\alpha)\big].
\end{aligned} \tag{3.2}$$

As $r \neq 0$ the functions F_1 and F_2 are analytical and they are 2π-periodic in θ, which will be the independent variable of system (2.9). After some computations we observe that the averaging theory of first order does not apply because the average functions of F_1 and F_2 vanish in the period

$$f_1(r,\alpha) = \int_0^{2\pi}(F_{11},F_{12})\,d\theta = (0,0).$$

On the other hand, the function f_1 in [7] is zero, then we proceed to calculate the function f_2 by applying the second order averaging theory. It is verified that f_2 is defined by

$$f_2(r,\alpha) = \int_0^{2\pi}[D_{r\alpha}F_1(\theta,r,\alpha).y_1(\theta,r,\alpha)+F_2(\theta,r,\alpha)]\,d\theta, \tag{3.3}$$

where

$$y_1(\theta,r,\alpha) = \int_0^{\theta}F_1(t,r,\alpha)\,dt.$$

In particular, the two components of the vector function y_1 are

$$\begin{aligned}
y_{11} &= \textstyle\int_0^\theta F_{11}(t,r,\alpha)\,dt\\
&= \tfrac{1}{6}\big[-2Ar^2(-1+\cos^3\theta)+Dr\sqrt{2h-r^2}(4\cos\alpha-3\cos(\alpha-\theta)-\\
&\quad \cos(\alpha+3\theta))+4Bh(3+\cos(2(\alpha+\theta))+2\cos(2\alpha+\theta])\sin^2(\tfrac{\theta}{2})-\\
&\quad 2Br^2(3+\cos(2(\alpha+\theta))+2\cos(2\alpha+\theta))\sin^2(\tfrac{\theta}{2})\big],
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
y_{12} &= \textstyle\int_0^\theta F_{12}(t,r,\alpha)\,dt\\
&= -\tfrac{1}{12}\big[Ar(9\sin\theta+\sin(3\theta))-\tfrac{4Dh}{\sqrt{2h-r^2}}\big(-4\sin\alpha-3\sin(\alpha-\theta)+\\
&\quad 6\sin(\alpha+\theta)+\sin(\alpha+3\theta)\big)+\tfrac{3Dr^2}{\sqrt{2h-r^2}}\big(-4\sin\alpha-3\sin(\alpha-\theta)+\\
&\quad 6\sin(\alpha+\theta)+\sin(\alpha+3\theta)\big)-\tfrac{2Bh}{r}\big(-4\sin\alpha+6\sin(\theta)+3\sin(2\alpha+\theta)+\\
&\quad \sin(2\alpha+3\theta)\big)+Br\big(-4\sin(2\alpha)+6\sin\theta+3\sin(2\alpha+\theta)+\\
&\quad \sin(2\alpha+3\theta)\big)\big].
\end{aligned} \tag{3.5}$$

Using Theorem 3.1 in [7] we arrive to the fact that the function $f_2 = (f_{21}, f_{22})$ is given by

$$
\begin{aligned}
f_{21} = {} & \tfrac{\sin\alpha}{12}\big[5D\sqrt{2h-r^2}\left(2Bh+Ar^2\right)+2\left(AB-6(B^2+D^2)\right)\cdot \\
& r(-2h+r^2)\cos\alpha\big] \\
f_{22} = {} & -\tfrac{1}{12r(-2h+r^2)}\big[r\big\{4h^2(6AB-B^2+4D^2)+2hr^2(5A^2-18AB-2B^2- \\
& 7D^2)-r^4(5A^2-12AB-3B^2-3D^2)\big\}+10Dh\sqrt{2h-r^2}\cdot \\
& \big(2Bh+r^2(3A-2B)\big)\cos\alpha+2r(-2h^2+3hr^2-r^4)(AB- \\
& 6(B^2+D^2))\cos(2\alpha)\big].
\end{aligned}
\tag{3.6}
$$

The next step is to find the zeros (r^*, α^*) of $f_2(r,\alpha)$, and then we need to check that the Jacobian determinant

$$|D_{r,\alpha} f_2(r^*, \alpha^*)| \tag{3.7}$$

is non null. We observe that the components of $D_{r,\alpha} f_2(r,\alpha)$ are

$$
\begin{aligned}
\tfrac{\partial f_{21}}{\partial r} = {} & -\tfrac{1}{12}\sin(\alpha)\big(4ABh\cos(\alpha)-6ABr^2\cos(\alpha)-24B^2h\cos(\alpha)+ \\
& 36B^2r^2\cos(\alpha)-24D^2h\cos(\alpha)+36D^2r^2\cos(\alpha)+ \\
& 10ADr\sqrt{2h-r^2}-\tfrac{5ADr^3}{\sqrt{2h-r^2}}-\tfrac{10BDhr}{\sqrt{2h-r^2}}\big) \\
\tfrac{\partial f_{21}}{\partial \alpha} = {} & -\tfrac{1}{12}\sin(\alpha)\big(-4ABhr\sin(\alpha)+2ABr^3\sin(\alpha)+24B^2hr\sin(\alpha)- \\
& 12B^2r^3\sin(\alpha)+24D^2hr\sin(\alpha)-12D^2r^3\sin(\alpha)\big)- \\
& \tfrac{1}{12}\cos(\alpha)\big(4ABhr\cos(\alpha)-2ABr^3\cos(\alpha)-24B^2hr\cos(\alpha)+ \\
& 12B^2r^3\cos(\alpha)-24D^2hr\cos(\alpha)+12D^2r^3\cos(\alpha)+5ADr^2\sqrt{2h-r^2}+ \\
& 10BDh\sqrt{2h-r^2}\big) \\
\tfrac{\partial f_{22}}{\partial r} = {} & \tfrac{1}{12r(2h-r^2)}\big[-4ABh^2\cos(2\alpha)+18ABhr^2\cos(2\alpha)-10ABr^4\cos(2\alpha)- \\
& 60ADhr\cos(\alpha)\sqrt{2h-r^2}-\tfrac{20ADr^5\cos(\alpha)}{\sqrt{2h-r^2}}+80ADr^3\cos(\alpha)\sqrt{2h-r^2}+ \\
& \tfrac{30ADhr^3\cos(\alpha)}{\sqrt{2h-r^2}}+24B^2h^2\cos(2\alpha)-108B^2hr^2\cos(2a)+ \\
& 60B^2r^4\cos(2a)+\tfrac{20BDh^2r\cos(\alpha)}{\sqrt{2h-r^2}}+40BDhr\cos(\alpha)\sqrt{2h-r^2}- \\
& \tfrac{20BDhr^3\cos(\alpha)}{\sqrt{2h-r^2}}+24D^2h^2\cos(2a)-108D^2hr^2\cos(2\alpha)+ \\
& 60D^2r^4\cos(2\alpha)+30A^2hr^2-25A^2r^4+24ABh^2-108ABhr^2+ \\
& 60ABr^4-4B^2h^2-12B^2hr^2+15B^2r^4+16D^2h^2-42D^2hr^2+ \\
& 15D^2r^4\big]+\tfrac{1}{6(2h-r^2)^2}\big[-4ABh^2r\cos(2\alpha)+6ABhr^3\cos(2\alpha)- \\
& 2ABr^5\cos(2a)-30ADhr^2\cos(a)\sqrt{2h-r^2}+ \\
& 20ADr^4\cos(\alpha)\sqrt{2h-r^2}+24B^2h^2r\cos(2\alpha)- \\
& 36B^2hr^3\cos(2\alpha)+12B^2r^5\cos(2\alpha)-20BDh^2\cos(\alpha)\sqrt{2h-r^2}+ \\
& 20BDhr^2\cos(\alpha)\sqrt{2h-r^2}+24D^2h^2r\cos(2\alpha)-36D^2hr^3\cos(2\alpha)+
\end{aligned}
$$

$$12D^2r^5\cos(2\alpha)+10A^2hr^3-5A^2r^5+24ABh^2r-36ABhr^3+12ABr^5-$$
$$4B^2h^2r-4B^2hr^3+3B^2r^5+16D^2h^2r-14D^2hr^3+3D^2r^5\big]-$$
$$\tfrac{1}{12r^2(2h-r^2)}\big[-4ABh^2r\cos(2\alpha)+6ABhr^3\cos(2\alpha)-2ABr^5\cos(2\alpha)-$$
$$30ADhr^2\cos(\alpha)\sqrt{2h-r^2}+20ADr^4\cos(\alpha)\sqrt{2h-r^2}+24B^2h^2r\cos(2\alpha)-$$
$$36B^2hr^3\cos(2\alpha)+12B^2r^5\cos(2\alpha)-20BDh^2\cos(\alpha)\sqrt{2h-r^2}+$$
$$20BDhr^2\cos(\alpha)\sqrt{2h-r^2}+24D^2h^2r\cos(2\alpha)-36D^2hr^3\cos(2\alpha)+$$
$$12D^2r^5\cos(2\alpha)+10A^2hr^3-5A^2r^5+24ABh^2r-36ABhr^3+$$
$$12ABr^5-4B^2h^2r-4B^2hr^3+3B^2r^5+16D^2h^2r-14D^2hr^3+3D^2r^5\big]$$

$$\tfrac{\partial f_{22}}{\partial\alpha}=\tfrac{1}{12r\left(2h-r^2\right)}\big(8ABh^2r\sin(2\alpha)-12ABhr^3\sin(2\alpha)+4ABr^5\sin(2\alpha)+$$
$$30ADhr^2\sin(\alpha)\sqrt{2h-r^2}-20ADr^4\sin(\alpha)\sqrt{2h-r^2}-48B^2h^2r\sin(2\alpha)+$$
$$72B^2hr^3\sin(2\alpha)-24B^2r^5\sin(2\alpha)+20BDh^2\sin(\alpha)\sqrt{2h-r^2}-$$
$$20BDhr^2\sin(\alpha)\sqrt{2h-r^2}-48D^2h^2r\sin(2\alpha)+72D^2hr^3\sin(2\alpha)-$$
$$24D^2r^5\sin(2\alpha)\big)$$

We verify that the system $f_2(r,\alpha)=(0,0)$ has 12 solutions but 9 of them satisfy $r^*>0$ which exist if and only if $A+B\neq 0$ and $D\neq 0$ and they are the following:

(1) For $D>0$ and $h>0$:

$$\begin{aligned}
P_1^+ &= (r^*=\tfrac{\sqrt{2h}D}{\sqrt{(A+B)^2+D^2}},\alpha=0)\\
P_2^+ &= (r^*=\tfrac{\sqrt{2h}D}{\sqrt{(A+B)^2+D^2}},\alpha^*=-\pi)\\
P_3^+ &= (r^*=\tfrac{\sqrt{2h}D}{\sqrt{(A+B)^2+D^2}},\alpha^*=\pi),
\end{aligned}\tag{3.8}$$

(2) For $D<0$ and $h>0$:

$$\begin{aligned}
P_1^- &= (r^*=-\tfrac{\sqrt{2h}D}{\sqrt{(A+B)^2+D^2}},\alpha^*=0)\\
P_2^- &= (r^*=-\tfrac{\sqrt{2h}D}{\sqrt{(A+B)^2+D^2}},\alpha^*=-\pi)\\
P_3^- &= (r^*=-\tfrac{\sqrt{2h}D}{\sqrt{(A+B)^2+D^2}},\alpha^*=\pi).
\end{aligned}\tag{3.9}$$

After some algebraic manipulations, we obtain that the Jacobian (we will use the notation $J(P)$ to say that the Jacobian of (3.7) is evaluated in P) in any case satisfies:

$$J(P_1^+)=d_1d_2d_3 \tag{3.10}$$

where

$$d_1 = -\frac{5h^2}{36}\,\frac{1}{|A+B|[(A+B)^2+D^2]^3},$$

$$d_2 = 5|A+B|(AB+B^2+D^2)+2[A^2B-A(5B^2+6D^2)-6B(B^2+D^2)]$$

$$\begin{aligned} d_3 = {} & 2D^2(A+B)|A+B|\big[A^4-2A^2(3B^2+D^2)-8AB(B^2+D^2)- \\ & 3(B^2+D^2)^2\big]+(A^2+2AB+B^2+D^2)\big[A^5B+A^4(5B^2-3D^2)+ \\ & 2A^3(5B^3-3BD^2)+2A^2(5B^4+3D^4)+A(5B^5+6B^3D^2+9BD^4)+ \\ & (B^2+D^2)^3\big] \end{aligned}$$

and

$$J(P_2^+) = d_1 d_4 d_5 \tag{3.11}$$

where

$$d_4 = 5|A+B|(AB+B^2+D^2)-2[A^2B-A(5B^2+6D^2)-6B(B^2+D^2)]$$

$$\begin{aligned} d_5 = {} & -2D^2(A+B)|A+B|\big[A^4-2A^2(3B^2+D^2)-8AB(B^2+D^2)- \\ & 3(B^2+D^2)^2\big]+(A^2+2AB+B^2+D^2)\big[A^5B+A^4(5B^2-3D^2)+ \\ & 2A^3(5B^3-3BD^2)+2A^2(5B^4+3D^4)+A(5B^5+6B^3D^2+9BD^4)+ \\ & (B^2+D^2)^3\big]. \end{aligned}$$

It is verified that

$$J(P_3^+) = J(P_1^-) = J(P_2^+), \tag{3.12}$$

and

$$J(P_2^-) = J(P_3^-) = J(P_1^+), \tag{3.13}$$

and the value does not depends on the sign on D since the Jacobian function is even in D.

In Figure 1 (resp.2) we show the region where $J(P_1^+) \neq 0$ with $A+B>0$ (resp. $A+B<0$) for $D>0$.

In Figure 3 (resp.4) we show the region where $J(P_2^+) \neq 0$ with $A+B>0$ (resp. $A+B<0$) for $D>0$.

Summarizing, for $D>0$ the solution P_1^+ of $f_2(r,\alpha)=0$ provides a periodic orbit of the system (2.9) (and consequently of the Hamiltonian system (2.2) on the Hamiltonian level $h>0$) if $J(P_1^+) \neq 0$. In particular, there is one periodic orbit if the parameters A,B,D satisfy $J(P_1^+) \neq 0$.

Analogously, again for $D>0$, the solution P_2^+ and P_3^+ of $f_2(r,\alpha)=0$ provides a periodic orbit of the system (2.9) (and consequently of the Hamiltonian system (2.2) on the Hamiltonian level $h>0$) if $J(P_2^+) \neq 0$. In particular, there are two periodic orbits if the parameters A,B,D satisfy $J(P_2^+) \neq 0$.

Under the conditions $J(P_1^+) \neq 0$ and $J(P_2^+) \neq 0$ (see Figure 5-6), with $A+B \neq 0$ and $D>0$, the solutions P_1^+, P_2^+ and P_3^+ provide three periodic orbits.

In a similar way for $D<0$ the solution P_1^- of $f_2(r,\alpha)=0$ provides a periodic orbit of system (2.9) (and consequently of the Hamiltonian system (2.2) on the Hamiltonian level $h>0$) if $J(P_2^+) \neq 0$. In particular, there is one periodic orbit if the parameters A,B,D satisfy $J(P_2^+) \neq 0$.

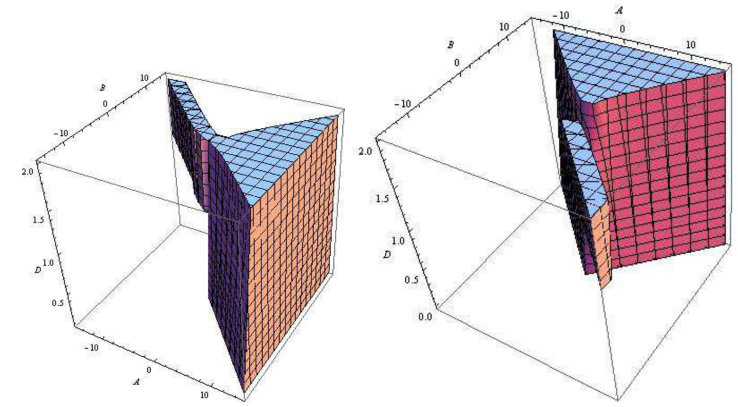

Fig. 1. Region where $J(P_1^+) \neq 0$ for $A+B>0$ and $D>0$ as function of the parameters A, B and D. Left: $J(P_1^+)>0$. Right: $J(P_1^+)<0$.

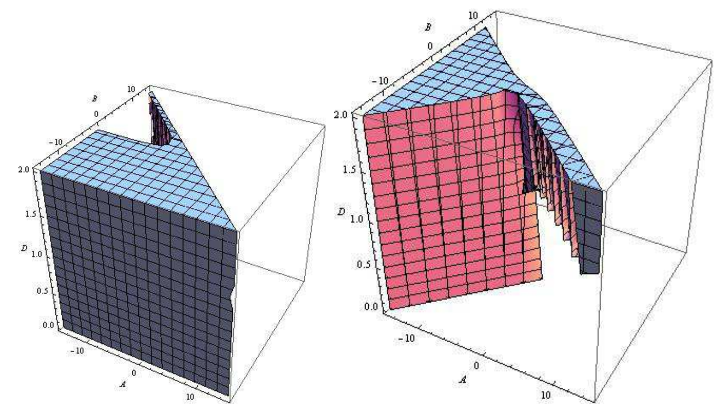

Fig. 2. Region where $J(P_1^+) \neq 0$ for $A+B<0$ and $D>0$ as function of the parameters A, B and D. Left: $J(P_1^+)>0$. Right: $J(P_1^+)<0$.

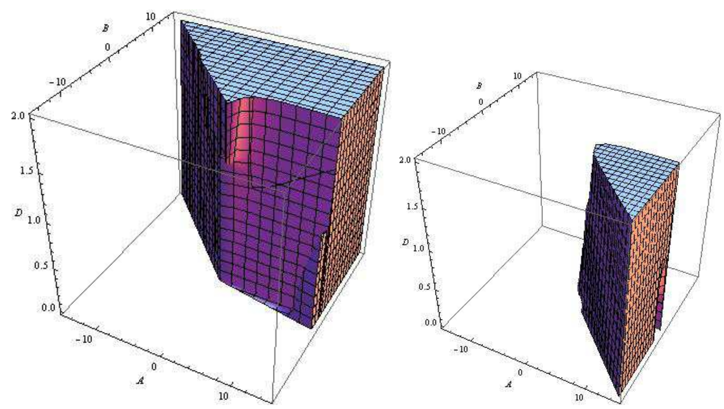

Fig. 3. Region where $J(P_2^+) \neq 0$ for $A+B>0$ and $D>0$ as function of the parameters A, B and D. Left: $J(P_2^+)>0$. Right: $J(P_2^+)<0$.

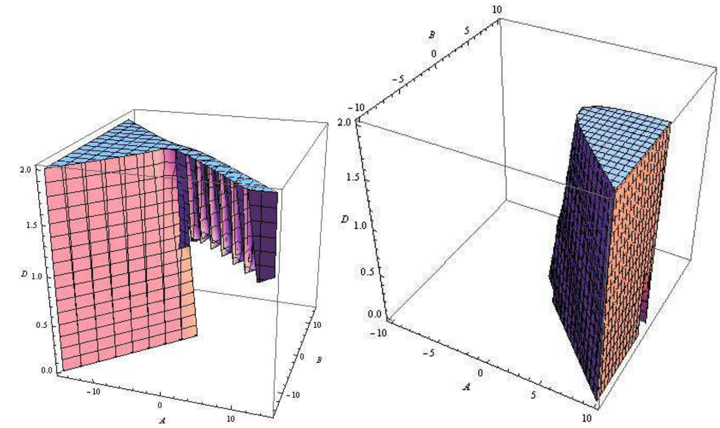

Fig. 4. Region where $J(P_2^+) \neq 0$ for $A+B>0$ and $D>0$ as function of the parameters A, B and D. Left: $J(P_2^+)>0$. Right: $J(P_2^+)<0$.

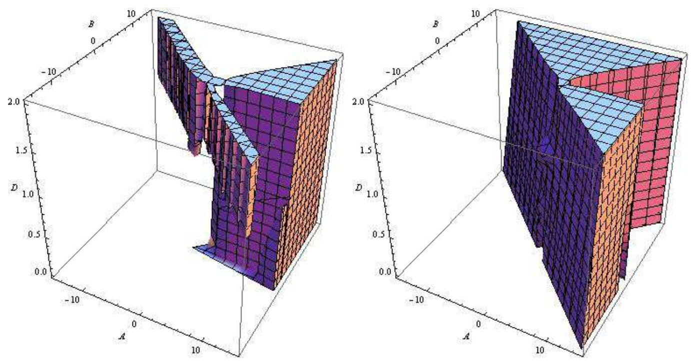

Fig. 5. Left: Region where $J(P_1^+)\cdot J(P_2^+)>0$. Right: Region where $J(P_1^+)\cdot J(P_2^+)<0$ in the case $A+B>0$.

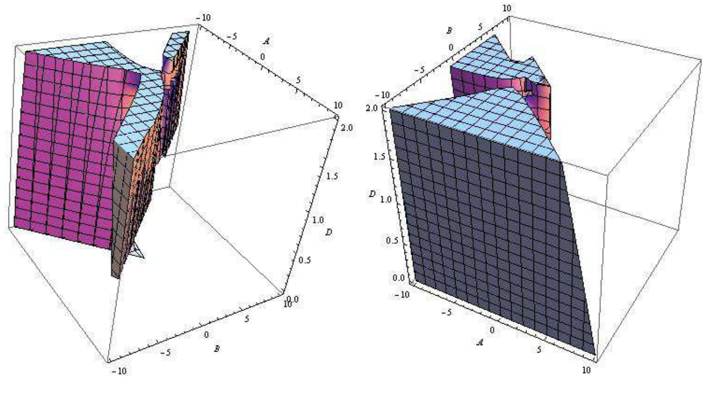

Fig. 6. Left: Region where $J(P_1^+)\cdot J(P_2^+)>0$. Right: Region where $J(P_1^+)\cdot J(P_2^+)<0$ in the case $A+B<0$.

Analogously, still in the case $D<0$, the solution P_2^- and P_3^- of $f_2(r,\alpha)=0$ provides a periodic orbit of system (2.9) (and consequently of the Hamiltonian system (2.2) on the Hamiltonian level

$h > 0$) if $J(P_1^+) \neq 0$. In particular, there are two periodic orbits if the parameters A, B, D satisfy $J(P_1^+) \neq 0$.

Theorem 3.1 in [7] guarantees for $\varepsilon \neq 0$ sufficiently small the existence of a 2π-periodic orbit corresponding to the point (r^*, α^*) of the form $(r(\theta, \varepsilon), \alpha(\theta, \varepsilon))$ for system (2.9) such that $(r(0, \varepsilon), \alpha(0, \varepsilon)) \to (r^*, \alpha^*)$ when $\varepsilon \to 0$. Then $\rho = \rho(\theta)$ defined in (2.8) is also 2π-periodic. Therefore system (2.2) has the 2π- periodic solution

$$\begin{aligned} X(\theta,\varepsilon) &= r(\theta,\varepsilon)\cos\theta, \\ Y(\theta,\varepsilon) &= \rho(\theta,\varepsilon)\cos(\theta+\alpha(\theta,\varepsilon)), \\ P_X(\theta,\varepsilon) &= r(\theta,\varepsilon)\sin\theta, \\ P_Y(\theta,\varepsilon) &= \rho(\theta,\varepsilon)\sin(\theta+\alpha(\theta,\varepsilon)), \end{aligned} \tag{3.14}$$

for ε small enough. Finally, for $\varepsilon \neq 0$ sufficiently small the Hamiltonian system associated the generalized Hénon–Heiles has a periodic solution by

$$(x(\theta,\varepsilon), y(\theta,\varepsilon), p_x(\theta,\varepsilon), p_y(\theta,\varepsilon)) = (\varepsilon X(\theta,\varepsilon), \varepsilon Y(\theta,\varepsilon), \varepsilon P_X(\theta,\varepsilon), \varepsilon P_Y(\theta,\varepsilon)), \tag{3.15}$$

which goes to the origin of coordinates when $\varepsilon \to 0$.

Thus, our main result on the periodic orbits of the generalized Hénon–Heiles system (1.2) is summarized as follows.

Theorem 3.1. *Assume that $D > 0$ and $A + B \neq 0$, then at every positive energy level, the generalized Hénon–Heiles Hamiltonian system* (1.2) *has at least*

(1) one periodic orbit if $J(P_1^+) \neq 0$ (see Figures 1-2),
(2) two periodic orbits if $J(P_2^+) \neq 0$ (see Figures 3-4),
(3) three periodic orbits if $J(P_1^+) \neq 0$ and $J(P_2^+) \neq 0$ (see Figures 5-6).

A similar result is true in the case $D < 0$.

Remark 3.1. It is known that if the Hamiltonian is of the form

$$\frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{A}{3}x^3 + Bxy^2 + Dx^2y + Ey^3,$$

there is a symplectic change of coordinates where in the potential we can eliminate the x^2y term, but it is verified that the term in y^3 appears explicitly. On the other hand, if $D = 0$ and $E \neq 0$ we have verified that our arguments cannot be applied, since the computations are very difficult.

4. Study of the stability of the periodic solutions

By Theorem 3.1 we have that the periodic solutions are of the form

$$\begin{aligned} x(t) &= \varepsilon r^* \cos(t) + O(\varepsilon^2), & p_x(t) &= \varepsilon r^* \sin(t) + O(\varepsilon^2) \\ y(t) &= \varepsilon \rho^* \cos(t + \alpha^*) + O(\varepsilon^2), & p_y(t) &= \varepsilon \rho^* \sin(t + \alpha^*) + O(\varepsilon^2), \end{aligned} \tag{4.1}$$

where r^*, ρ^*, α^* are defined in (3.8)-(3.9), so they are functions of the parameters h, A, B, D, and ε is a small parameter. Our objective in this section is to give information about the stability or instability of each family of periodic solutions obtained in Theorem 3.1.

Since in our case $f_1 \equiv 0$, the stability of each family of periodic solutions is the same as the singular point $p = (r^*, \alpha^*)$ associated to the Poincaré map of the limit cycle $\varphi(t, \varepsilon)$ of the averaged system of second order in coordinates (r, α). Therefore, we will study the eigenvalues of the linearization associated to the averaged system (2.9) associated to the point (r^*, α^*). Firstly, we observe that the eigenvalues are of the form $\lambda_1, \lambda_2 = -\lambda_1$. Secondly, the Jacobian of this equilibrium point must satisfy $J(r^*, \alpha^*) = \lambda_1 \cdot \lambda_2 = -\lambda_1^2$. Therefore, the eigenvalues are pure imaginary if $J(r^*, \alpha^*) > 0$ and the eigenvalues are real if $J(r^*, \alpha^*) < 0$. Thus, in the first case the corresponding family of periodic orbits is linearly stable and in the second case is unstable.

Thus, our main result on the stability of the periodic orbits obtained in Theorem 3.1 of the generalized Hénon–Heiles system (1.2) is summarized as follows.

Theorem 4.1. *Assume that $D > 0$ and $A + B \neq 0$, then the stability or instability of the families of periodic orbits in Theorem 3.1 on each positive energy level h of the generalized Hénon–Heiles Hamiltonian system* (1.2) *is as follows:*

(i) The family in (a) is linearly stable if $J(P_1^+) > 0$ and unstable if $J(P_1^+) < 0$ (see Figures 1-2),
(ii) The two families in (b) are linearly stable if $J(P_2^+) > 0$ and unstable if $J(P_2^+) < 0$ (see Figures 3-4),
(iii) The three families in (c) are linearly stable if $J(P_1^+) \cdot J(P_2^+) > 0$ (see Figures 5) and unstable if $J(P_1^+) \cdot J(P_2^+) < 0$ (see Figure 6).

A similar result is true in the case where $D < 0$.

5. Liouville–Arnol'd non-integrability

A Hamiltonian system with Hamiltonian H of two degrees of freedom is *integrable in the sense of Liouville–Arnol'd* if it has a first integral C independent with H (i.e. the gradient vectors of H and C are *independent* in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with H (i.e., the Poisson parenthesis of H and C is zero). For Hamiltonian systems with two degrees of freedom the involution condition is redundant, because the fact that C is a first integral of the Hamiltonian system, implies that the mentioned Poisson parenthesis is always zero.

Using the notation of [4] when a connected component I_{hc}^* of $I_{hc} = \{p \in M \quad H(p) = h, \text{and} C(p) = c\}$ is diffeomorphic to a torus, either all orbits on this torus are periodic if the rotation number associated to this torus is rational, or they are quasi-periodic (i.e., every orbit is dense in the torus) if the rotation number associated to this torus is not rational. It is known that a periodic solution of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and another is 1 due to the existence of the first integral given by the Hamiltonian. Remember the result proved in [26], Section 36, that is, if a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnol'd integrable, and C is a second first integral such that the gradients of H and C are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to 1. This result gives us a tool to study the non Liouville–Arnol'd integrability, independently of the class of differentiability of the second first integral. The main problem for applying this theorem is to find periodic orbits having multipliers different from 1.

Theorem 5.1. *Assume that our generalized Hénon–Heiles Hamiltonian system* (1.2) *satisfies the assumptions of one of the statements of Theorem 3.1, and denote by (*) this statement. Then, under the assumption of statement (*),*

(a) either the generalized Hénon–Heiles Hamiltonian system is Liouville–Arnol'd integrable and the gradients of the two constants of motion are linearly dependent on some points of the periodic orbits found in statement (*) of Theorem 3.1,
(b) or the generalized Hénon–Heiles system is not Liouville–Arnol'd integrable with any second first integral of class $\mathscr{C}^1$.

Proof. We assume that $D > 0$, $A + B \neq 0$, so there are solutions in (3.8) and we will assume that $J(P_1^+) \neq 0$ or $J(P_2^+) \neq 0$, therefore we are under the assumptions of Theorem 3.1, and that one of the three found periodic orbits corresponding to the solutions in (3.8) exists. Now, we compute the associated Jacobians (3.10) and (3.11) and we observe that they are different from 1 moving the energy level h. As these Jacobians are the product of the four multipliers of these periodic orbits with two of them always equal to 1, the remainder two multipliers cannot be equal to 1. Thus, under the assumptions of Theorem 3.1, by result proved in [26], Section 36, either our generalized Hénon–Heiles systems cannot be Liouville–Arnol'd integrable with any second first integral C, or the system is Liouville-Arnol'd integrable and the differentials of H and C are linearly dependent on some points of these periodic orbits. Thus we have concluded the proof of the theorem. □

In the previous result based on the study of the periodic orbits via the averaging method for any level $H = h$, we have proved that for any second first integral of class C^1 many generalizations of the Hénon-Heiles system are not integrable in the sense of Liouville–Arnol'd. Since in Theorem 5.1, $D \neq 0$, our result is a complement of the result obtained in [19] because there was considered the case $D = 0$. On other hand, as far as we known, the results of the previous theorems are new in the case $D \neq 0$.

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