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## Equilibria of a solvable $N$ -body problem and related properties of the $N$ numbers $x_n$ at which the Jacobi polynomial of order $N$ has the same value

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The class of solvable  $N$ -body problems of “goldfish” type has been recently extended by including (the additional presence of) *three-body* forces. In this paper we show that the equilibria of some of these systems are simply related to the  $N$  roots  $x_n$  of the polynomial equation  $P_N^{(\alpha,\beta)}(x) = w$ , where  $P_N^{(\alpha,\beta)}(x)$  is the Jacobi polynomial of order  $N$ , the parameters  $\alpha$  and  $\beta$  are related to parameters of the  $N$ -body problem (which can be *arbitrarily* assigned) and  $w$  is an *arbitrary* number. By investigating the behavior of these solvable  $N$ -body systems in the infinitesimal neighborhood of these equilibria, the eigenvalues associated to certain  $N \times N$  matrices explicitly given in terms of the  $N$  numbers  $x_n$  (and of additional free parameters of the  $N$ -body problem) are explicitly identified. In some cases—corresponding to *isochronous*  $N$ -body problems—these findings have a *Diophantine* connotation, inasmuch as these eigenvalues are then *rational* numbers.

### 1. Introduction

Over three decades ago [1] a class of  $N$ -body problems featuring several free parameters (“coupling constants”) were introduced by identifying the coordinates  $z_n(t)$  of the  $N$  (unit-mass, equal) particles—moving in the complex  $z$ -plane—with the  $N$  zeros of a time-dependent (monic) polynomial  $\psi(z, t)$  (of degree  $N$  in the variable  $z$ ),

$$\psi(z, t) = \prod_{n=1}^N [z - z_n(t)] = z^N + \sum_{m=1}^N c_m(t) z^{N-m}, \quad (1.1)$$

itself evolving according to a *linear* Partial Differential Equation (PDE)—suitably restricted to guarantee that it feature a polynomial solution of degree  $N$  in  $z$ . This implies that these  $N$ -body problems are *solvable*—as shown below, for arbitrary (positive integer)  $N$ —by *algebraic* operations: essentially by finding (generalized) eigenvalues associated to two given  $N \times N$  matrices, or equivalently by finding the  $2N$  zeros of polynomials of degree  $2N$ . The simplest dynamical system belonging to this class displays remarkably neat properties and was therefore considered a “goldfish” [2]. Subsequently this terminology has been often employed to identify  $N$ -body models belonging to this class. The behavior of the solutions of these  $N$ -body problems have been variously investigated and also employed to arrive at related mathematical results, such as finite-dimensional representations of differential operators and Diophantine properties of the zeros of certain polynomials: see the two

monographs [3] and [4] and the references quoted there (including the more recent ones added to the 2012 paperback version of [4]), and the more recent papers [5].

The *linear* PDEs satisfied by the polynomial  $\psi(z, t)$  were originally restricted to feature derivatives up to *second* order [1,3,4]. Indeed, the restriction to *time*-differentiations of *second* order was motivated by the interest in  $N$ -body problems featuring equations of motion of *Newtonian* type (i.e., “acceleration equal force”); while the restriction to  $z$ -differentiations of no more than *second* order implied that the  $N$ -body problems under consideration only involved *one-body* and *two-body* (velocity-dependent) forces. Recently consideration has been extended to *third* order  $z$ -derivatives—while maintaining the restriction to *second* order *time*-differentiations so as to treat  $N$ -body problems of *Newtonian* type; the corresponding *solvable*  $N$ -body problems thereby identified involve, additionally, *three-body* forces [6].

In this paper we investigate the *equilibria* of a subclass of these solvable  $N$ -body problems, such that the behavior of their solutions can be more explicitly ascertained (see below). It turns out that the values of the  $N$  coordinates  $\hat{z}_n$  characterizing the *equilibrium configurations* of these systems are simply related to the  $N$  roots  $x_n$  of the polynomial equation  $P_N^{(\alpha, \beta)}(x) = w$ , where  $P_N^{(\alpha, \beta)}(x)$  is the Jacobi polynomial of degree  $N$  and  $w$  is an *arbitrary* number. We then consider the behavior of these solvable systems in the infinitesimal neighborhood of their equilibria. This behavior is of course characterized by an appropriately *linearized* version of the Newtonian equations of motion, the solution of which is therefore reduced to the purely algebraic task of determining the *eigenvalues* of certain generalized eigenvalue problems for  $N \times N$  matrices explicitly defined in terms of the  $N$  coordinates characterizing the equilibria. Since the behavior of a system near its equilibria is merely a special case of its general behavior, for the *solvable* systems under consideration it is thereby possible to explicitly obtain these eigenvalues. These findings are the main results of this paper, as reported in the following Section 3; in some cases they have a *Diophantine* connotation, inasmuch as these eigenvalues are *all rational numbers*. In Section 4 these findings are proven by the procedure we just outlined above, while Section 2 reports relevant results of [6], thereby establishing notation and making this paper self-sufficient. A terse Section 5 entitled “Outlook” outlines possible future developments.

## 2. Review of relevant results of Ref. [6]

In this Section 2 we report relevant results of [6] and introduce notation used below.

### 2.1. Notation

The  $N$  coordinates (“dependent variables”) of the point, unit-mass, moving particles which are the protagonists of the  $N$ -body problem treated in [6] are generally denoted as  $z_n \equiv z_n(t)$ , with  $t$  (“time”) the “independent variable”. As usual differentiation with respect to time is denoted by a superimposed dot, hence  $\dot{z}_n \equiv dz_n/dt$ ,  $\ddot{z}_n \equiv d^2z_n/dt^2$ . As we just did here, often the indication of the  $t$ -dependence is not explicitly displayed. We generally assume that these coordinates  $z_n$  are *complex* numbers, so that the points with coordinates  $z_n(t)$  move in the *complex*  $z$ -plane. Here and hereafter  $i$  is the imaginary unit,  $i^2 = -1$ ,  $N$  is an *arbitrary* positive integer (generally  $N \geq 2$ ), and it is understood that subscripts such as  $n$  (and also  $m$ ,  $k$ ,  $\ell$ , but not  $j$ ; see below) run over the positive integers from 1 to  $N$  (unless otherwise specified); the reader is often (but not always) explicitly reminded of this fact. We also use occasionally the notation  $\underline{z}$  to denote the (generally *complex*)  $N$ -vector of components  $z_n$ ,  $\underline{z} \equiv (z_1, \dots, z_N)$ ; and likewise for other underlined letters (see below). A key role in our

treatment is played by the time-dependent (monic) polynomial  $\psi(z, t)$  of degree  $N$  in the (scalar, generally complex) variable  $z$  which features the  $N$  coordinates  $z_n(t)$  as its  $N$  zeros, see (1.1).  $\square$

The solvable  $N$ -body problem treated in [6] is characterized by the following Newtonian equations of motion:

$$\begin{aligned} \ddot{z}_n + E \dot{z}_n = & B_0 + B_1 z_n - (N-1) [2 A_3 + 3 (N-2) F_4] z_n^2 \\ & + (N-1) (N-2) G_3 \dot{z}_n z_n \\ & + \sum_{\ell=1; \ell \neq n}^N \left\{ (z_n - z_\ell)^{-1} [2 \dot{z}_n \dot{z}_\ell + 2 (A_0 + A_1 z_n + A_2 z_n^2 + A_3 z_n^3) \right. \\ & \left. - (\dot{z}_n + \dot{z}_\ell) (D_0 + D_1 z_n) - D_2 z_n (\dot{z}_n z_\ell + \dot{z}_\ell z_n)] \right\} \\ & + \sum_{k, \ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left[ \frac{3 (F_0 + F_1 z_n + F_2 z_n^2 + F_3 z_n^3 + F_4 z_n^4)}{(z_n - z_\ell) (z_n - z_k)} \right. \\ & \left. - \left( \frac{G_0 + G_1 z_n + G_2 z_n^2 + G_3 z_n^3}{z_n - z_\ell} \right) \left( \frac{\dot{z}_n + \dot{z}_k}{z_n - z_k} + \frac{\dot{z}_\ell + \dot{z}_k}{z_\ell - z_k} \right) \right], \\ n = & 1, \dots, N. \end{aligned} \tag{2.1}$$

Here and hereafter the 19 upper-case letters  $A_j, B_j, D_j, E, F_j, G_j$  denote time-independent parameters (“coupling constants”). They are *a priori arbitrary* (possibly complex) numbers; special assignments are considered below.

This Newtonian  $N$ -body system is solvable by algebraic operations because—as shown in [6]—the coordinates  $z_n(t)$  evolving according to this system of  $N$  Ordinary Differential Equations (ODEs) coincide with the  $N$  zeros of the time-dependent polynomial (1.1) of degree  $N$  in  $z$ , itself evolving according to the following linear PDE:

$$\begin{aligned} \psi_{tt} + \{E - (N-1) [D_2 + (N-2) G_3] z\} \psi_t & \\ + (G_0 + G_1 z + G_2 z^2 + G_3 z^3) \psi_{zxt} & \\ + (D_0 + D_1 z + D_2 z^2) \psi_{xt} & \\ + (F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4) \psi_{zzz} & \\ + (A_0 + A_1 z + A_2 z^2 + A_3 z^3) \psi_{zz} & \\ + \{B_0 + B_1 z - (N-1) [2 A_3 + 3 (N-2) F_4] z^2\} \psi_z & \\ + N \{-B_1 - (N-1) [A_2 + (N-2) F_3] & \\ + (N-1) [A_3 + 2 (N-2) F_4] z\} \psi = 0. & \end{aligned} \tag{2.2}$$

Here the 19 upper-case letters are of course the same time-independent parameters featured by the Newtonian equations of motion (2.1); and (here and hereafter) subscripted variables denote partial differentiations.

This PDE implies that the  $N$  coefficients  $c_m \equiv c_m(t)$ , see (1.1), evolve according to the following system of  $N$  linear ODEs:

$$\begin{aligned} & \ddot{c}_m + (N + 2 - m) (N + 1 - m) G_0 \dot{c}_{m-2} \\ & + (N + 1 - m) [D_0 + (N - m) G_1] \dot{c}_{m-1} \\ & + \{E + (N - m) [D_1 + (N - 1 - m) G_2]\} \dot{c}_m \\ & - m [D_2 + (2N - 3 - m) G_3] \dot{c}_{m+1} \\ & + (N + 3 - m) (N + 2 - m) (N + 1 - m) F_0 c_{m-3} \\ & + (N + 2 - m) (N + 1 - m) [A_0 + (N - m) F_1] c_{m-2} \\ & + (N + 1 - m) \{(N - m) [A_1 + (N - 1 - m) F_2] + B_0\} c_{m-1} \\ & - m \{(2N - 1 - m) A_2 + B_1 \\ & + [3N^2 - 6N + 2 - 3(N - 1)m + m^2] F_3\} c_m \\ & + m(m + 1) [A_3 + (3N - 5 - m) F_4] c_{m+1} = 0, \quad m = 1, \dots, N, \end{aligned} \tag{2.3}$$

of course with  $c_0 = 1$  and  $c_m = 0$  for  $m < 0$  and for  $m > N$  (see (1.1)).

This system, (2.3), of  $N$  autonomous linear ODEs is of course solvable by algebraic operations. Indeed its general solution reads

$$\underline{c}(t) = \sum_{m=1}^N \left[ b_m^{(+)} \underline{v}^{(+)(m)} \exp\left(\lambda_m^{(+)} t\right) + b_m^{(-)} \underline{v}^{(-)(m)} \exp\left(\lambda_m^{(-)} t\right) \right], \tag{2.4}$$

where the  $N$ -vector  $\underline{c}(t)$  has the  $N$  components  $c_m(t)$ , the  $2N$  (time-independent) coefficients  $b_m^{(\pm)}$  are *a priori* arbitrary—to be fixed *a posteriori* in order to satisfy the  $2N$  initial conditions  $c_m(0)$  and  $\dot{c}_m(0)$ —while the  $2N$  (time-independent)  $N$ -vectors  $\underline{v}^{(\pm)(m)}$  respectively the  $2N$  (time-independent) numbers  $\lambda_m^{(\pm)}$  are the  $2N$  eigenvectors, respectively the  $2N$  eigenvalues, of the following (time-independent) generalized matrix-vector eigenvalue problem:

$$(\lambda^2 \underline{I} + \lambda \underline{U} + \underline{V}) \underline{v} = 0. \tag{2.5a}$$

Here of course  $\underline{I}$  is the  $N \times N$  identity matrix ( $I_{mn} = \delta_{m,n}$  where, here and below,  $\delta_{m,n}$  is the Kronecker symbol) and the two  $N \times N$  matrices  $\underline{U}$  and  $\underline{V}$  are defined componentwise as follows:

$$\begin{aligned} U_{mn} &= (N + 1 - m) (N + 2 - m) G_0 \delta_{n,m-2} \\ &+ (N + 1 - m) [D_0 + (N - m) G_1] \delta_{n,m-1} \\ &+ \{E + (N - m) [D_1 + (N - 1 - m) G_2]\} \delta_{n,m} \\ &- m [D_2 + (2N - 3 - m) G_3] \delta_{n,m+1}, \quad n, m = 1, \dots, N; \end{aligned} \tag{2.5b}$$

$$\begin{aligned} V_{mn} &= (N + 3 - m) (N + 2 - m) (N + 1 - m) F_0 \delta_{n,m-3} \\ &+ (N + 2 - m) (N + 1 - m) [A_0 + (N - m) F_1] \delta_{n,m-2} \\ &+ (N + 1 - m) \{(N - m) [A_1 + (N - 1 - m) F_2] + B_0\} \delta_{n,m-1} \\ &- m \{(2N - 1 - m) A_2 + B_1 \\ &+ [3N^2 - 6N + 2 - 3(N - 1)m + m^2] F_3\} \delta_{n,m} \\ &+ m(m + 1) [A_3 + (3N - 5 - m) F_4] \delta_{n,m+1} \quad n, m = 1, \dots, N. \end{aligned} \tag{2.5c}$$

This implies of course that the  $2N$  eigenvalues  $\lambda_m^{(\pm)}$  are the  $2N$  roots of the following polynomial equation (of degree  $2N$  in  $\lambda$ ):

$$\det(\lambda^2 \underline{I} + \lambda \underline{U} + \underline{V}) = 0. \quad (2.5d)$$

These findings show that the solution of system (2.3) is achieved by the *algebraic* operation of determining the eigenvalues and eigenvectors of the matrix-vector generalized eigenvalue problem (2.5a).

The algebraic equation (2.5d) can be *explicitly* solved (for arbitrary  $N$ ) in the two special cases—on which we shall mainly focus—in which the two  $N \times N$  matrices  $\underline{U}$  and  $\underline{V}$  are *triangular*.

The *first* of these two special cases obtains if, of the 19 parameters in (2.3), the following 4 vanish:

$$A_3 = D_2 = F_4 = G_3 = 0. \quad (2.6a)$$

The *second* of these two special cases obtains instead if, of the 19 parameters in (2.3), the following 9 vanish:

$$A_0 = A_1 = B_0 = D_0 = F_0 = F_1 = F_2 = G_0 = G_1 = 0. \quad (2.6b)$$

It is then plain that—in both these two cases—the  $2N$  eigenvalues  $\lambda_m^{(\pm)}$  read

$$\lambda_m^{(\pm)} = -\frac{1}{2} \{E + (N - m) [D_1 + (N - 1 - m) G_2] \pm \Delta_m\}, \quad (2.7a)$$

$$\begin{aligned} \Delta_m^2 = & \{E + (N - m) [D_1 + (N - 1 - m) G_2]\}^2 \\ & + 4m \{(2N - 1 - m) A_2 + B_1 \\ & + [3N^2 - 6N + 2 - 3(N - 1)m + m^2] F_3\}. \end{aligned} \quad (2.7b)$$

These findings imply that—at least in these cases—predictions on the specific behavior of the solutions of the Newtonian  $N$ -body problem (2.1) can be made. And it is in particular plain that this Newtonian  $N$ -body problem is *isochronous* provided the 6 parameters  $E, D_1, G_2, A_2, B_1, F_3$  satisfy the following restrictions:

$$E = -2i r_1 \omega, \quad D_1 = -2i r_2 \omega, \quad G_2 = -2i r_3 \omega, \quad (2.8a)$$

$$A_2 = a_2 \omega^2, \quad B_1 = b_1 \omega^2, \quad F_3 = f_3 \omega^2, \quad (2.8b)$$

with

$$f_3 = 2 r_3 (s_2 r_4 - R_2), \quad (2.8c)$$

$$a_2 = 2 (s_1 s_2 - 1) R_1 r_3 - R_2^2 + r_4^2 - 3 (N - 1) f_3, \quad (2.8d)$$

$$b_1 = 2 R_1 (s_1 r_4 - R_2) - (2N - 1) a_2 - (3N^2 - 6N + 2) f_3, \quad (2.8e)$$

$$R_1 = r_1 + N [r_2 + (N - 1) r_3], \quad R_2 = r_2 + (2N - 1) r_3. \quad (2.8f)$$

Here and hereafter  $r_1, r_2, r_3, r_4$  are 4 arbitrary *rational* numbers,  $s_1$  and  $s_2$  are two *arbitrary signs* (+ or -) and  $\omega$  is a positive parameter. Indeed with these assignments the  $2N$  eigenvalues  $\lambda_m^{(\pm)}$  read

as follows,

$$\lambda_m^{(\pm)} = i r_m^{(\pm)} \omega, \quad r_m^{(\pm)} = (1 \pm s_1) R_1 - (R_2 \pm r_4) m + (1 \pm 1) r_3 m^2 \equiv \frac{p_m^{(\pm)}}{q_m^{(\pm)}}, \quad (2.9)$$

with the  $N$  numbers  $q_m^{(\pm)}$  all positive integers, the  $N$  numbers  $p_m^{(\pm)}$  all integers (positive, negative or vanishing, with  $p_m^{(\pm)}$  and  $q_m^{(\pm)}$  coprimes). Consequently (see (2.4))—if the rational numbers  $r_m^{(\pm)}$  are all different among themselves, as we hereafter assume for simplicity—the  $N$  coefficients  $c_m(t)$  evolve *isochronously*,

$$c_m(t + T) = c_m(t), \quad m = 1, \dots, N, \quad (2.10a)$$

with the period

$$T = \frac{2 \pi q}{\omega} \quad (2.10b)$$

independent of the initial data. Here of course  $q$  is the *minimum common multiple* of the  $2N$  denominators  $q_m^{(\pm)}$  of the  $2N$  rational numbers  $r_m^{(\pm)} \equiv p_m^{(\pm)} / q_m^{(\pm)}$ . And it is plain that the same property of *isochrony* is then shared by the  $N$  coordinates  $z_n(t)$ , namely the Newtonian  $N$ -body problem (2.1) is then *isochronous* as well; with the possibility that in some open regions of its phase space the periodicity only hold for a period which is a (generally *small*) integer multiple of  $T$  due to the fact that some of the zeros of the polynomial  $\psi(z, t)$ —itself evolving *isochronously* with period  $T$ , see (1.1) and (2.10)—might “exchange their roles” over the time evolution. For an analysis of this phenomenology, also explaining the meaning of the assertion made above that the *integer multiple* in question is generally *small*, see [7]; note however that this phenomenology is not relevant to the situations of main interest in this paper, when the evolution is characterized by coordinates  $z_n(t)$  infinitesimally close to their equilibrium values  $\bar{z}_n$ , hence unable to “exchange their roles” over the time evolution.

Other *isochronous* variants of the Newtonian  $N$ -body problem (2.1) are identified in [6] (see eqs. (19) there).

In this paper we restrict consideration to the subcase of the solvable  $N$ -body problem (2.1) characterized by the condition (2.6b), sufficient to allow the *explicit* evaluation of the  $2N$  eigenvalues  $\lambda_m^{(\pm)}$ , see (2.7); and moreover restricted by an additional condition on the coupling constants, see below, which is required in order that this  $N$ -body problem feature *genuine* equilibria, characterized by the requirement that the equilibrium coordinates  $\hat{z}_n$  be *all different among themselves*,  $\hat{z}_n \neq \hat{z}_m$  if  $n \neq m$ . This definition of *genuine* equilibria is introduced to avoid the ambiguities associated with the singularities appearing in the (right-hand side of the) equations of motion (2.1) whenever the values coincide of two coordinates  $z_n$  and  $z_m$  with  $n \neq m$ .

### 3. Main results

In this section we report the main results of this paper, which are then proven in the following Section 4.

Firstly let us discuss the *genuine equilibria* of the solvable  $N$ -body problem (2.1) with the “coupling constants” restricted by the conditions (2.6b) and (3.1)—the first of which, (2.6b), allows to obtain *explicitly* the  $2N$  eigenvalues  $\lambda_m^{(\pm)}$  characterizing the time evolution of this model (see (2.7)

and (2.4)), while the second,

$$B_1 = -(N-1) [A_2 + (N-2) F_3] , \quad (3.1)$$

is necessary and sufficient (see Section 4) for the existence of *genuine equilibria*. Hence the equations of motion characterizing the relevant time evolution now read as follows:

$$\begin{aligned} \ddot{z}_n + E \dot{z}_n = & -(N-1) \{A_2 + (N-2) F_3 - [2 A_3 + 3 (N-2) F_4] z_n\} z_n \\ & + (N-1) (N-2) G_3 \dot{z}_n z_n \\ & + \sum_{\ell=1; \ell \neq n}^N \{ (z_n - z_\ell)^{-1} [2 \dot{z}_n \dot{z}_\ell + 2 (A_2 + A_3 z_n) z_n^2 \\ & - D_1 (\dot{z}_n + \dot{z}_\ell) z_n - D_2 (\dot{z}_n z_\ell + \dot{z}_\ell z_n) z_n] \} \\ & + \sum_{k,\ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left\{ \frac{3 (F_3 + F_4 z_n) z_n^3}{(z_n - z_\ell) (z_n - z_k)} \right. \\ & \left. - \left[ \frac{(G_2 + G_3 z_n) z_n^2}{z_n - z_\ell} \right] \left( \frac{\dot{z}_n + \dot{z}_k}{z_n - z_k} + \frac{\dot{z}_\ell + \dot{z}_k}{z_\ell - z_k} \right) \right\} , \quad n = 1, \dots, N . \end{aligned} \quad (3.2)$$

Note that these equations of motion feature only the 9 coupling constants  $A_2, A_3, E, D_1, D_2, F_3, F_4, G_2, G_3$ .

Let now the  $N$  (time-independent) numbers  $\hat{z}_n$  indicate the *equilibrium* coordinates of this  $N$ -body problem, hence the solutions of the system of  $N$  algebraic equations

$$\begin{aligned} & -(N-1) [A_2 + (N-2) F_3] \hat{z}_n - (N-1) [2 A_3 + 3 (N-2) F_4] \hat{z}_n^2 \\ & + \sum_{\ell=1; \ell \neq n}^N \left[ \frac{2 \hat{z}_n^2 (A_2 + A_3 \hat{z}_n)}{\hat{z}_n - \hat{z}_\ell} \right] \\ & + \sum_{k,\ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left[ \frac{3 \hat{z}_n^3 (F_3 + F_4 \hat{z}_n)}{(\hat{z}_n - \hat{z}_\ell) (\hat{z}_n - \hat{z}_k)} \right] = 0 , \quad n = 1, \dots, N . \end{aligned} \quad (3.3)$$

Note that this system features now only the 4 coupling constants  $A_2, A_3, F_3, F_4$ .

**Proposition 3.1.** *The  $N$  equilibrium coordinates  $\hat{z}_n$  are given by the formula*

$$\hat{z}_n = \frac{2 a}{1 - x_n} , \quad n = 1, \dots, N , \quad (3.4)$$

where the  $N$  numbers  $x_n$  are the  $N$  roots  $x = x_n$  of the polynomial equation (of degree  $N$  in  $x$ )

$$P_N^{(\alpha, \beta)}(x) = w , \quad (3.5a)$$

where  $P_N^{(\alpha, \beta)}(x)$  is the standard Jacobi polynomial (see for instance [8]),

$$\alpha = 4 - 3 N - \frac{A_3}{F_4} , \quad \beta = -2 + \frac{A_3}{F_4} - \frac{A_2}{F_3} , \quad a = -\frac{F_3}{F_4} , \quad (3.5b)$$

and  $w$  is an arbitrary parameter.

**Remark 3.1.** This result implies that the  $N$  equilibrium coordinates  $\hat{z}_n$  depend on  $N$  and on the 3 coupling constant ratios  $A_3/F_4, A_2/F_3, F_3/F_4$ ; and moreover on the *arbitrary* parameter  $w$ , so that there is in fact a one-parameter family of equilibria for any given  $N$ -body problem (3.2).



Next, let us report the findings obtained by investigating the behavior of the solvable  $N$ -body problem (4.3b) in the infinitesimal neighborhood of its *genuine* equilibrium configuration described just above, see Proposition 3.1.

**Proposition 3.2.** *Let the  $N$  numbers  $x_n$  be the  $N$  roots of (3.5a) with (3.5b)—hence functions of the integer  $N$ , of the 3 ratios  $A_3/F_4$ ,  $A_2/F_4$  and  $F_3/F_4$ , and of the arbitrary parameter  $w$ —and let the two  $N \times N$  matrices  $\tilde{U}$  and  $\tilde{V}$  be defined componentwise, in terms of these numbers  $x_n$ , of the integer  $N$  and of the 10 arbitrary parameters  $A_2, A_3, B_1, D_1, D_2, E, F_3, F_4, G_2, G_3$  as follows:*

$$\tilde{U}_{nm} = E - \frac{2(N-1)(N-2)G_3 a}{1-x_n} + \sum_{\ell=1; \ell \neq n}^N \left[ \frac{D_1(1-x_\ell) + 2D_2 a}{x_n - x_\ell} \right] + \left( G_2 + \frac{2G_3 a}{1-x_n} \right) X_n, \quad n = 1, \dots, N; \tag{3.6a}$$

$$\begin{aligned} \tilde{U}_{nm} &= \left( D_1 + \frac{2D_2 a}{1-x_n} \right) \frac{(1-x_m)}{(x_n - x_m)} \\ &+ 2 \left( G_2 + \frac{2G_3 a}{1-x_n} \right) \frac{(1-x_m)}{(x_n - x_m)} \sum_{\ell=1; \ell \neq n, \ell \neq m}^N \left( \frac{1-x_\ell}{x_n - x_\ell} \right), \\ &n, m = 1, \dots, N, \quad n \neq m; \end{aligned} \tag{3.6b}$$

$$\begin{aligned} \tilde{V}_{nn} &= -B_1 + \frac{4(N-1)[2A_3 + 3(N-2)F_4] a}{1-x_n} \\ &- 4 \left( A_2 + \frac{3A_3 a}{1-x_n} \right) \sum_{\ell=1; \ell \neq n}^N \left( \frac{1-x_\ell}{x_n - x_\ell} \right) \\ &+ 2 \left( A_2 + \frac{2A_3 a}{1-x_n} \right) \sum_{\ell=1; \ell \neq n}^N \left( \frac{1-x_\ell}{x_n - x_\ell} \right)^2 + 3F_3 \left( \frac{5+3x_n}{1-x_n} \right) X_n \\ &- 6F_3 \left( \frac{1+x_n}{1-x_n} \right) \sum_{k, \ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left[ \frac{(1-x_\ell)(1-x_k)^2}{(x_n - x_\ell)(x_n - x_k)^2} \right], \end{aligned} \tag{3.6c}$$

$$n = 1, \dots, N; \tag{3.6d}$$

$$\begin{aligned} \tilde{V}_{nm} &= 2 \left( \frac{1-x_m}{x_n - x_m} \right)^2 \left\{ - \left( A_2 + \frac{2A_3 a}{1-x_n} \right) \right. \\ &+ 3F_3 \left( \frac{1+x_n}{1-x_n} \right) \sum_{\ell=1; \ell \neq n, \ell \neq m}^N \left( \frac{1-x_\ell}{x_n - x_\ell} \right) \left. \right\}, \\ &n, m = 1, \dots, N, \quad n \neq m; \end{aligned} \tag{3.6e}$$

where

$$X_n = \sum_{k, \ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left[ \frac{(1-x_\ell)(1-x_k)}{(x_n - x_\ell)(x_n - x_k)} \right], \quad n = 1, \dots, N, \tag{3.6f}$$

or equivalently (via (4.3a) and (3.4))

$$X_n = \frac{1}{3 F_3} \left( \frac{1-x_n}{1+x_n} \right) \left\{ B_1 - (N-1) [2 A_3 + 3 (N-2) F_4] \frac{2 a}{1-x_n} + 2 \left( A_2 + \frac{2 A_3 a}{1-x_n} \right) \sum_{\ell=1; \ell \neq n}^N \left( \frac{1-x_\ell}{x_n-x_\ell} \right) \right\}, \quad n = 1, \dots, N; \quad (3.6g)$$

and (see (3.5b))  $a = -F_3/F_4$ .

Then the  $2N$  eigenvalues  $\lambda_m^{(\pm)}$  of the generalized eigenvalue equation

$$(\lambda^2 \underline{I} + \lambda \underline{\tilde{U}} + \underline{\tilde{V}}) \underline{v} = 0, \quad (3.7a)$$

hence correspondingly the  $2N$  zeros of the following polynomial equation (of degree  $2N$ ) in  $\lambda$ ,

$$\det(\lambda^2 \underline{I} + \lambda \underline{\tilde{U}} + \underline{\tilde{V}}) = 0, \quad (3.7b)$$

are given by the explicit expression (2.7).

**Remark 3.2.** The  $2N$  quantities  $\lambda_m^{(\pm)}$  depend of course on  $N$  and  $m$ , and moreover (only) on the 6 parameters  $A_2, B_1, D_1, E, F_3, G_2$ : see (2.7). This implies that the eigenvalue problem (3.7a) is *isospectral* with respect to variations of the 5 parameters  $A_3, D_2, F_4, G_3$  and  $w$ , which feature (in addition of course to  $N$  and  $A_2, B_1, D_1, E, F_3, G_2$ ) in the definition of the two  $N \times N$  matrices  $\underline{\tilde{U}}$  and  $\underline{\tilde{V}}$ , see (3.6) and the definition (in Proposition 3.1) of the  $N$  numbers  $x_n$ .

**Remark 3.3.** If the 6 parameters  $A_2, B_1, D_1, E, F_3, G_2$  are restricted by the conditions (2.8), the  $2N$  numbers  $\lambda_m^{(\pm)}$  are all *rational* numbers, see (2.9), up to the common rescaling constant  $\omega$ .

Note the *Diophantine* character of this last finding.

The findings concerning Jacobi polynomials implied by Proposition 3.2 and by the Remarks 3.2 and 3.3 are somewhat similar to, but—due to the presence of *several arbitrary* parameters—much more general than previous results concerning the zeros of Jacobi polynomials (see in particular [9]). It is likely, indeed certain, that they could be also arrived at by techniques not as closely connected to  $N$ -body problems as the approach adopted in the present paper, but rather analogous to those employed in [9], or based on finite-dimensional representations of differential operators (see Section 2.4 entitled “Finite-dimensional representations of differential operators, Lagrangian interpolation and all that” and Appendix D entitled “Remarkable matrices and related identities” of [3]). It is indeed generally the case for findings concerning special functions that, after their initial discovery, several alternative ways to prove them can be developed.

#### 4. Proofs

In this section we prove the findings reported in the preceding Section 3.

Our first task is to investigate the equilibrium configurations,  $z_n(t) = \hat{z}_n$  (with  $\hat{z}_n$  of course time-independent) of the  $N$ -body problem characterized by the equations of motion (2.1). Clearly they

satisfy the following system of  $N$  algebraic equations (see (2.1))

$$\begin{aligned}
 & B_0 + B_1 \hat{z}_n - (N - 1) [2 A_3 + 3 (N - 2) F_4] \hat{z}_n^2 \\
 & + \sum_{\ell=1; \ell \neq n}^N \left[ \frac{2 (A_0 + A_1 \hat{z}_n + A_2 \hat{z}_n^2 + A_3 \hat{z}_n^3)}{\hat{z}_n - \hat{z}_\ell} \right] \\
 & + \sum_{k, \ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left[ \frac{3 (F_0 + F_1 \hat{z}_n + F_2 \hat{z}_n^2 + F_3 \hat{z}_n^3 + F_4 \hat{z}_n^4)}{(\hat{z}_n - \hat{z}_\ell) (\hat{z}_n - \hat{z}_k)} \right] = 0, \\
 & n = 1, \dots, N;
 \end{aligned} \tag{4.1a}$$

likewise, they are the  $N$  zeros of the (time-independent, monic) polynomial (see (1.1))

$$\hat{\psi}(z) = \prod_{n=1}^N (z - \hat{z}_n) = z^N + \sum_{m=1}^N \hat{c}_m z^{N-m}, \tag{4.1b}$$

which satisfies the third-order ODE (see (2.2))

$$\begin{aligned}
 & (F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4) \hat{\psi}''' \\
 & + (A_0 + A_1 z + A_2 z^2 + A_3 z^3) \hat{\psi}'' \\
 & + \{B_0 + B_1 z - (N - 1) [2 A_3 + 3 (N - 2) F_4] z^2\} \hat{\psi}' \\
 & + N \{-B_1 - (N - 1) [A_2 + (N - 2) F_3] \\
 & + (N - 1) [A_3 + 2 (N - 2) F_4] z\} \hat{\psi} = 0,
 \end{aligned} \tag{4.1c}$$

(here and hereafter appended primes in (4.1c) indicate of course differentiations with respect to the variable  $z$ ), and whose  $N$  (time-independent) coefficients  $\hat{c}_m$ , see (4.1b), satisfy the recursion relations (see (2.3))

$$\begin{aligned}
 & (N + 3 - m) (N + 2 - m) (N + 1 - m) F_0 \hat{c}_{m-3} \\
 & + (N + 2 - m) (N + 1 - m) [A_0 + (N - m) F_1] \hat{c}_{m-2} \\
 & + (N + 1 - m) \{(N - m) [A_1 + (N - 1 - m) F_2] + B_0\} \hat{c}_{m-1} \\
 & \quad - m \{(2 N - 1 - m) A_2 + B_1 \\
 & \quad + [3 N^2 - 6 N + 2 - 3 (N - 1) m + m^2] F_3\} \hat{c}_m \\
 & + m (m + 1) [A_3 + (3 N - 5 - m) F_4] \hat{c}_{m+1} = 0, \quad m = 1, \dots, N,
 \end{aligned} \tag{4.2a}$$

with the boundary conditions

$$\hat{c}_0 = 1 \quad \text{and} \quad \hat{c}_m = 0 \quad \text{for} \quad m < 0 \quad \text{and} \quad m > N. \tag{4.2b}$$

**Remark 4.1.** Of course an equilibrium configuration of the  $N$ -body problem (2.1) exists only if the differential equation (4.1c) possesses a *polynomial* solution of degree  $N$ . Conditions for this are discussed below. And recall that we focus on the identification of *genuine equilibria*, characterized by the requirement that the  $N$  coordinates  $\hat{z}_n$  be *all different among themselves*,  $\hat{z}_n \neq \hat{z}_m$  for  $n \neq m$ .

As indicated above, in this paper we deal only with the subcase of the class of  $N$ -body problems (2.1) characterized by the restriction (2.6b), which kills 9 of the 19 *a priori arbitrary* coupling

constants. Then (4.1a) simplifies to read

$$\begin{aligned}
 & B_1 - (N - 1) [2 A_3 + 3 (N - 2) F_4] \hat{z}_n \\
 & + \hat{z}_n \sum_{\ell=1; \ell \neq n}^N \left[ \frac{2 (A_2 + A_3 \hat{z}_n)}{\hat{z}_n - \hat{z}_\ell} \right] \\
 & + \hat{z}_n^2 \sum_{k, \ell=1; k \neq n, \ell \neq n, k \neq \ell}^N \left[ \frac{3 (F_3 + F_4 \hat{z}_n)}{(\hat{z}_n - \hat{z}_\ell) (\hat{z}_n - \hat{z}_k)} \right] = 0, \\
 & n = 1, \dots, N;
 \end{aligned} \tag{4.3a}$$

note that, to obtain this equation, we assumed (for simplicity, but without any significant loss of generality, see below) that none of the  $N$  equilibrium coordinates  $\hat{z}_n$  vanishes. Likewise (4.1c) simplifies substantially, reading

$$\begin{aligned}
 & (F_3 z^3 + F_4 z^4) \hat{\psi}''' + (A_2 z^2 + A_3 z^3) \hat{\psi}'' + \{B_1 z - (N - 1) [2 A_3 + 3 (N - 2) F_4] z^2\} \hat{\psi}' \\
 & + N \{-B_1 - (N - 1) [A_2 + (N - 2) F_3] + (N - 1) [A_3 + 2 (N - 2) F_4] z\} \hat{\psi} = 0;
 \end{aligned} \tag{4.3b}$$

and correspondingly (4.2a) becomes the following two-term recursion relation to be satisfied for  $m = 1, \dots, N$  (with the boundary condition  $\hat{c}_{N+1} = 0$ , clearly entailing  $\hat{c}_m = 0$  for  $m > N$ ):

$$\begin{aligned}
 & \{(2 N - 1 - m) A_2 + B_1 + [3 N^2 - 6 N + 2 - 3 (N - 1) m + m^2] F_3\} \hat{c}_m \\
 & = (m + 1) [A_3 + (3 N - 5 - m) F_4] \hat{c}_{m+1}, \quad m = 1, \dots, N.
 \end{aligned} \tag{4.3c}$$

**Remark 4.2.** These equations admit of course the trivial solution  $\hat{c}_m = 0$  for  $m = 1, \dots, N$ , implying  $\hat{\psi}(z) = z^N$ , see (4.1b) (and the diligent reader will verify that this is indeed a solution of (4.3b)). Of course all the zeros of this polynomial vanish, so they do not provide a *genuine* equilibrium configuration. To find a *genuine* equilibrium another solution of (4.3c), hence another polynomial solution of (4.3b), must be found (see below). But such a solution, if it exists, shall contain an *arbitrary* constant—in addition to the arbitrary overall multiplicative constant implied by the *linearity* of (4.3b) and (4.3c), which can itself be fixed by requiring the polynomial solution  $\hat{\psi}(z)$  to be *monic*—because if  $\hat{\psi}(z)$  is a solution of (4.3b), then  $\hat{\psi}(z) + L z^n$  with  $L$  an arbitrary parameter is clearly also a solution.

It is plain from (4.3c) with  $m = N$  that the boundary condition  $\hat{c}_{N+1} = 0$  entails the condition (3.1), which is therefore necessary and sufficient to guarantee that the  $N$ -body problem under consideration possess (for generic values of the *a priori arbitrary* constants  $A_2, A_3, F_3, F_4$ ) a *genuine* equilibrium configuration. It is then a standard task to solve the two-term recursion relation (4.3c), yielding (via (3.1))

$$\hat{c}_m = \hat{c}_1 a_{m-1} \frac{(1 - N)_{m-1} (N + 2 + \alpha + \beta)_{m-1}}{m! (2 + \alpha)_{m-1}}, \quad m = 1, \dots, N, \tag{4.3d}$$

with  $\hat{c}_1$  arbitrary. Here  $\alpha, \beta$  and  $a$  are defined by (3.5b) and  $(\eta)_m$  is the Pochhammer symbol,

$$(\eta)_0 = 1, \quad (\eta)_m = \eta (\eta + 1) \cdots (\eta + m - 1), \quad m = 1, 2, 3, \dots \tag{4.3e}$$

Note that the quantity  $\hat{c}_1$  in (4.3d) can now be assigned *arbitrarily* (consistently with Remark 4.2).

It is now a matter of trivial algebra—by inserting (4.3d) in (4.1b) and using the definition of the hypergeometric function (see for instance eq. 2.1.1(2) of [10]), then relating the resulting expression

of the hypergeometric function to Jacobi polynomials (see for instance eq. 10.8(16) of [8]), and also utilizing the standard differentiation formula for Jacobi polynomials (see for instance eq. 10.8(17) of [8] with  $m = 1$ )—to arrive at the formula

$$\hat{\psi}(z) = z^N \left[ K + L P_N^{(\alpha, \beta)}(x) \right], \quad x = 1 - \frac{2a}{z}, \quad (4.4)$$

with  $\alpha$ ,  $\beta$  and  $a$  defined by (3.5b). Here the letters  $K$  and  $L$  denote *constant* parameters whose values can be *arbitrarily assigned*; or only one of them can be *arbitrarily assigned*, and then the other can be adjusted to make  $\hat{\psi}(z)$  monic. In any case clearly this formula proves, via (4.1b), Proposition 3.1.

Our next task is to prove Proposition 3.2. The idea is to investigate the behavior of the equations of motion (3.2) in the infinitesimal neighborhood of its *genuine* equilibrium configuration  $\hat{z}_n$ , i.e. by setting (in (3.2))

$$z_n(t) = \hat{z}_n + \varepsilon \tilde{z}_n(t), \quad (4.5)$$

with  $\varepsilon$  infinitesimal. Then of course the  $N$  coordinates  $\tilde{z}_n(t)$  satisfy *linear* equations of motion, which can be written (in matrix-vector notation) as follows:

$$\ddot{\tilde{z}} + \tilde{U} \dot{\tilde{z}} + \tilde{V} \tilde{z} = 0. \quad (4.6a)$$

The two  $N \times N$  matrices  $\tilde{U}$  and  $\tilde{V}$  turn out—after a standard if somewhat tedious computation—to be defined componentwise by (3.6). The *general* solution of this *linear* system is then given by a standard formula analogous—*mutatis mutandis*—to (2.4), say

$$\tilde{z}(t) = \sum_{m=1}^N \left[ \beta_m^{(+)} \tilde{v}^{(+)(m)} \exp\left(\tilde{\lambda}_m^{(+)} t\right) + \beta_m^{(-)} \tilde{v}^{(-)(m)} \exp\left(\tilde{\lambda}_m^{(-)} t\right) \right], \quad (4.6b)$$

with the  $2N$  quantities  $\tilde{\lambda}_m^{(\pm)}$  being of course now the  $2N$  roots of the polynomial equation, of degree  $N$  in  $\tilde{\lambda}$ ,

$$\det\left(\tilde{\lambda}^2 \underline{I} + \tilde{\lambda} \tilde{U} + \tilde{V}\right) = 0. \quad (4.6c)$$

But the behavior of the  $N$ -body system (3.2) in the immediate vicinity of its equilibria cannot differ from its general behavior, which is characterized by the  $2N$  exponentials  $\exp\left(\lambda_m^{(\pm)} t\right)$ , see (2.4) and (1.1). Hence the set of the  $2N$  quantities  $\tilde{\lambda}_m^{(\pm)}$  must coincide with the set of the  $2N$  quantities  $\lambda_m^{(\pm)}$ , see (2.7). Proposition 3.2 is thus proven.

And the validity of the following Remarks 3.2 and 3.3 is sufficiently obvious not to require any further discussion.

## 5. Outlook

A natural follow-up to the results reported in this paper is to perform analogous investigations of the solvable  $N$ -body problem (2.1) in cases *not* satisfying the restriction (2.6b); in particular cases characterized instead by the alternative restriction (2.6a), or consisting of the 7 *isochronous* models identified in [6] (see eqs. (9) there). A very preliminary investigation suggests that these extensions—especially the first one—are far from trivial, requiring a considerable amount of further work, that we shall undertake if circumstances shall permit.

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