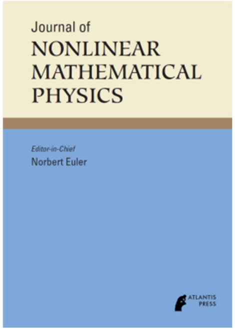

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In this paper we express the matrix coefficients of the Fock representation of a q-oscillator algebra in terms of the d-orthogonal Al-Salam Carlitz polynomials. Also, we derive a generating functions, recurrence relations and q-difference equations for these d-orthogonal polynomials.

Keywords: Coherent states; Quantum algebra; Basic orthogonal polynomials.

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1. Introduction

Let $\mathscr{P}$ be the linear space of polynomials with complex coefficients and let $\mathscr{P}'$ be its algebraic dual. A polynomials sequence $\{P_n\}_n$ is called a polynomial set if and only if $\deg(P_n) = n$ for all nonnegative integer n. We denote by $< u, f >$ the effect of the linear functional $u \in \mathscr{P}'$ on the polynomial $f \in \mathscr{P}$.
Let $\{P_n\}_n$ be a polynomials set in $\mathscr{P}$. The corresponding dual sequence (u_n) is defined by

$$< u_n, P_m >= \delta_{nm},\ n,m = 0,1,\dots,$$

where δ_{nm} being the Kronecker symbol.
A natural extension of the notion of orthogonality was introduced by Van Iseghem [7] and Maroni

[15] as follows:
Let d be a positive integer and let $\{P_n\}_n$ be a polynomials set in $\mathscr{P}$.
$\{P_n\}_n$ is called a d-orthogonal polynomials set(d-OPS for shorter) with respect to the d-dimensional functional vector $\mathscr{U} = {}^t(u_0, u_1, \ldots, u_{d-1})$ if it verifies the following conditions:

$$\langle u_k, P_m P_n \rangle = 0, \;\; n > md + k + 1,$$

$$\langle u_k, P_n P_{nd+k} \rangle \neq 0, \;\; n \geq 0.$$

For each integer $k \in \{0, 1, \ldots, d-1\}$.
For the particular case $d = 1$, we meet the well known notion of orthogonality.
Recall that $\{P_n\}_n$ is d-OPS if and only if it satisfies a recurrence relation of order $d+1$ of the type

$$xP_n(x) = \beta_{n+1} P_{n+1}(x) - \sum_{k=0}^{d} \alpha_{k,n-k} P_{n-k}(x),$$

where $\beta_{n+1}\alpha_{0,n-d} \neq 0$ and the convention $P_{-n} = 0, \; n \geq 1$. The result for $d = 1$ is reduced to the so-called Favard Theorem. During the past two decades, the d-OPS have been the subject of numerous investigations and applications. In particular they are connected with the study of vector padé approximants, simultaneous padé approximants and other problems such as vectorial continued fractions and polynomials solutions of the higher order differential equations. We mention also the appearance of multiple orthogonal polynomials is some problems of modern mathematical physics. The d-OPS can be obtained from general multiple orthogonal polynomials under some restrictions upon their parameters [1]. We mention also that numerous explicit examples of such polynomials have "good properties" that's to say explicit expression in terms of generalized hypergeometric functions or possessing some "classical properties" (see, [15]). A new applications of the d-OPS was presented recently in [17] by L.vinet and A.Zhedanov is connected with nonlinear automorphisms of the Weyl algebra.
In the same context, we would like to present a q-analogue of this work. In fact, we will consider an operator S which is no longer unitary and the corresponding matrix coefficients of this operator with respect to the initial basis give arise to a system of polynomials, which essentially coincides with a q-Charlier polynomials d-OPS. We show that almost all nontrivial properties the d-OPS q-Charlier polynomials can be derived directly from their definition as matrix elements of the Fock representation of the q-oscillator algebra.

2. The q-Oscillator algebra

In this section we consider a form of the q-oscillator algebra and we discuss some of its basic properties. Let us first review a few basic notions of q-calculus; the interested reader may consult [5].
Let q be a real number $0 < q < 1$. The q-shifted factorial are defined by

$$(a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \tag{2.1}$$

$$(a_1, \ldots, a_r; q)_n := (a_1;q)_n \ldots (a_r;q)_n, n = 0, 1, \ldots. \tag{2.2}$$

We denote also

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q,q)_k(q;q)_{n-k}}. \tag{2.3}$$

The q-exponentials functions are defined by [5]

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1-q)z;q)_\infty},\ |z| < \frac{1}{1-q},$$
$$E_q(z) := \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{z^n}{[n]_q!} = (-(1-q)z;q)_\infty,\ z \in \mathbb{C},$$

where

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

and

$$[n]_q = \frac{1-q^n}{1-q},\ \ [n]_q! = \prod_{k=1}^{n}[k]_q.$$

The q-difference operator

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

We have

$$D_q(fg)(x) = g(qx)D_q f(x) + f(x)D_q g(x).$$

2.1. *The Fock representations of the q-Oscillator algebra*

In the literature there are several forms of the q-deformed oscillator algebra, see [11, Ch.5]. In this work, we consider the q-oscillator algebra denoted by $\mathscr{A}_A$, which is the associative algebra over $\mathbb{C}$ generated by A_-, A_+, q_0^A, q^{-A_0} and relations (see [11])

$$[A_-,A_+]_q = 1,\ q^{A_0}A_+ = qA_+q^{A_0},\ q^{A_0}A_- = q^{-1}A_-q^{A_0},\ q^{A_0}q^{-A_0} = q^{-A_0}q^{A_0} = 1. \tag{2.4}$$

where

$$[A,B]_q := AB - qBA.$$

In the case $q = 1$, this algebra represents the one–dimensional harmonic oscillator algebra generated by three generators a, a^* and 1 with relations

$$aa^* - a^*a = 1,\ 1a = a1,\ 1a^* = a^*1.$$

Let $\mathscr{H}$ the be the Hilbert space with orthonormal basis $\{|n>\}_{n\in\mathbb{N}}$ and let $\mathscr{D}$ be the linear dense subspace of $\mathscr{H}$ spanned by $\{|n>\}_{n\in\mathbb{N}}$. Here we have used the standard Dirac notation (see [16]).

In this notation an state $|\psi>$ has the decomposition

$$|\psi>=\sum_{n=0}^{\infty}<n|\psi\rangle|n>,$$

where $<n|\psi>$ means the scalar product of the two states $|n>$ and $|\varphi>$.
The Fock representation of the q-oscillator algebra $\mathscr{A}_A$ is given by

$$A_+|n>=\sqrt{[n+1]_q}|n+1>, \tag{2.5}$$

$$A_-|n>=\sqrt{[n]_q}|n-1>, \tag{2.6}$$

$$q^{A_0}|n>=q^n|n>. \tag{2.7}$$

From (2.5) we get

$$|n>=\frac{A_+^n}{\sqrt{[n]_q!}}|0>, \tag{2.8}$$

where the vector $|0>$ is normalized by the condition

$$A_-|0>=0.$$

It's clear that from (2.5) and (2.6) the operator A_+A_- is hermitian and has for $n=0,1,\ldots,$ the q-numbers $[n]_q$ as eigenvalues

$$A_+A_-\mid n>=[n]_q\mid n>.$$

We denote by $|z>$ the q-coherent state defined by

$$|z>=e_q(zA_+)|0>=\sum_{n=0}^{+\infty}\frac{z^n}{\sqrt{[n]!_q}}|n>. \tag{2.9}$$

The state $|z>$ can be looked upon as an eigenstate of the operator A_- such that

$$A_-|z>=z|z>. \tag{2.10}$$

For q-coherent states $|z_1>$ and $|z_2>$, we have

$$<z_1|z_2>=e_q(z_1\overline{z_2}).$$

In addition, if

$$\psi(z)=<\overline{z}|\psi>,$$

then

$$D_q\psi(z)=<z|A_-|\psi>\ \ and\ \ z\psi(z)=<z|A_+|\psi>.$$

Let $S(A_-,A_+,A_0)$ be an operator constructed from operators A_-, A_+, A_0 . We assume that this operator is invertible, i.e there exists an operator $S^{-1}(A_-,A_+,A_0)$ such that

$$SS^{-1}=S^{-1}S=1. \tag{2.11}$$

Consider two systems of matrix coefficients:

$$\psi_{nk} =< k|S|n > \quad \text{and} \quad \phi_{nk} =< n|S^{-1}|k > . \tag{2.12}$$

It is assumed that the functions ψ_{nk} and ϕ_{nk} do exist. A simple computation shows that

$$S^{-1}|n >= \sum_{k=0}^{\infty} < k|S^{-1}|n > \ |k >$$

and

$$SS^{-1}|n >= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty} < k|S^{-1}|n >< r|S|k > |r > .$$

Then

$$< m|SS^{-1}|n >= \sum_{k=0}^{\infty} < m|S|k >< k|S^{-1}|n >,$$

and by (2.11) we obtain the identities

$$\sum_{k=0}^{\infty} < m|S|k >< k|S^{-1}|n >=< m|SS^{-1}|n >=< m|n >= \delta_{mn}.$$

Similarly,

$$\sum_{n=0}^{\infty} < k|S^{-1}|n >< n|S|s >=< k|S^{-1}S|s >=< k|s >= \delta_{ks}.$$

Hence, the matrix elements ψ_{nk}, ϕ_{nk} satisfy the bi-orthogonality relations

$$\sum_{k=0}^{\infty} \psi_{kn}\phi_{km} = \delta_{mn} \text{ and } \sum_{n=0}^{\infty} \psi_{sn}\phi_{kn} = \delta_{ks}. \tag{2.13}$$

2.2. *Identities in q-oscillator algebra*

The theory of quantum algebra and in particular q-oscillator algebra has been successful in producing identities for q-special functions (see [14]) and further references given there. From [14, Proposition 3.1] we have

$$e_q(q^{A_0}+A_+) = e_q(A_+)e_q(q^{A_0}),\ E_q(q^{A_0}+A_+) = E_q(q^{A_0})E_q(A_+),$$

$$e_q(A_- + q^{A_0}) = e_q(q^{A_0})e_q(A_-),\ E_q(A_- + q^{A_0}) = E_q(A_-)E_q(q^{A_0}).$$

Proposition 2.1. *For $n = 0,1,2,\ldots,$ we have*

$$[A_-,A_+^n] = [n]_q A_+^{n-1} q^{A_0}, \tag{2.14}$$

$$[A_-^n,A_+] = [n]_q q^{A_0} A_-^{n-1}. \tag{2.15}$$

Moreover, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a formal power series, we have

$$[A_-, f(A_+)] = D_q f(A_+)q^{A_0},\ \ [f(A_-),A_+] = q^{A_0} D_q f(A_-). \tag{2.16}$$

Proof. Let $n \in \mathbb{N}$, from (2.4) we have

$$\begin{aligned}[A_-^n, A_+] &= \sum_{i=0}^{n-1} A_-^i [A_-, A_+] A_-^{n-1-i} \\ &= \sum_{i=0}^{n-1} A_-^i q^{A_0} A_-^{n-1-i}, \\ &= \sum_{i=0}^{n-1} q^i q^{A_0} A_-^{n-1}, \\ &= [n]_q q^{A_0} A_-^{n-1}.\end{aligned}$$

The identity (2.16), follows from (2.14) and (2.15) and the fact that

$$[A_+, f(A_-)] = \sum_{n\geq 0} a_n [A_-^n, A_+] \text{ and } \quad [f(A_-), A_+] = \sum_{n\geq 0} a_n [A_-^n, A_+]. \tag{2.17}$$

□

Proposition 2.2. *Let P be a polynomial and t a complex number, we have*

$$\begin{aligned} e_q(tA_-)P(A_+)E_q(-tA_-) &= P(A_+ + tq^{A_0}), \\ E_q(tA_+)P(A_-)e_q(-tA_+) &= P(A_- - tq^{A_0}).\end{aligned}$$

Proposition 2.3. *Let $N \geq 0$. For all complex numbers $a_0, \ldots, a_N$, we have*

$$\prod_{i=0}^{N} e_q(a_iA_-)A_+A_- \prod_{i=0}^{N} E_q(-a_iA_-) = q^{A_0}[1 - \prod_{i=0}^{N}(1 - a_iA_-)] + A_+A_- \tag{2.18}$$

$$\prod_{i=0}^{N} e_q(a_iA_-)q^{A_0} \prod_{i=0}^{N} E_q(-a_iA_-) = q^{A_0} \prod_{i=0}^{N}(1 - a_i(1-q)A_-). \tag{2.19}$$

Proof. We will prove the formula (2.18) by recurrence.
For $N = 0$, we have

$$e_q(a_0A_-)A_+E_q(-a_0A_-) = a_0q^{A_0}A_- + A_+A_-.$$

We suppose that this expression is true for N , stay it true for the order $N+1$?
We have

$$\begin{aligned}\prod_{i=0}^{N+1} e_q(a_iA_-)A_+ \prod_{i=0}^{N+1} E_q(-a_iA_-) &= e_q(a_{N+1}A_-)\big(\sum_{p=1}^{N} \sum_{i_1<\cdots<i_p} a_{i_1} \ldots a_{i_p} q^{A_0} A_-^{p-1} + A_+\big)E_q(-a_{d+1}A_-) \\ &= (\sum_{p=1}^{N} \sum_{i_1<\cdots<i_p} a_{i_1} \ldots a_{i_p} e_q(a_{d+1}A_-) q^{A_0} A_-^{p-1} E_q(-a_{N+1}A_-) \\ &+ e_q(a_{N+1}A_-)A_+E_q(-a_{N+1}A_-).\end{aligned}$$

On the other hand

$$e_q(a_{N+1}A_-)q^{A_0} = q^{A_0} \sum_{n=0}^{+\infty} \frac{(1-q)^n (a_{N+1}qA_-)^n}{(q;q)_n} = q^{A_0} e_q(qa_{N+1}A_-).$$

Then we obtain

$$e_q(a_{N+1}A_-)q^{A_0} = q^{A_0}e_q(qa_{N+1}A_-) = q^{A_0}(1-a_{N+1}A_-)e_q(a_{N+1}A_-).$$

Hence

$$\prod_{i=0}^{N+1} e_q(a_iA_-)A_+\prod_{i=0}^{N+1} E_q(-a_iA_-) = \sigma_1 q^{A_0} - \sigma_2 q^{A_0}A_- + \cdots + (-1)^N\sigma_{N+1}q^{A_0}A_-^N + A_+.$$

Then, the formula (2.18) follows from the fact that

$$\sigma_1 q^{A_0}A_- - \sigma_2 q^{A_0}A_-^2 + \cdots + (-1)^N\sigma_{N+1}q^{A_0}A_-^{N+1} = q^{A_0}[1-\prod_{i=0}^{N}(1-a_iA_-)].$$

The proof of (2.19) is similar to (2.18). □

3. Properties

3.1. *Generating functions*

In this section we calculate the generating functions of the matrix coefficients ψ_{nk} and ϕ_{nk} related to the operator S given by

$$S = E_q(\beta A_+)\prod_{i=1}^{d} e_q(a_iA_-). \tag{3.1}$$

The method is very similar to the one used in [17].
We have according to (2.9)

$$\sum_{n=0}^{\infty}\psi_{nk}\frac{z^n}{\sqrt{[n]_q!}} = \sum_{n=0}^{\infty} < k|S|n > \frac{z^n}{\sqrt{[n]_q!}} =< k|S|\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_q!}}|n >=< k|S|z > .$$

Taking into account of formula (2.10), we have

$$< k|S|z >=< k|E_q(\beta A_+)H_d(A_-)|z >= H_d(z) < k|E_q(\beta A_+)e_q(zA_+)|0 >,$$

where

$$H_d(z) := H_d(z,a_1,\ldots,a_d) = \prod_{i=1}^{d} e_q(a_iz).$$

On the other hand, we have successively by means of the (2.8) and q-binomial formula (see [5])

$$E_q(\beta A_+)e_q(zA_+)|0\rangle = \sum_{n=0}^{\infty}\frac{\theta_n(z,\beta;q)}{[n]_q!}A_+^n|0\rangle = \sum_{n=0}^{\infty}\frac{\theta_n(z,\beta;q)}{\sqrt{[n]_q!}}|n >,$$

and

$$< k|E_q(\beta A_+)e_q(zA_+)|0 >= \frac{\theta_k(z,\beta;q)}{\sqrt{[k]_q!}},$$

where

$$\theta_k(z,\beta;q) = z^k(-\beta/z;q)_k. \tag{3.2}$$

Hence, the matrix coefficients ψ_{nk} are generated by

$$F(z,k) := \frac{\theta_k(z,\beta;q)}{\sqrt{[k]_q!}} H_d(z) = \sum_{n=0}^{\infty} \psi_{nk} \frac{z^n}{\sqrt{[n]_q!}}. \tag{3.3}$$

3.2. *Recurrence relations*

If we apply the q-difference operator D_q to each member of (3.3) and we use the following formulas

$$\theta_k(qz,\beta;q) = q^k(z+\beta/q)\theta_{k-1}(z,\beta;q),\ D_q\theta_k(z,\beta;q) = [k]\theta_{k-1}(z,\beta;q),$$

$$D_qH(z) = Q(z)H(z),\ \ Q(z) = \frac{1-\prod_{i=1}^{d}(1-(1-q)a_iz)}{(1-q)z}.$$

We get

$$\begin{aligned}\sum_{n=1}^{\infty}\sqrt{[n]_q}\psi_{kn}\frac{z^{n-1}}{\sqrt{[n-1]_q!}} &= \frac{1}{\sqrt{(k]_q!}}\Big([k]_q\theta_{k-1}(z,\beta;q)H_d(qz) + \theta_k(z,\beta;q)Q(z)H_d(z)\Big)\\ &= \frac{1}{\sqrt{(k]_q!}}\Big(\frac{[k]_q\theta_k(qz,\beta;q)}{q^k(z+\beta/q)}H_d(qz) + \theta_k(z,\beta;q)Q(z)H_d(z)\Big),\\ &= \frac{q^{-k}[k]_q}{q^k(z+\beta/q)}F(qz,k) + \big(\sum_{i=0}^{d}\alpha_iz^i\big)F(z,k),\end{aligned}$$

where

$$(z+\beta/q)Q(z) = \sum_{i=0}^{d}\alpha_iz^i. \tag{3.4}$$

Consequently

$$\begin{aligned}(z+\beta/q)\sum_{n=1}^{\infty}\sqrt{[n]_q}\psi_{nk}\frac{z^{n-1}}{\sqrt{[n-1]_q!}} &= q^{-k}[k]_q\sum_{n=0}^{\infty}q^n\psi_{nk}\frac{z^n}{\sqrt{[n]_q!}}\\ &+ \sum_{n=0}^{\infty}\Big(\sum_{i=0}^{d}\alpha_i\sqrt{[n]_q\ldots[n-i+1]_q}\psi_{n-ik}\Big)\frac{z^n}{\sqrt{[n]_q!}}.\end{aligned}$$

Comparing now the coefficients of z^n, we get

Proposition 3.1. *The matrix coefficients ψ_{nk} satisfy the recurrence relation*

$$\frac{\beta}{q}\sqrt{[n+1]_q}\psi_{n+1\,k} = -[n-k]_q\psi_{n\,k} + \sum_{i=0}^{d}\alpha_i\sqrt{[n]_q\ldots[n-i+1]_q}\psi_{n-ik}. \tag{3.5}$$

Now, from (3.5) one can express $\psi_{n,k}$ recursively, starting from ψ_{0k}. Indeed, putting $n=0$ we obtain

$$\psi_{1\,k} = \frac{q}{\beta}(\alpha_0 - \frac{1-q^{-k}}{1-q})\psi_{0\,k},$$

and for $n=1$, we have

$$\psi_{2\,k} = \Big(\frac{q^2}{\beta^2}\frac{1-q}{1-q^2}(\alpha_0 - \frac{1-q^{-k}}{1-q})(\alpha_0 - \frac{1-q^{1-k}}{1-q}) + \alpha_1\Big)\psi_{0\,k}.$$

Repeating this process we arrive at the following.

Proposition 3.2. *The matrix elements $\psi_{n,k}$ are expressed as*

$$\psi_{nk} = \psi_{0k} V_n^{(a_1,\ldots,a_d)}(q^{-k})$$

where $V_n^{(a_1,\ldots,a_d)}(q^{-k})$ is a polynomial of degree n in q^{-k} and satisfying the recurrence relation of order $(d+1)$

$$\frac{\beta}{q}\sqrt{[n+1]_q} V_{n+1}^{(a_1,\ldots,a_d)}(q^{-k}) = -[n-k]_q V_n^{(a_1,\ldots,a_d)}(q^{-k}) + \sum_{i=0}^{d} \alpha_i \sqrt{[n]_q \ldots [n-i+1]_q} V_{n-i}^{(a_1,\ldots,a_d)}(q^{-k}),$$

with initial conditions

$$V_0^{(a_1,\ldots,a_d)}(q^{-k}) = 1,\ V_n^{(a_1,\ldots,a_d)}(q^{-k}) = 0,\ n < 0.$$

Consequently $\{V_n^{(a_1,\ldots,a_d)}(q^{-k})\}_{n\geq 0}$ is d-orthogonal.
The associated monic polynomial $\widetilde{V}_n^{(a_1,\ldots,a_d)}(q^{-k})$ is defined by

$$\widetilde{V}_n^{(a_1,\ldots,a_d)}(q^{-k}) = q^{-\binom{n}{2}} \beta^n (1-q)^n \sqrt{[n]_q!} V_n^{(a_1,\ldots,a_d)}(q^{-k})_n.$$

The polynomial $\widetilde{V}_n^{(a_1,\ldots,a_d)}$ is generated by

$$\sum_{n=0}^{\infty} (-1)^n \frac{\widetilde{V}_n^{(a_1,\ldots,a_d)}(q^{-k})}{(q;q)_n} z^n = \theta_k(z) H_d(z).$$

3.3. *Orthogonality relations*

Proposition 3.3. *The matrix coefficients ϕ_{nk} satisfy the difference equation*

$$-[n-k]_q \phi_{nk} = \beta \phi_{n-1\,k} - \sum_{i=0}^{d} \alpha_i q^{-i} \sqrt{[n+1]_q \ldots [n+i]_q} \phi_{n+i\,k}.$$

Proof. From the bi-orthogonality relations (2.13) and the generating function (3.3) the matrix coefficients ϕ_{nk} have the following generating function

$$G(z,k) := \frac{z^n}{\sqrt{[n]_q!} H_d(z)} = \sum_{k=0}^{\infty} \phi_{kn} \frac{\theta_k(z,\beta;q)}{\sqrt{[k]_q!}}. \tag{3.6}$$

Applying the operator D_q to each members of (3.6) we obtain

$$\sum_{k=1}^{\infty} [k]_q \phi_{nk} \frac{\theta_{k-1}(z)}{\sqrt{[k]_q!}} = \frac{[n]_q z^{n-1}}{\sqrt{[n]_q!} H_d(qz)} - \frac{z^n}{\sqrt{[n]_q!} H_d(z)} Q(z).$$

So that

$$\sum_{k=1}^{\infty} q^{-k} [k]_q \phi_{nk} \frac{\theta_k(qz)}{\sqrt{[k]_q!}} = \beta q^{-n} \sqrt{[n]_q} \frac{(qz)^{n-1}}{\sqrt{[n-1]_q!} H_d(qz)} + \frac{[n]_q z^n}{\sqrt{[n]_q!} H_d(qz)} \tag{3.7}$$

$$- \frac{1}{\sqrt{[n]_q!}} \sum_{i=0}^{d} \alpha_i \frac{z^{n+i}}{H_d(qz)}.$$

The result is finished by comparing the coefficients of $\theta_k(qz)$ in each members of (3.7). □

If $d > 1$ it is possible to express ϕ_{nk} in terms of polynomials of argument q^{-k}. According to the above proposition, the coefficient ϕ_{nk} can be expressed as

$$\phi_{nk} = \sum_{i=0}^{d-1} \phi_{ik} R_n^{(i)}(q^{-k}),$$

where $R_n^{(i)}(q^{(-k)}$ are polynomials of argument q^{-k}. The degrees of these polynomials depend on n in the following manner. Assume that $n = dj + r$ where $r = 0, \dots, d-1$. Then

$$\deg R_n^{(i)} = j \text{ if } i \leq r, \ \deg R_n^{(i)} = j-1 \text{ if } i > r.$$

In connection with the above result we introduce the functionals vector $(L_1, L_2, \dots, L_{d-1})$ defined by

$$L_i(f(x)) = \sum_{k=0}^{\infty} f(q^{-k}) q^{\binom{k}{2}} \frac{\beta^k}{\sqrt{[k]_q!}} \phi_{ik}.$$

Then we have the following.

Proposition 3.4. *The system of polynomials $\{P_n(x)\}_{n\in\mathbb{N}}$ satisfies the following vector orthogonality relation*

$$L_i(x^m \hat{P}_n(x)) = 0, \ \ n \geq md + i + 1, \ \ i = 0, \dots, d-1, \tag{3.8}$$

$$L_i(x^m \hat{P}_n(x)) \neq 0, \ \ n = md + i, \ \ i = 0, \dots, d-1. \tag{3.9}$$

Proof. Relations (3.8) and (3.9) are direct consequence of (2.13). □

4. Explicit expression of the d-OPS of q-Charlier type.

The Al-Salam Carlitz II polynomials $V_n^{(a)}(x;q)$ are defined by [12]

$$V_n^{(a)}(x;q) = (-a)^n q^{-\binom{n}{2}} {}_2\phi_0 \left(\left. \begin{matrix} q^{-n}, x \\ - \end{matrix} \right| q, \frac{q^n}{a} \right). \tag{4.1}$$

The polynomial $y(x) = V_n^{(a)}(x;q)$ is an eigenfunction of the following second order q-difference operator

$$(1-x)(a-x)y(qx) - [(1-x)(a-x) + aq]y(x) + aqy(q^{-1}x) = -(1-q^n)x^2 y(x).$$

The Al-Salam Carlitz II polynomials are closely related to the q-Charlier polynomials [12]

$$C_n(q^{-k}|\, q) = V_n(q^{-k}|\, q).$$

The three-term recurrence relation for the polynomials (4.1) is as follows (see [12])

$$xV_n^{(a)}(x;q) = V_{n+1}^{(a)}(x;q) + (a+1)q^{-n}V_n^{(a)}(x;q) + aq^{-2n+1}(1-q^n)V_{n-1}^{(a)}(x;q).$$

In this section we calculate the matrix coefficients ψ_{nk} and ϕ_{nk} associated to the operator S given by

$$S = E_q(\beta A_+) \prod_{i=1}^{d} e_q(a_i A_-), \tag{4.2}$$

in terms of the By means of technic based on the notion of a generating function, we express in this section the matrix elements ψ_{nk} in terms of a d-OPS where $V_n(q^{-k}|\, q)$ are the Al-Salam Carlitz II

polynomials.
Let $\omega = e^{2i\pi/d}$, $a \in \mathbb{C}$ and we suppose for $i = 0, 1, ..., d-1$, $a_i = a\omega^i$. From [5, I.30], we have

$$(a^d; q^d)_n = (a, a\omega, ..., a\omega^{d-1}; q)_n. \tag{4.3}$$

If we let $n \to \infty$ in (4.3), we get

$$\prod_{k=0}^{d-1} e_q(az\omega^k) = e_{q^d}(a^d z^d). \tag{4.4}$$

Hence the operator S becomes

$$S = E_q(\beta A_+) e_{q^d}(a^d A_-^d).$$

We denote by

$$V_n^{(a,d)}(x) = V_n^{(a,a\omega,\ldots,a\omega^{d-1})}(x).$$

Now, by means of technic based on the notion of a generating function, we express the matrix elements ψ_{nk} in terms of a d-OPS of Al-Salam Carlitz *II* polynomials evaluate the matrix coefficients ϕ_{nk} in terms of basic hypergeometric series. Expanding $e_q(q^{-k}z))^{-1}$ and $e_{q^d}(az^d)$ in terms of z^n, we find

$$\sum_{n=0}^{\infty} (-1)^n \frac{V_n^{(a,d)}(q^{-k})}{(q;q)_n} z^n = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{[n/d]} \frac{a^i (1-q)^{n-id} (1-q^d)^i (q^{-k};q)_{n-id}}{(q^d;q^d)_i (q;q)_{n-id}} \right) z^n.$$

Consequently

$$V_n^{(a,d)}(q^{-k}) = (-1)^n (q;q)_n = \sum_{i=0}^{[n/d]} \frac{a^i (1-q)^{n-id} (1-q^d)^i (q^{-k};q)_{n-id}}{(q^d;q^d)_i (q;q)_{n-id}}.$$

According to the following identity

$$\frac{(q^{-k};q)_{n-id}}{(q;q)_{n-id}} = q^{-ikd} \frac{(q^{-k};q)_n}{(q;q)_n} \frac{\prod_{j=1}^d (q^{-1-n+j};q^d)_i}{\prod_{j=1}^d (q^{k-n+j};q^d)_i},$$

we obtain

$$V_n^{(a,d)}(q^{-k}) = (-1)^n (q^{-k};q)_n (1-q)^n \sum_{i=0}^{\infty} \frac{\prod_{j=1}^d (q^{-1-n+j};q^d)_i}{\prod_{j=1}^d (q^{k-n+j};q^d)_i} \frac{a^i (1-q^d)^i q^{-ikd}}{(q^d;q^d)_i (1-q)^{id}}.$$

Finally

$$V_n^{(a,d)}(q^{-k}) = (-1)^n (q^{-k};q)_n (1-q)^n {}_{d+1}\varphi_d \left(\begin{matrix} \Delta(-n,d;q^d), 0 \\ \Delta(-n,d;q^d) \end{matrix} \middle| q^d; \frac{a(1-q^d)^i q^{-kd}}{(1-q)^d} \right),$$

where

$$\Delta(\lambda, m; q) = q^{\lambda/m}, q^{(\lambda+1)/m}, \ldots, q^{(\lambda+m-1)/m}.$$

4.1. *Lowering and raising operators*

According to Proposition 2.1 and Proposition 2.2, we get

$$SA_- = (A_- - \beta q^{A_0})S.$$

Then

$$\sqrt{[n]_q}\psi_{n-1,k} = \langle k|SA_-|n\rangle = \langle k|(A_- - \beta q^{A_0})S|n\rangle = \langle k|A_-S|n\rangle - \beta\langle k|q^{A_0}S|n\rangle$$

and

$$\sqrt{[n]_q}\psi_{n-1,k} = \sqrt{[k+1]_q}\psi_{n,k+1} - \beta q^k\psi_{nk}. \tag{4.5}$$

Recall that

$$\psi_{0k} =< k|S|0 >=< k|E_q(\beta A_+)|0 >=< k|\beta >= \frac{\beta^k q^{k(k-1)/2}}{\sqrt{[k]_q!}}.$$

Dividing the two members of (4.5) by ψ_{0k} we get

$$\beta^k q^k\big(V_n^{(a_1,\dots,a_d)}(q^{-k-1})(q^{-(k+1)}) - V_n^{(a_1,\dots,a_d)}(q^{-k})\big) = \sqrt{[n]_q}V_{n-1}^{(a_1,\dots,a_d)}(q^{-k}).$$

Since

$$\widetilde{V}_n^{(a_1,\dots,a_d)}(q^{-k}) = q^{-\binom{n}{2}}\beta^n(1-q)^n\sqrt{[n]_q!}V_n^{(a_1,\dots,a_d)}(q^{-k})_n.$$

So that

$$q^k\big(\widetilde{V}_n^{(a_1,\dots,a_d)}(q^{-k-1}) - \widetilde{V}_n^{(a_1,\dots,a_d)}(q^{-k})\big) = (q^n-1)\widetilde{V}_{n-1}^{(a_1,\dots,a_d)}(q^{-k}).$$

On other words

$$(D_{q^{-1}}\widetilde{V}_n^{(a_1,\dots,a_d)})(q^{-k}) = [n]_q\widetilde{V}_n^{(a_1,\dots,a_d)}(q^{-k}).$$

From Proposition 2.2 and Proposition 2.3, we can write

$$\begin{aligned} SA_+S^{-1} &= E_q(\beta A_+)(A_+q^{A_0}Q(A_-))e_q(-\beta A_+) \\ &= A_+ + (1+(1-q)\beta A_+)q^{A_0}Q(A_- - (1-q)\beta q^{A_0}). \end{aligned}$$

Hence

$$SA_+ = (A_+ + (1+(1-q)\beta A_+)q^{A_0})(A_- - (1-q)\beta q^{A_0})^{d-1}S. \tag{4.6}$$

The operator A_- and q^{A_0} satisfy the q-commutation relation

$$A_-q^{A_0} = qq^{A_0}A_-.$$

Then from the well know q-binomial Newton formula for q-commuting variables (see [14]) we get

$$(A_- - (1-q)\beta q^{A_0})^{d-1} = \sum_{s=0}^{d-1}\begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-(1-q)\beta)^{d-s-1}q^{(d-s-1)A_0}A_-^s. \tag{4.7}$$

From (4.6), we have

$$\begin{aligned}
\sqrt{[n+1]_q}\psi_{n+1\,k} &= < k|SA_+|n > \\
&= < k|A_+ + (1+(1-q)\beta A_+)q^{A_0}Q(A_- - (1-q)\beta q^{A_0})|n > \\
&= \sqrt{[n+1]_q}(1-(-(1-q)\beta)^{d-1}q^{kd})\psi_{n\,k+1} + \sum_{s=0}^{d-1}\begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-(1-q)\beta)^{d-s-1} \\
&\times q^{(d-s-1)k}\sqrt{\frac{[k]_q!}{[k-s]_q!}}\psi_{n\,k-s} + \sum_{s=0}^{d-2}\begin{bmatrix} d-1 \\ s+1 \end{bmatrix}_q (-(1-q)\beta)^{d-s-1}q^{(d-s-1)k} \\
&\times [k+1]_q\sqrt{\frac{[k]_q!}{[k-s]_q!}}\psi_{n\,k-s}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sqrt{[n+1]_q}\psi_{n+1\,k} = \sqrt{[n+1]_q}&(1-(-(1-q)\beta)^{d-1})q^{kd}\psi_{n\,k+1} + \sqrt{\frac{[k]_q!}{[k-d-1]_q!}} \\
&\times \psi_{n\,k-d+1} + \sum_{s=0}^{d-1}\begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-(1-q)\beta)^{d-s-1}q^{(d-s-1)k}\sqrt{\frac{[k]_q!}{[k-s]_q!}} \\
&\times (1+\frac{[d-s-2]_q[k+1]_q}{[s+1]_q})\psi_{n\,k-s}.
\end{aligned}$$

Henceforth

$$\begin{aligned}
\sqrt{[n+1]_q}V_{n+1}^{(ad)}(q^{-k}) = \sqrt{[n+1]_q}&(1-(-(1-q)\beta)^{d-1})q^{kd} \\
&\times V_n^{(ad)}(q^{-k-1}) + \sqrt{\frac{[k]_q!}{[k-d-1]_q!}}V_n^{(ad)}(q^{-k+d-1}) \\
&+ \sum_{s=0}^{d-1}\begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-(1-q)\beta)^{d-s-1}q^{(d-s-1)k}\sqrt{\frac{[k]_q!}{[k-s]_q!}} \\
&\times (1+\frac{[d-s-2]_q[k+1]_q}{[s+1]_q})V_n^{(ad)}(q^{-k+s}).
\end{aligned}$$

Let introduce the operator R_q

$$\begin{aligned}
R_q = &\sqrt{[n+1]_q}(1-(-(1-q)\beta)^{d-1})q^{kd} \\
&\times T_{q^{-1}}^{k+1} + \sqrt{\frac{[k]_q!}{[k-d-1]_q!}}T_{q^{-1}}^{k-d+1} \\
&+ \sum_{s=0}^{d-1}\begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-(1-q)\beta)^{d-s-1}q^{(d-s-1)k}\sqrt{\frac{[k]_q!}{[k-s]_q!}} \\
&\times (1+\frac{[d-s-2]_q[k+1]_q}{[s+1]_q})T_{q^{-1}}^{k-s}.
\end{aligned}$$

Here $T_{q^{-1}}$ is the q-shift operator defined by $(T_{q^{-1}}P)(x) = P(q^{-1}x)$. Then

$$(R_q V_n^{(ad)})(q^{-k}) = \sqrt{[n+1]_q} V_{n+1}^{(ad)}(q^{-k}).$$

Note that the operators $D_{q^{-1}}$ and R_q satisfy the relation

$$R_q D_{q^{-1}} - q D_{q^{-1}} R_q = 1.$$

In order to find the dual function ϕ_{nk}, we need the following Lemma.

Lemma 4.1. *If* $f(z) = \sum_{n=0}^{\infty} a_k \theta_k(z)$, *then*

$$a_k = \frac{1}{[k]_q!}\left[D_q^k f(z)\right]_{z=-\beta}.$$

If $d = 1$, by Lemma 4.1, we can write

$$\phi_{0k} = \frac{1}{\sqrt{[k]_q!}}\left[D_q^k(E_q(-az))\right]_{z=-\beta} = \frac{1}{\sqrt{[k]_q!}}(-a)^k q^{\binom{k}{2}}\left[E_q(-azq^k)\right]_{z=-\beta}$$

$$= \frac{1}{\sqrt{[k]_q!}}(-a)^k q^{\binom{k}{2}} E_q\left(\frac{-aq^{k+1}}{1-q}\right) = \frac{1}{\sqrt{[k]_q!}}(-a)^k q^{\binom{k}{2}} \frac{(a,q)_\infty}{(aq;q)_k}.$$

On the other hand,

$$\phi_{0k}\psi_{0k} = \frac{a^k q^{k^2}}{(aq;q)_k (q;q)_k}.$$

Consequently

$$L_0(V_n^{(a)}(x)V_m^{(a)}(x)) = \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(aq;q)_k (q;q)_k} V_n^{(a)}(q^{-k}) V_m^{(a)}(q^{-k}) = 0,\ n \neq m.$$

If $d \geq 2$, then according to Lemma 4.1, we get

$$\begin{aligned}
\phi_{nk} &= \frac{1}{\sqrt{[k]_q!}}\left[D_q^k\left(\frac{z^n}{\sqrt{[n]_q!}} E_{q^d}(-a^d z^d)\right)\right]_{z=-\beta} \\
&= \frac{q^{n-k}(a^d;q^d)_\infty}{(1-q)^n\sqrt{[k]_q![n]_q!}} \sum_{i=0}^{\infty} \frac{q^{i(n+1)}(q^{-k};q)_i}{(q;q)_i (a^d;q^d)_i} \\
&= \frac{q^{n-k}(a^d;q^d)_\infty}{(1-q)^n\sqrt{[k]_q![n]_q!}}\, {}_{d+1}\varphi_d\left(\begin{matrix} q^{-k}, 0, 0, \ldots, 0 \\ a, a\omega, \ldots, a\omega^{d-1}\end{matrix}\,\middle|\, q; q^{n+1}\right).
\end{aligned}$$

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