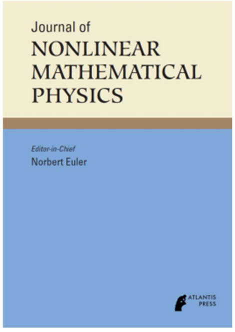

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Four-Wave Semidiscrete Nonlinear Integrable System with $\mathscr{P}\mathscr{T}$-Symmetry

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The new type of third-order spectral operator suitable to generate new multifield semidiscrete nonlinear systems with two coupling parameters in the framework of zero-curvature equation is proposed. The evolution operator corresponding to the first integrable system in an infinite hierarchy is explicitly recovered and the general form of first integrable system is isolated. The generalized procedure for the direct recursive development of infinite hierarchy of local conservation laws is presented and several lowest local conservation laws and local conserved densities are found. The reduction to the real field amplitudes in general system with unfixed sampling functions is made and the symmetric parametrization of field amplitudes allowing to exclude the redundant field function and to resolve the problem of sampling fixation for the particular realization of reduced integrable system is considered. This parametrization gives rise to the four field nonlinear model which in certain intervals of adjustable coupling parameter could serve as a semidiscrete analogy to the beam-plasma interaction system.

Keywords: integrable nonlinear system; local conservation laws; $\mathscr{P}\mathscr{T}$-symmetry; beam-plasma oscillations.

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1. Introduction

Initiated by the discovery of first integrable nonlinear dynamical models on a regular one-dimensional lattice [2–4, 17, 20, 28, 37, 38] the interest to the construction of new integrable semidiscrete nonlinear systems has been permanently supported by the wide range of physical problems, where the spatial discreteness and regularity play a crucial role. Among the most typical physical objects, where the semidiscrete nonlinear systems find their applications are the semiconductor superlattices [10, 26, 34, 51], electric superstructures [29], optical waveguide arrays [15], photonic crystal fibers [7] as well as the regular macromolecular structures of both natural [16] and synthetic [31] origin. The extensive bibliography on experimental realizations of discrete soliton-like entities is presented, e.g. in two review articles on discrete solitons in optics [24] and on optical spatial solitons [14]. Another motivation for the development of new semidiscrete nonlinear integrable models is to discretize the already known continuous nonlinear integrable models as it has been done, e.g. for the Maxwell-Bloch system, which describes resonant interaction of light with a medium of two-level atoms [9].

Obviously, the more complex nonlinear physical phenomenon requires the more rich nonlinear model for its adequate treatment. The richness of semidiscrete integrable nonlinear system is dictated by the order of auxiliary spectral operator consistent with some evolution operator in the framework of system zero-curvature representation.

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According to Caudrey definition [12, 13] the order of spectral operator associated with some integrable evolutionary nonlinear system is determined by the number of *distinct* eigenvalues of respective limiting spectral problem. In this terminology the auxiliary spectral problems linked with the known multicomponent semidiscrete nonlinear Schrödinger systems [5, 18, 39, 43, 44] taken at vanishing boundary conditions must be treated as second-order ones despite being rather sophisticated matrix generalizations of the basic Ablowitz-Ladik problem [2–4]. The similar phenomenon is typical of the auxiliary spectral problems associated with the matrix generalizations [11, 45] of nonlinear Toda system [17, 28, 37, 38]. In general, however, the high rank of the spectral matrix is able to provide the higher order of the spectral problem.

Thus ten years ago [46] we have suggested the spectral operator of third order and used it in obtaining the early unknown integrable semidiscrete nonlinear dynamical system, consisting of three coupled subsystems where two subsystems are the Toda-like ones. The respective auxiliary third-order spectral problem turned out to be distinct from the third-order discrete spectral problem discussed by Levi and Grundland [25]. This observation presumes the existence of several distinct operators of the same order leading to distinct nonlinear integrable systems.

Considering the above perspective the main idea of our present work was to find a new third-order spectral operator allowing to isolate early unknown integrable semidiscrete multifield nonlinear systems, i.e. nonlinear evolution systems embedded into one or another quasionedimensional lattice.

Another key issue was to adjust the procedure for the direct generation of local conservation laws to be applicable to the multicomponent semidiscrete integrable systems and to obtain the lowest conservation laws from the infinite hierarchy for the proposed nonlinear systems.

2. Auxiliary spectral and evolution operators

In our searchings we used an approach based upon the zero-curvature equation [36]

$$\dot{L}(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \tag{2.1}$$

and tested a number of candidates on the spectral $L(n|z)$ and evolution $A(n|z)$ operators regarding their ability to convert zero-curvature equation (2.1) into the zero-curvature representation for some new semidiscrete nonlinear systems (i.e. regarding the mutual consistency of $L(n|z)$ and $A(n|z)$). Here the dot written over the operator $L(n|z)$ in the left-hand side of zero-curvature equation (2.1) means the differentiation with respect to time τ, the integer n denotes the discrete spatial coordinate running from minus to plus infinity, while z stands for the auxiliary spectral parameter independent on time.

Assuming the operator $L(n|z)$ to be the 3×3 nonsingular matrix with matrix elements $L_{jk}(n|z)$ being some Laurent polynomial functions of z and z^{-1} we revealed that the matrix

$$L(n|z) = \begin{pmatrix} r_{11}(n)z^2 + t_{11}(n) & \beta s_{12}(n)z + \alpha s_{12}(n) & s_{13}(n)z + u_{13}(n)z^{-1} \\ \alpha s_{21}(n)z + \beta s_{21}(n) & 0 & \alpha u_{23}(n) + \beta u_{23}(n)z^{-1} \\ s_{31}(n)z + u_{31}(n)z^{-1} & \beta u_{32}(n) + \alpha u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} \end{pmatrix} \tag{2.2}$$

gives rise to the result consistent with the zero-curvature equation (2.1) provided the matrix $A(n|z)$ is sought in the form

$$A(n|z) = \begin{pmatrix} a_{11}(n)z^2 + c_{11}(n) & b_{12}(n)z + c_{12}(n) & b_{13}(n)z + d_{13}(n)z^{-1} \\ b_{21}(n)z + c_{21}(n) & 0 & c_{23}(n) + d_{23}(n)z^{-1} \\ b_{31}(n)z + d_{31}(n)z^{-1} & c_{32}(n) + d_{32}(n)z^{-1} & c_{33}(n) + e_{33}(n)z^{-2} \end{pmatrix} \tag{2.3}$$

and the time independent fitting parameters α and β are subjected to the constraint

$$\alpha^2 + \beta^2 = 0. \tag{2.4}$$

Indeed, inserting the expressions (2.2) and (2.3) for $L(n|z)$ and $A(n|z)$ into the zero-curvature equation (2.1) we are able to specify almost all matrix elements $A_{jk}(n|z)$ of operator $A(n|z)$ in terms of functions entered into operator $L(n|z)$. Precisely the specifiable functions involved into operator $A(n|z)$ were found to be

$$a_{11}(n) = a_{11} \tag{2.5}$$

$$b_{12}(n) = \beta a_{11} s_{12}(n)/r_{11}(n) \tag{2.6}$$

$$c_{12}(n) = \alpha a_{11} s_{12}(n)/r_{11}(n) \tag{2.7}$$

$$b_{13}(n) = a_{11} s_{13}(n)/r_{11}(n) \tag{2.8}$$

$$b_{21}(n) = s_{21}(n-1) a_{11} \alpha / r_{11}(n-1) \tag{2.9}$$

$$c_{21}(n) = s_{21}(n-1) a_{11} \beta / r_{11}(n-1) \tag{2.10}$$

$$b_{31}(n) = s_{31}(n-1) a_{11} / r_{11}(n-1) \tag{2.11}$$

and

$$e_{33}(n) = e_{33} \tag{2.12}$$

$$d_{32}(n) = \alpha e_{33} u_{32}(n)/v_{33}(n) \tag{2.13}$$

$$c_{32}(n) = \beta e_{33} u_{32}(n)/v_{33}(n) \tag{2.14}$$

$$d_{31}(n) = e_{33} u_{31}(n)/v_{33}(n) \tag{2.15}$$

$$d_{23}(n) = u_{23}(n-1) e_{33} \beta / v_{33}(n-1) \tag{2.16}$$

$$c_{23}(n) = u_{23}(n-1) e_{33} \alpha / v_{33}(n-1) \tag{2.17}$$

$$d_{13}(n) = u_{13}(n-1) e_{33} / v_{33}(n-1), \tag{2.18}$$

where parameters a_{11} and e_{33} can be arbitrary functions of time. The functions $c_{11}(n)$ and $c_{33}(n)$ referred to as the sampling ones remain arbitrary for the time being. The similar situation with the unfixed sampling is typical of other integrable models [40,47] and can be resolved relying upon the local conservation laws [48,49] dictated by the matrix structure of proposed spectral operator (2.2).

3. General form of semidiscrete integrable nonlinear system

Except of the explicit presentation of auxiliary evolution operator $A(n|z)$ the zero-curvature equation (2.1) yields the system of semidiscrete nonlinear equations for the prototype field amplitudes $r_{11}(n)$, $t_{11}(n)$, $s_{12}(n)$, $s_{13}(n)$, $u_{13}(n)$, $s_{21}(n)$, $u_{23}(n)$, $s_{31}(n)$, $u_{31}(n)$, $u_{32}(n)$, $t_{33}(n)$, $v_{33}(n)$. In the most general case of unfixed sampling functions $c_{11}(n)$ and $c_{33}(n)$ these equations read as follows

$$\begin{aligned}\dot{r}_{11}(n) = {} & c_{11}(n+1)r_{11}(n) - r_{11}(n)c_{11}(n) + \\ & + \alpha\beta a_{11}s_{12}(n+1)s_{21}(n)/r_{11}(n+1) - \alpha\beta a_{11}s_{12}(n)s_{21}(n-1)/r_{11}(n-1) + \\ & + a_{11}s_{13}(n+1)s_{31}(n)/r_{11}(n+1) - a_{11}s_{13}(n)s_{31}(n-1)/r_{11}(n-1)\end{aligned} \tag{3.1}$$

$$\begin{aligned}\dot{t}_{11}(n) = {} & c_{11}(n+1)t_{11}(n) - t_{11}(n)c_{11}(n) + \\ & + \alpha\beta a_{11}s_{12}(n+1)s_{21}(n)/r_{11}(n+1) - \alpha\beta a_{11}s_{12}(n)s_{21}(n-1)/r_{11}(n-1) + \\ & + a_{11}s_{13}(n+1)u_{31}(n)/r_{11}(n+1) - a_{11}u_{13}(n)s_{31}(n-1)/r_{11}(n-1) + \\ & + u_{13}(n)e_{33}s_{31}(n)/v_{33}(n) - s_{13}(n)e_{33}u_{31}(n)/v_{33}(n)\end{aligned} \tag{3.2}$$

$$\begin{aligned}\dot{s}_{12}(n) = {} & c_{11}(n+1)s_{12}(n) - a_{11}t_{11}(n)s_{12}(n)/r_{11}(n) + \\ & + a_{11}s_{13}(n+1)u_{32}(n)/r_{11}(n+1) - s_{13}(n)e_{33}u_{32}(n)/v_{33}(n)\end{aligned} \tag{3.3}$$

$$\begin{aligned}\dot{s}_{13}(n) = {} & c_{11}(n+1)s_{13}(n) - s_{13}(n)c_{33}(n) + \\ & + \alpha\beta a_{11}s_{12}(n+1)u_{23}(n)/r_{11}(n+1) - \alpha\beta s_{12}(n)u_{23}(n-1)e_{33}/v_{33}(n-1) + \\ & + a_{11}s_{13}(n+1)t_{33}(n)/r_{11}(n+1) - a_{11}t_{11}(n)s_{13}(n)/r_{11}(n) + \\ & + a_{11}u_{13}(n) - r_{11}(n)u_{13}(n-1)e_{33}/v_{33}(n-1)\end{aligned} \tag{3.4}$$

$$\begin{aligned}\dot{u}_{13}(n) = {} & c_{11}(n+1)u_{13}(n) - u_{13}(n)c_{33}(n) + \\ & + \alpha\beta a_{11}s_{12}(n+1)u_{23}(n)/r_{11}(n+1) - \alpha\beta s_{12}(n)u_{23}(n-1)e_{33}/v_{33}(n-1) + \\ & + u_{13}(n)e_{33}t_{33}(n)/v_{33}(n) - t_{11}(n)u_{13}(n-1)e_{33}/v_{33}(n-1) + \\ & + a_{11}s_{13}(n+1)v_{33}(n)/r_{11}(n+1) - s_{13}(n)e_{33}\end{aligned} \tag{3.5}$$

$$\begin{aligned}\dot{s}_{21}(n) = {} & s_{21}(n)t_{11}(n)a_{11}/r_{11}(n) - s_{21}(n)c_{11}(n) + \\ & + u_{23}(n)e_{33}s_{31}(n)/v_{33}(n) - u_{23}(n)s_{31}(n-1)a_{11}/r_{11}(n-1)\end{aligned} \tag{3.6}$$

$$\begin{aligned}\dot{u}_{23}(n) = {} & u_{23}(n)t_{33}(n)e_{33}/v_{33}(n) - u_{23}(n)c_{33}(n) + \\ & + s_{21}(n)a_{11}u_{13}(n)/r_{11}(n) - s_{21}(n)u_{13}(n-1)e_{33}/v_{33}(n-1)\end{aligned} \tag{3.7}$$

$$\begin{aligned}\dot{s}_{31}(n) = {} & c_{33}(n+1)s_{31}(n) - s_{31}(n)c_{11}(n) + \\ & + \alpha\beta e_{33}u_{32}(n+1)s_{21}(n)/v_{33}(n+1) - \alpha\beta u_{32}(n)s_{21}(n-1)a_{11}/r_{11}(n-1) + \\ & + s_{31}(n)t_{11}(n)a_{11}/r_{11}(n) - t_{33}(n)s_{31}(n-1)a_{11}/r_{11}(n-1) + \\ & + e_{33}u_{31}(n+1)r_{11}(n)/v_{33}(n+1) - u_{31}(n)a_{11}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\dot{u}_{31}(n) = {} & c_{33}(n+1)u_{31}(n) - u_{31}(n)c_{11}(n) + \\ & + \alpha\beta e_{33}u_{32}(n+1)s_{21}(n)/v_{33}(n+1) - \alpha\beta u_{32}(n)s_{21}(n-1)a_{11}/r_{11}(n-1) + \\ & + e_{33}u_{31}(n+1)t_{11}(n)/v_{33}(n+1) - e_{33}t_{33}(n)u_{31}(n)/v_{33}(n) + \\ & + e_{33}s_{31}(n) - v_{33}(n)s_{31}(n-1)a_{11}/r_{11}(n-1)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\dot{u}_{32}(n) = {} & c_{33}(n+1)u_{32}(n) - t_{33}(n)e_{33}u_{32}(n)/v_{33}(n) + \\ & + e_{33}u_{31}(n+1)s_{12}(n)/v_{33}(n+1) - u_{31}(n)a_{11}s_{12}(n)/r_{11}(n)\end{aligned} \tag{3.10}$$

$$\begin{aligned}\dot{t}_{33}(n) = {} & c_{33}(n+1)t_{33}(n) - t_{33}(n)c_{33}(n) + \\ & + \alpha\beta e_{33}u_{32}(n+1)u_{23}(n)/v_{33}(n+1) - \alpha\beta e_{33}u_{32}(n)u_{23}(n-1)/v_{33}(n-1) + \\ & + e_{33}u_{31}(n+1)s_{13}(n)/v_{33}(n+1) - e_{33}s_{31}(n)u_{13}(n-1)/v_{33}(n-1) + \\ & + s_{31}(n)a_{11}u_{13}(n)/r_{11}(n) - u_{31}(n)a_{11}s_{13}(n)/r_{11}(n)\end{aligned} \tag{3.11}$$

$$\begin{aligned}\dot{v}_{33}(n) = {} & c_{33}(n+1)v_{33}(n) - v_{33}(n)c_{33}(n) + \\ & + \alpha\beta e_{33}u_{32}(n+1)u_{23}(n)/v_{33}(n+1) - \alpha\beta e_{33}u_{32}(n)u_{23}(n-1)/v_{33}(n-1) + \\ & + e_{33}u_{31}(n+1)u_{13}(n)/v_{33}(n+1) - e_{33}u_{31}(n)u_{13}(n-1)/v_{33}(n-1).\end{aligned} \tag{3.12}$$

According to the very method of their construction the obtained equations (3.1) – (3.12) are said to possess the zero-curvature representation (2.1) with the spectral and evolution operators $L(n|z)$ and $A(n|z)$ given by the formulas (2.2) and (2.3) respectively, where the constraint (2.4) imposed onto the fitting parameters α and β as well as the expressions (2.5) – (2.18) for the constituent parts of evolution operator $A(n|z)$ have been taken into account. This property proves to be the key indication on an integrability [36] of the system under consideration (3.1) – (3.12) in Lax sense.

To our knowledge the suggested matrix pair of spectral and evolution operators (2.2) and (2.3) has never been considered by the other authors.

4. Local conservation laws

By definition any local conservation law of the infinite set linked with some semidiscrete integrable system on quasionedimensional infinite lattice can be written in the form

$$\dot{\rho}(n) = J(n|n-1) - J(n+1|n), \tag{4.1}$$

where the quantities $\rho(n)$ and $J(n+1/2|n-1/2)$ are referred to as the local density and the local current, respectively.

According to the general rule [48] some of the lowest local conservation laws are obtainable from the equation

$$\frac{d}{d\tau}\ln[\det L(n|z)] = \mathrm{Sp}A(n+1|z) - \mathrm{Sp}A(n|z) \tag{4.2}$$

which follows directly from the zero-curvature equation (2.1) by virtue of identity

$$\mathrm{Sp}\left[L^{-1}(n|z)\dot{L}(n|z)\right] \equiv \frac{d}{d\tau}\ln[\det L(n|z)] \tag{4.3}$$

valid provided $\det L(n|z) \neq 0$. For the system under study (3.1) – (3.12) the right-hand side $\mathrm{Sp}A(n+1|z) - \mathrm{Sp}A(n|z)$ of universal local conservation law (4.2) does not contain the spectral parameter z and is equal to $c_{11}(n+1)+c_{33}(n+1)-c_{11}(n)-c_{33}(n)$. On the contrary, the left-hand side $d\{\ln[\det L(n|z)]\}/d\tau$ contains the truncated Laurent series with respect to the spectral parameter z both in its nominator and denominator inasmuch as

$$\begin{aligned}\det L(n|z) = {} & \alpha\beta(z^2+1)\Big[s_{21}(n)s_{13}(n)u_{32}(n)+u_{23}(n)s_{31}(n)s_{12}(n)- \\ & - u_{23}(n)r_{11}(n)u_{32}(n)-s_{21}(n)t_{33}(n)s_{12}(n)\Big]+ \\ & + \alpha\beta(1+z^{-2})\Big[s_{21}(n)u_{13}(n)u_{32}(n)+u_{23}(n)u_{31}(n)s_{12}(n)- \\ & - u_{23}(n)t_{11}(n)u_{32}(n)-s_{21}(n)v_{33}(n)s_{12}(n)\Big].\end{aligned} \tag{4.4}$$

In so doing the universal local conservation law (4.2) can be readily split into the three equations, one of which turns out to be the linear combination of another two ones. As a result we obtain two independent local conservation laws

$$\begin{aligned}\frac{d}{d\tau}\ln\Big[s_{21}(n)s_{13}(n)u_{32}(n) & + u_{23}(n)s_{31}(n)s_{12}(n)- \\ & - u_{23}(n)r_{11}(n)u_{32}(n)-s_{21}(n)t_{33}(n)s_{12}(n)\Big] = \\ & = c_{11}(n+1)+c_{33}(n+1)-c_{11}(n)-c_{33}(n)\end{aligned} \tag{4.5}$$

$$\begin{aligned}\frac{d}{d\tau}\ln\Big[s_{21}(n)u_{13}(n)u_{32}(n) & + u_{23}(n)u_{31}(n)s_{12}(n)- \\ & - u_{23}(n)t_{11}(n)u_{32}(n)-s_{21}(n)v_{33}(n)s_{12}(n)\Big] = \\ & = c_{11}(n+1)+c_{33}(n+1)-c_{11}(n)-c_{33}(n).\end{aligned} \tag{4.6}$$

At $z=\pm i$ the matrix $L(n|z)$ becomes singular: $\det L(n|\pm i)=0$ (see expression (4.4)). We exploit this fact to find one more local conservation law. Thus, taking into account the zero-curvature equation (2.1) and relying upon the identity

$$\frac{d}{d\tau}L(n|z) \equiv \frac{d}{d\tau}[L(n|z)-\lambda I] \tag{4.7}$$

we obtain

$$\begin{aligned}\frac{d}{d\tau}\ln\{\det[L(n|\pm i)-\lambda I]\} = {} & \mathrm{Sp}A(n+1|\pm i)-\mathrm{Sp}A(n|\pm i)+ \\ & + \mathrm{Sp}\left\{[L(n|\pm i)-\lambda I]^{-1}\lambda[A(n+1|\pm i)-A(n|\pm i)]\right\},\end{aligned} \tag{4.8}$$

where λ is some auxiliary time independent parameter and I stands for the 3×3 unity matrix. At $\lambda\to 0$ the last term in the right-hand side of above equation (4.8) tends to zero, while the left-hand side term gives the finite result. As a consequence we come to the following local conservation law

$$\begin{aligned}\frac{d}{d\tau}\ln\Big\{[r_{11}(n)-t_{11}(n)]\,[v_{33}(n)-t_{33}(n)]+[s_{13}(n)-u_{13}(n)]\,[s_{31}(n)-u_{31}(n)]\Big\} & = \\ = c_{11}(n+1)+c_{33}(n+1)-c_{11}(n)-c_{33}(n). & \end{aligned} \tag{4.9}$$

The previous two approaches are able to produce only three local conservation laws (4.5), (4.6) and (4.9).

There are, however, the generalized direct procedure [48] permitting to develop the infinite set of local conservation laws recursively without any reference on the scattering data of auxiliary spectral problem as well as on the hamiltonian structure underlying the hierarchy of integrable systems (assuming it exists) linked with our spectral operator (2.2). The approach is based upon the recursive presentation of auxiliary quantities $\Gamma_{jk}(n|z)$ subjected to the following set of spatial Riccati equations

$$\Gamma_{jk}(n+1|z)\sum_{i=1}^{3} L_{ki}(n|z)\Gamma_{ik}(n|z) = \sum_{i=1}^{3} L_{ji}(n|z)\Gamma_{ik}(n|z) \tag{4.10}$$

and upon the subsequent substitution of the obtained series into the collection of generating equations

$$\frac{d}{d\tau}\ln M_{jj}(n|z) = B_{jj}(n+1|z) - B_{jj}(n|z). \tag{4.11}$$

Here the shorthands $M_{jk}(n|z)$ and $B_{jk}(n|z)$ are defined by the expressions

$$M_{jk}(n|z) = \sum_{i=1}^{3} L_{ji}(n|z)\Gamma_{ik}(n|z) \tag{4.12}$$

and

$$B_{jk}(n|z) = \sum_{i=1}^{3} A_{ji}(n|z)\Gamma_{ik}(n|z), \tag{4.13}$$

respectively, while the quantities $\Gamma_{jk}(n|z)$ are assumed to comply with the rule

$$\Gamma_{ji}(n|z)\Gamma_{ik}(n|z) = \Gamma_{jk}(n|z). \tag{4.14}$$

Any summation, whenever it appears, is always marked by the summation symbol $\sum$.

The quantities $\ln M_{jj}(n|z)$ are determined exclusively by the spectral operator $L(n|z)$ (through $\Gamma_{jk}(n|z)$ and $L_{jk}(n|z)$) and should be treated as the generating functions of local densities. The quantities $-B_{jj}(n|z)$ are determined both by the spectral operator $L(n|z)$ (through $\Gamma_{jk}(n|z)$) and the evolution operator $A(n|z)$ (through $A_{jk}(n|z)$) and should be treated as the generating functions of local currents. In this context the true sense of generating equations (4.11) consists not in the finding of auxiliary functions $\Gamma_{jk}(n|z)$ (which is the prerogative of spatial Riccati equations (4.10)) but in correct combination of expansion terms into infinite collection of continuity equations referred to as the hierarchy of local conservation laws.

The necessity to operate with the set of several Riccati equations (4.10) and with the collection of several generating equations (4.11) is dictated by the complicated matrix structures of spectral and evolution operators (see formulas (2.2) and (2.3), respectively). Namely the extended number of Riccati equations and generating equations distinguishes our consideration from the similar approaches of other authors [22, 41, 42, 50, 53].

The general property (4.14) of auxiliary functions $\Gamma_{jk}(n|z)$ says that only two of them can be treated as truly independent. We call such two independent functions as the basic ones.

Thus, having taken the functions $\Gamma_{12}(n|z)$ and $\Gamma_{23}(n|z)$ as the basic ones it is reasonable to use substitutions

$$\Gamma_{12}(n|z) = (\alpha+\beta z)\gamma^{+}_{12}(n|z) \tag{4.15}$$

$$\Gamma_{23}(n|z) = (\alpha+\beta z^{-1})\gamma^{+}_{23}(n|z). \tag{4.16}$$

Then according to the original Riccati equations (4.10) the set of two coupled equations for $\gamma_{12}^{+}(n|z)$ and $\gamma_{23}^{+}(n|z)$ read as follows

$$\begin{aligned}
&(\alpha\beta)^2 z\left(z+z^{-1}\right)^2 \gamma_{12}^{+}(n+1|z)s_{21}(n)\gamma_{12}^{+}(n|z)\gamma_{23}^{+}(n|z)+\\
&+\alpha\beta\left(z+z^{-1}\right)\gamma_{12}^{+}(n+1|z)u_{23}(n)=\\
&=\alpha\beta\left(z+z^{-1}\right)\left[r_{11}(n)z^2+t_{11}(n)\right]\gamma_{12}^{+}(n|z)\gamma_{23}^{+}(n|z)+\\
&+\alpha\beta\left(z+z^{-1}\right)s_{12}(n)\gamma_{23}^{+}(n|z)+s_{13}(n)z+u_{13}(n)z^{-1}
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&\alpha\beta\left(z+z^{-1}\right)\gamma_{23}^{+}(n+1|z)\left[s_{31}(n)z+u_{31}(n)z^{-1}\right]\gamma_{12}^{+}(n|z)\gamma_{23}^{+}(n|z)+\\
&+\alpha\beta z^{-1}\left(z+z^{-1}\right)\gamma_{23}^{+}(n+1|z)u_{32}(n)\gamma_{23}^{+}(n|z)+\\
&+\gamma_{23}^{+}(n+1|z)\left[t_{33}(n)+v_{33}(n)z^{-2}\right]=\\
&=\alpha\beta z\left(z+z^{-1}\right)s_{21}(n)\gamma_{12}^{+}(n|z)\gamma_{23}^{+}(n|z)+u_{23}(n).
\end{aligned} \tag{4.18}$$

These equations permit at least two types of recursive solutions based upon two types of expansions. Namely at $|z|\to 0$ we should adopt the expansions

$$\gamma_{12}^{+}(n|z)=\sum_{i=0}^{\infty}x_{12}^{+}(n|i|0)z^{2i} \tag{4.19}$$

$$\gamma_{23}^{+}(n|z)=\sum_{i=0}^{\infty}x_{23}^{+}(n|i|0)z^{2i+2} \tag{4.20}$$

while at $|z|\to\infty$ the constructive expansions should be assumed as follows

$$\gamma_{12}^{+}(n|z)=\sum_{i=0}^{\infty}x_{12}^{+}(n|i|\infty)z^{-2i} \tag{4.21}$$

$$\gamma_{23}^{+}(n|z)=\sum_{i=0}^{\infty}x_{23}^{+}(n|i|\infty)z^{-2i}. \tag{4.22}$$

In the case when the two basic functions are chosen as $\Gamma_{32}(n|z)$ and $\Gamma_{21}(n|z)$ it is reasonable to use substitutions

$$\Gamma_{32}(n|z)=\left(\beta+\alpha z^{-1}\right)\gamma_{32}^{-}(n|z) \tag{4.23}$$

$$\Gamma_{21}(n|z)=(\beta+\alpha z)\gamma_{21}^{-}(n|z). \tag{4.24}$$

Then according to the original Riccati equations (4.10) the set of two coupled equations for $\gamma_{32}^{-}(n|z)$ and $\gamma_{21}^{-}(n|z)$ read as follows

$$\begin{aligned}
&(\alpha\beta)^2 z^{-1}\left(z+z^{-1}\right)^2 \gamma_{32}^{-}(n+1|z)u_{23}(n)\gamma_{32}^{-}(n|z)\gamma_{21}^{-}(n|z)+\\
&+\alpha\beta\left(z+z^{-1}\right)\gamma_{32}^{-}(n+1|z)s_{21}(n)=\\
&=\alpha\beta\left(z+z^{-1}\right)\left[t_{33}(n)+v_{33}(n)z^{-2}\right]\gamma_{32}^{-}(n|z)\gamma_{21}^{-}(n|z)+\\
&+\alpha\beta\left(z+z^{-1}\right)u_{32}(n)\gamma_{21}^{-}(n|z)+s_{31}(n)z+u_{31}(n)z^{-1}
\end{aligned} \tag{4.25}$$

$$
\begin{aligned}
&\alpha\beta\left(z+z^{-1}\right)\gamma_{21}^{-}(n+1|z)\left[s_{13}(n)z+u_{13}(n)z^{-1}\right]\gamma_{32}^{-}(n|z)\gamma_{21}^{-}(n|z)+\\
&+\alpha\beta z\left(z+z^{-1}\right)\gamma_{21}^{-}(n+1|z)s_{12}(n)\gamma_{21}^{-}(n|z)+\\
&+\gamma_{21}^{-}(n+1|z)\left[r_{11}(n)z^{2}+t_{11}(n)\right]=\\
&=\alpha\beta z^{-1}\left(z+z^{-1}\right)u_{23}(n)\gamma_{32}^{-}(n|z)\gamma_{21}^{-}(n|z)+s_{21}(n).
\end{aligned}
\tag{4.26}
$$

These equations permit at least two types of recursive solutions based upon two types of expansions. Namely, at $|z|\to\infty$ we should adopt the expansions

$$\gamma_{32}^{-}(n|z)=\sum_{i=0}^{\infty}x_{32}^{-}(n|i|\infty)z^{-2i} \tag{4.27}$$

$$\gamma_{21}^{-}(n|z)=\sum_{i=0}^{\infty}x_{21}^{-}(n|i|\infty)z^{-2i-2} \tag{4.28}$$

while at $|z|\to 0$ the constructive expansions should be assumed as follows

$$\gamma_{32}^{-}(n|z)=\sum_{i=0}^{\infty}x_{32}^{-}(n|i|0)z^{2i} \tag{4.29}$$

$$\gamma_{21}^{-}(n|z)=\sum_{i=0}^{\infty}x_{21}^{-}(n|i|0)z^{2i}. \tag{4.30}$$

On the whole, each of four sets of expansions (4.19), (4.20); (4.21), (4.22); (4.27), (4.28); (4.29), (4.30) maintains a particular recursive procedure to build up the auxiliary functions $\Gamma_{jk}(n|z)$ and then the generating functions of local densities $\ln M_{jj}(n|z)$ and local currents $-B_{jj}(n|z)$ via the use of definitions (4.12) and (4.13) for $M_{jk}(n|z)$ and $B_{jk}(n|z)$, respectively. As a consequence we receive four variants to develop infinite series of local conservation laws. In each of variants there is three subvariants originated from three admissible generating equations (4.11). The concrete calculations show that some local conservation laws in one of twelve announced infinite series have twins in other infinite series. However, in order to extract the complete collection of local conservation laws all twelve series seem to be required.

Below we present six lowest local conserved densities found in the framework of above described scheme:

$$\rho_{11}(n)=\ln r_{11}(n) \tag{4.31}$$

$$\rho_{11}^{-}(n)=\frac{t_{11}(n)}{r_{11}(n)}+\frac{\alpha\beta s_{12}(n)s_{21}(n-1)+s_{13}(n)s_{31}(n-1)}{r_{11}(n)r_{11}(n-1)} \tag{4.32}$$

$$\rho_{11}^{+}(n)=\frac{t_{11}(n)}{r_{11}(n)}+\frac{\alpha\beta s_{12}(n+1)s_{21}(n)+s_{13}(n+1)s_{31}(n)}{r_{11}(n+1)r_{11}(n)} \tag{4.33}$$

$$\rho_{33}(n)=\ln v_{33}(n) \tag{4.34}$$

$$\rho_{33}^{-}(n)=\frac{t_{33}(n)}{v_{33}(n)}+\frac{\alpha\beta u_{32}(n)u_{23}(n-1)+u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n-1)} \tag{4.35}$$

$$\rho_{33}^{+}(n)=\frac{t_{33}(n)}{v_{33}(n)}+\frac{\alpha\beta u_{32}(n+1)u_{23}(n)+u_{31}(n+1)u_{13}(n)}{v_{33}(n+1)v_{33}(n)}\,. \tag{4.36}$$

The unrolled records of local conservation laws corresponding to the densities $\rho_{11}(n)$ and $\rho_{33}(n)$ are evident respectively from the equations (3.1) and (3.12) of general semidiscrete nonlinear system (3.1) – (3.12). As for the local conservation laws corresponding to the densities $\rho_{11}^{-}(n)$, $\rho_{11}^{+}(n)$ and $\rho_{33}^{-}(n)$, $\rho_{33}^{+}(n)$ we do not write them down in view of rather huge expressions for their local currents. The interested reader can readily reproduce the formulas for the respective local conservation laws relying upon the evolution equations (3.1) – (3.12) for the general semidiscrete nonlinear system.

5. General semidiscrete system in the reduction to the real-valued prototype field amplitudes

The matrix structures (2.2) and (2.3) of original spectral $L(n|z)$ and evolution $A(n|z)$ operators permit us to make the following mutually consistent reductions

$$r_{11}(n) = h(n) = v_{33}(n) \tag{5.1}$$

$$t_{11}(n) = h(n)T(n) = t_{33}(n) \tag{5.2}$$

$$s_{12}(n) = h(n)F_{+}(n) = u_{32}(n) \tag{5.3}$$

$$s_{21}(n) = h(n)F_{-}(n) = u_{23}(n) \tag{5.4}$$

$$s_{13}(n) = h(n)G_{+}(n) = u_{31}(n) \tag{5.5}$$

$$s_{31}(n) = h(n)G_{-}(n) = u_{13}(n) \tag{5.6}$$

and

$$a_{11} = k = e_{33} \tag{5.7}$$

$$c_{11}(n) = c(n) = c_{33}(n) \tag{5.8}$$

with $h(n)$, $T(n)$, $F_{+}(n)$, $F_{-}(n)$, $G_{+}(n)$, $G_{-}(n)$ and $c(n)$ being the purely real functions of spatial coordinate n and time τ, while k being purely real function of time. The coupling parameter $\alpha\beta$ should also be treated as the real one.

Then taking into account the on-cell local conservation laws found in the previous section the general set of semidiscrete nonlinear equations (3.1) – (3.12) rewritten in terms of real-valued prototype field amplitudes $h(n)$, $F_{+}(n)$, $F_{-}(n)$, $G_{+}(n)$, $T(n)$, $G_{-}(n)$ acquires the form

$$\begin{aligned}\frac{d}{d\tau}\ln h(n) &= c(n+1) + k\alpha\beta F_{+}(n+1)F_{-}(n) + kG_{+}(n+1)G_{-}(n) - \\ &- c(n) - k\alpha\beta F_{+}(n)F_{-}(n-1) - kG_{+}(n)G_{-}(n-1)\end{aligned} \tag{5.9}$$

$$\begin{aligned}\frac{d}{d\tau}\ln F_{+}(n) &= kG_{+}(n+1) - kG_{+}(n) - kT(n) - \\ &- k\alpha\beta F_{+}(n+1)F_{-}(n) - kG_{+}(n+1)G_{-}(n) + \\ &+ k\alpha\beta F_{+}(n)F_{-}(n-1) + kG_{+}(n)G_{-}(n-1) + c(n)\end{aligned} \tag{5.10}$$

$$\begin{aligned}\frac{d}{d\tau}\ln F_{-}(n) &= kG_{-}(n) - kG_{-}(n-1) + kT(n) + \\ &+ k\alpha\beta F_{+}(n)F_{-}(n-1) + kG_{+}(n)G_{-}(n-1) - \\ &- k\alpha\beta F_{+}(n+1)F_{-}(n) - kG_{+}(n+1)G_{-}(n) - c(n+1)\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{d}{d\tau}\ln\Big[1-T(n)+G_+(n)-G_-(n)\Big] &= kG_+(n)-kG_+(n+1)+\\ &+kG_-(n)-kG_-(n-1)-\\ &-k\alpha\beta F_+(n+1)F_-(n)-kG_+(n+1)G_-(n)+\\ &+k\alpha\beta F_+(n)F_-(n-1)+kG_+(n)G_-(n-1)\end{aligned} \tag{5.12}$$

$$\begin{aligned}\frac{d}{d\tau}\ln\Big[1+T(n)-G_+(n)-G_-(n)\Big] &= kG_+(n)-kG_+(n+1)-\\ &-kG_-(n)+kG_-(n-1)-\\ &-k\alpha\beta F_+(n+1)F_-(n)-kG_+(n+1)G_-(n)+\\ &+k\alpha\beta F_+(n)F_-(n-1)+kG_+(n)G_-(n-1)\end{aligned} \tag{5.13}$$

$$\begin{aligned}\frac{d}{d\tau}\ln\Big[1-T(n)-G_+(n)+G_-(n)\Big] &= kG_+(n+1)-kG_+(n)\\ &+kG_-(n-1)-kG_-(n)-\\ &-k\alpha\beta F_+(n+1)F_-(n)-kG_+(n+1)G_-(n)+\\ &+k\alpha\beta F_+(n)F_-(n-1)+kG_+(n)G_-(n-1).\end{aligned} \tag{5.14}$$

Here only five of six involved equations (5.9) – (5.14) ought to be considered as independent in view of the natural constraint

$$\frac{d}{d\tau}\ln[F_+(n)h(n)F_-(n)] = \frac{d}{d\tau}\ln\left\{\frac{[1-T(n)]^2-[G_+(n)-G_-(n)]^2}{1+T(n)-G_+(n)-G_-(n)}\right\} \tag{5.15}$$

emanated from the on-cell local conservation laws. Thus, no more than five prototype field amplitudes may claim to be independent. One more field amplitude can be excluded by some additional constraint fixing the sampling function $c(n)$.

The structure of reduced integrable equations (5.9) – (5.14) clearly demonstrates that the functions $\ln h(n)$, $\ln[F_+(n)F_-(n)]$, $\ln[1-T(n)+G_+(n)-G_-(n)]$, $\ln[1+T(n)-G_+(n)-G_-(n)]$, $\ln[1-T(n)-G_+(n)+G_-(n)]$ and their linear combinations exhibit all features of conserved densities prescribed by the standard definition (4.1) of a local conservation law.

6. Exclusion of superfluous field variables

Nominally there exists a number of variants how to select one of admissible additional constraints [49]. However, we prefer the way allowing to define the sampling function $c(n)$ directly through some redundant quantity $q(n|n-1)$ and to exclude both of them simultaneously from further consideration.

The approach consists in the following symmetric parametrization

$$h(n) = h\exp[+q(n+1|n)-q(n|n-1)] \tag{6.1}$$

$$F_+(n) = F_+\exp[+x_+(n)-y_+(n)+q(n|n-1)] \tag{6.2}$$

$$F_-(n) = F_-\exp[-x_-(n)+y_-(n)-q(n+1|n)] \tag{6.3}$$

$$1-T(n)+G_+(n)-G_-(n) = \exp[-x_+(n)-y_+(n)-x_-(n)+y_-(n)] \tag{6.4}$$

$$1+T(n)-G_+(n)-G_-(n) = \exp[-x_+(n)-y_+(n)+x_-(n)+y_-(n)] \tag{6.5}$$

$$1 - T(n) - G_+(n) + G_-(n) = \exp\left[+x_+(n) - y_+(n) + x_-(n) + y_-(n)\right] \tag{6.6}$$

where $\dot{F}_+ = 0 = \dot{F}_-$ and $\dot{h} = 0$. Without the loss of generality we can adopt $F_+F_- = 1$.

The equations of motion for the new field variables $x_+(n)$, $y_+(n)$ and $x_-(n)$, $y_-(n)$ read as follows

$$\dot{x}_+(n) = kG_+(n+1) - kG_+(n) \tag{6.7}$$

$$\dot{y}_+(n) = kT(n) + k\alpha\beta F_+(n+1)F_-(n) + kG_+(n+1)G_-(n) - k\alpha\beta F_+F_- \tag{6.8}$$

$$\dot{x}_-(n) = kG_-(n-1) - kG_-(n) \tag{6.9}$$

$$\dot{y}_-(n) = kT(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1) - k\alpha\beta F_+F_-. \tag{6.10}$$

These four-wave equations are essentially selfconsistent since the expressions

$$G_+(n) = 1 - \exp\left[+x_-(n) + y_-(n) - y_+(n)\right]\cosh[x_+(n)] \tag{6.11}$$

$$T(n) = 1 - \exp\left[-y_+(n) + y_-(n)\right]\cosh[x_+(n) + x_-(n)] \tag{6.12}$$

$$G_-(n) = 1 - \exp\left[-x_+(n) - y_+(n) + y_-(n)\right]\cosh[x_-(n)] \tag{6.13}$$

$$F_+(n+1)F_-(n) = \exp\left[+x_+(n+1) - y_+(n+1) - x_-(n) + y_-(n)\right] \tag{6.14}$$

comprise neither the redundant quantity $q(n|n-1)$ no the sampling function $c(n)$. In this context the equation

$$\dot{q}(n|n-1) = c(n) + k\alpha\beta F_+(n)F_-(n-1) + kG_+(n)G_-(n-1) - k\alpha\beta F_+F_- \tag{6.15}$$

could be treated as the collateral definition of $\dot{q}(n|n-1)$ through $c(n)$ and vice versa, imposing seemingly no influence upon the dynamics of true field variables $x_+(n)$, $y_+(n)$ and $x_-(n)$, $y_-(n)$. Nevertheless, the right choice of this definition (6.15) ensures the correct frame of reference for the field variables $x_+(n)$, $y_+(n)$ and $x_-(n)$, $y_-(n)$ via the presence of last terms in the right-hand sides of equations (6.8) and (6.10) for $\dot{y}_+(n)$ and $\dot{y}_-(n)$.

In order to avert unnecessary complications when dealing with the integration of nonlinear equations (6.7) – (6.10) by the inverse scattering transform it is reasonable to assume that the redundant function $q(n|n-1)$ is independent both on time τ and coordinate n. As a consequence the function $h(n)$ is reduced to the constant value h (see formula (6.1)) which can be safely equalized to unity. Evidently the same assumption allows to fix the sampling function $c(n)$ by the expression

$$c(n) = k\alpha\beta F_+F_- - k\alpha\beta F_+(n)F_-(n-1) - kG_+(n)G_-(n-1). \tag{6.16}$$

It would be interesting to find the Hamiltonian representation for the set of obtained semidiscrete nonlinear equations (6.7) – (6.10). However, this task turned out to be highly nontrivial and requires a separate investigation.

7. $\mathscr{PT}$-symmetry and remarks on possible physical applications

Considering the reduced four-field semidiscrete system (6.7) – (6.10) we clearly observe its $\mathscr{PT}$-symmetry (i.e. the symmetry under the space and time reversal), implying that the transformed functions $x'_+(n|\tau)$, $x'_-(n|\tau)$, $y'_+(n\tau)$, $y'_-(n|\tau)$ defined by the equalities

$$x'_+(n|\tau) = -x_-(-n|-\tau) \tag{7.1}$$

$$x'_-(n|\tau) = -x_+(-n|-\tau) \tag{7.2}$$

$$y'_+(n|\tau) = -y_-(-n|-\tau) \tag{7.3}$$

$$y'_-(n|\tau) = -y_+(-n|-\tau) \tag{7.4}$$

are governed by the same set of equations (6.7)–(6.10) as the original ones $x_+(n|\tau)$, $x_-(n|\tau)$, $y_+(n|\tau)$, $y_-(n|\tau)$. Here we have tacitly stipulated for the agreement that indices $+$ and $-$ should mark the sites respectively on the right and left legs of some infinite ladder lattice whose unit cells are numbered by the integer n.

Presently the $\mathscr{PT}$-symmetric models become increasingly demanded by the physical scientific community [1, 8, 19, 27] (especially for the application to nonlinear optics [1, 19, 27]) inasmuch as they permit to obtain physically meaningful results without invoking the more restrictive condition of Dirac Hermiticity [8].

Another interesting feature of the reduced four-field system (6.7)–(6.10) concerns the spectrum of its low-amplitude ($|x_\pm(n)| \ll 1$, $|y_\pm(n)| \ll 1$) excitations. The respective dispersion law $\Omega = \Omega(\varkappa)$ is determined by the following quartic equation

$$\begin{aligned}
&\Omega^4 - 2\alpha\beta\sin(\varkappa)\Omega^3 - 2[1-\cos(\varkappa)]\Omega^2 + \\
&+ 2\alpha\beta[1-2\cos(\varkappa)][1-\cos(\varkappa)]\Omega^2 + \\
&+ 8\alpha\beta\sin(\varkappa)[1-\cos(\varkappa)]\Omega - 4\alpha\beta[1-\cos(\varkappa)]^2 = 0,
\end{aligned} \tag{7.5}$$

where Ω serves for the dimensionless frequency while $\varkappa$ denotes the wave number (wave vector) within the first Brillouin zone $-\pi \le \varkappa \le +\pi$. For the sake of convenience we have equalized the coupling parameter k to unity: $k = 1$. In certain intervals of adjustable coupling parameter $\alpha\beta$ (namely, at $0.3731 < \alpha\beta < 0.5$ and at $0.5 < \alpha\beta < +\infty$) the low-amplitude excitations exhibit the peculiar loop-like structure of dispersion curves (see, lower row on the Figure 1) closely resembling that for the linearized beam-plasma oscillations in hydrodynamic plasma [6, 32], which are modelled by a sort of four-wave equations [6, 32]. In this respect we could expect the applicability of proposed four-wave semidiscrete nonlinear equations (6.7) – (6.10) to the description of nonlinear hydrodynamic plasma at least as a toy model.

The Figure 1 illustrates the typical dispersion curves calculated outside the peculiarity intervals (upper row) and inside one of the peculiarity intervals (lower row).

As a matter of fact there are several regimes of low-amplitude oscillations corresponding to the several intervals of adjustable coupling parameter $\alpha\beta$, i.e. the coupling parameter $\alpha\beta$ can be understood as a some bifurcation parameter. The detailed report on this subject will be published elsewhere.

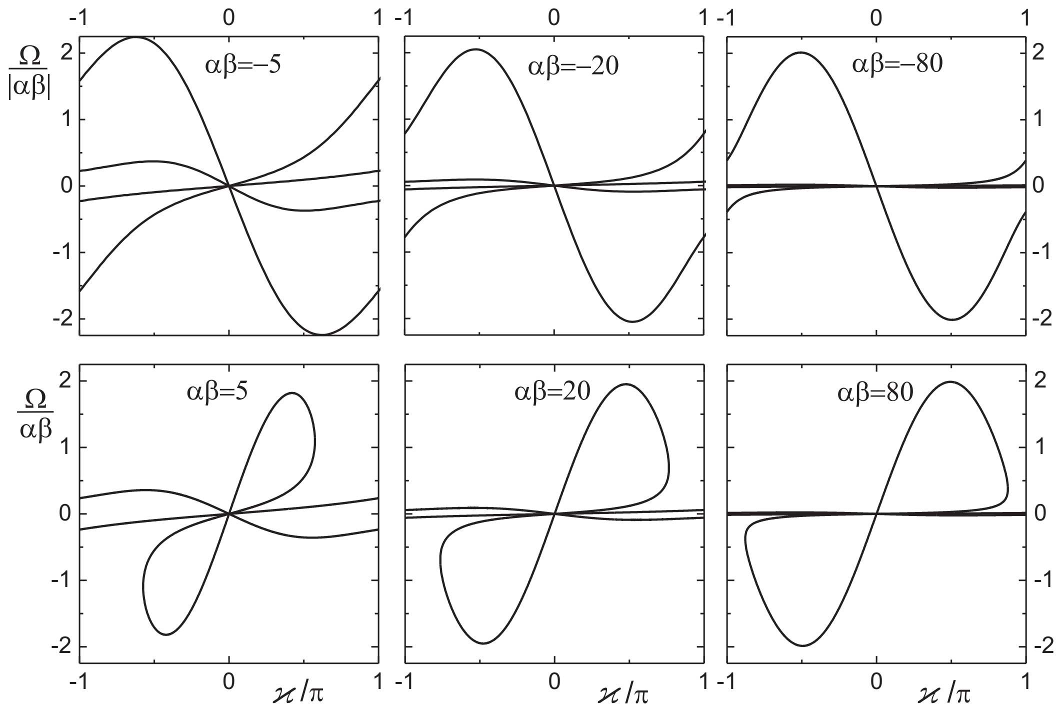

Fig. 1. Real-valued normal cyclic frequencies as the functions of wave vector at some negative (upper row) and positive (lower row) values of adjustable coupling parameter.

8. Conclusion

In this paper we have proposed the new type of third-order spectral operator on one-dimensional infinite lattice which turned out to be consistent with a particular sort of evolution operators, i.e. the type allowing to generate the hierarchy of new semidiscrete nonlinear integrable systems in the framework of zero-curvature equation. We have explicitly presented the general form of evolution operator corresponding to the first integrable system in hierarchy and have found the general form of first integrable system itself. On the whole, in order to obtain the K-th integrable system in hierarchy we must use a modified ansatz for the evolution operator. The question how to organize the powers of spectral parameter z in K-th evolution operator requires a special investigation.

The starting point of our present activity has been prompted by the close inspection of semidiscrete nonlinear integrable systems associated with the fourth-order spectral operator that had been suggested in our previous articles [48,49]. The experience gained with the selection of proper fourth-order spectral operator allowed us to succeed in obtaining the constructive form of third-order spectral operator too. The essential guiding principle in our searchings was the Caudrey definition of the order of spectral operator [12, 13, 46].

The key indication on a system integrability in Liouville sense is known to be the existence of infinite set of conservation laws [36]. We have applied this test to our general multifield nonlinear system relying upon early developed generalized direct recursive procedure based upon a set of spatial Riccati equations supplemented by a collection of generating equations [48]. In so doing we have found explicitly several local conservation laws and conserved densities.

It is worth noticing that the idea to use the direct recursive procedure for generating conservation laws was originated in the pioneering works by Konno, Sanuki and Ichikawa [22] and Wadati, Sanuki and Konno [50] dealing with the continuous nonlinear integrable systems. When having been applied to the semidiscrete nonlinear integrable systems the approach has been reformulated by Tsuchida, Ujino and Wadati [42]. However, in contrast to our present consideration, all these works [22, 42, 50] have suggested the simplest version of direct recursive procedure relying only upon a single spatial Riccati equation accompanied by a single generating equation.

We have also managed to reveal somewhat unexpected local conservation law connected with the singularity of spectral matrix at two particular values of the spectral parameter.

Finally, we have considered the reduction of general set of obtained nonlinear semidiscrete equations to the case of real field amplitudes. Although the general set of reduced nonlinear semidiscrete equations possesses the natural constraint the set remains underdetermined due to the presence of arbitrary sampling function $c(n)$. Such an underdeterminedness proves to be in lines with the basic statements of Mikhailov theory on the reduction group method [30]. We overcame both the natural constraint and the inherent underdeterminedness by the proper symmetric parametrization of prototype field functions giving rise to the set of four selfconsistent semidiscrete nonlinear equations.

The reduced integrable four-wave system exhibits the loop-like low-amplitude dispersion law typical of beam-plasma oscillations in hydrodynamic plasma [6, 32], but has nothing to do with the integrable Leble–Salle two-field system [23] invented as the semidiscrete analog of continuous Silin–Tykhonchuk equations for the parametric absorption and Langmuir turbulence in a plasma with spatially inhomogeneous density [35].

Concerning the general form of semidiscrete integrable nonlinear system given in Section 3, it permits also another interesting reductions. For example, the general system contains in itself the nonlinear Schrödinger-like ladder system with background-controlled intersite resonant coupling [47] enriched by the additional satellite fields. The question whether or not such a reduction is capable to extend our notions about integrability of $\mathscr{P}\mathscr{T}$-symmetric models [21, 33, 52] applicable to arrays of nonlinear optical waveguides with gains and losses requires a separate consideration.

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