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## Bäcklund transformations for certain rational solutions of Painlevé VI

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We introduce certain Bäcklund transformations for rational solutions of the Painlevé VI equation. These transformations act on a family of Painlevé VI tau functions. They are obtained from reducing the Hirota bilinear equations that describe the relation between certain points in the 3 component polynomial KP Grassmannian. In this way we obtain transformations that act on the root lattice of  $A_5$ . We also show that this  $A_5$  root lattice can be related to the  $F_4^{(1)}$  root lattice. We thus obtain Bäcklund transformations that relate Painlevé VI tau functions, parametrized by the elements of this  $F_4^{(1)}$  root lattice.

*Keywords:* Painlevé equations; Bäcklund; Grassmannian, KP

2000 Mathematics Subject Classification: 34M55, 14M15, 37K10

### 1. Introduction

In [1], which was a generalization of [2] (see also [8]), we showed that there is a connection between certain homogeneous solutions of the 3-component KP hierarchy and certain rational solutions (cf. [9]) of the Painlevé VI equation:

$$\begin{aligned} \frac{d^2 y}{dt^2} = & \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \\ & + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left\{ \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right\}. \end{aligned} \quad (1.1)$$

In this publication we focus on the Bäcklund transformations for the solutions of [1]. See also [7] for connections of  $gl_3$  KP hierarchy to Painlevé VI. Instead of obtaining Bäcklund transformations for the Painlevé VI equation, we obtain such transformations for the so-called Jimbo-Miwa-Okamoto  $\sigma$ -form of the Painlevé VI equation [4]:

$$\frac{d\sigma}{dt} \left( t(t-1) \frac{d^2 \sigma}{dt^2} \right)^2 + \left( \frac{d\sigma}{dt} \left[ 2\sigma - (2t-1) \frac{d\sigma}{dt} \right] + v_1 v_2 v_3 v_4 \right)^2 = \prod_{k=1}^4 \left( \frac{d\sigma}{dt} + v_k^2 \right), \quad (1.2)$$

where

$$v_1 + v_2 = \sqrt{-2\beta}, \quad v_1 - v_2 = \sqrt{2\gamma}, \quad v_3 + v_4 + 1 = \sqrt{1 - 2\delta}, \quad v_3 - v_4 = \sqrt{2\alpha}. \quad (1.3)$$

This  $\sigma$  is related via some choice of variables to the 3-component KP tau-function  $T$  by

$$\sigma(t) = t(t-1) \frac{d \log T}{dt} - at - b,$$

for certain constants  $a, b$ . In this paper we show that there exists such a tau-function for certain elements in the root lattice of  $sl_6$ :

$$Q(A_5) = \left\{ \underline{\alpha} = \sum_{i=1}^6 \alpha_i \underline{\delta}_i \mid \sum_{i=1}^6 \alpha_i = 0 \right\}, \quad (1.4)$$

where  $(\underline{\delta}_i)_j = \delta_{ij}$  and where we choose

$$v_i = \frac{\alpha_1 + \alpha_3}{2} + \alpha_{3+i} \quad (i = 1, 2, 3), \quad v_4 = \frac{\alpha_1 - \alpha_3}{2}. \quad (1.5)$$

The equations of the 3-component KP and modified 3-component KP produce Bäcklund transformations on the above tau-functions

$$T_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k} \partial_j (T_{\underline{\alpha}}) - T_{\underline{\alpha}} \partial_j (T_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k}) + n_j(\underline{\alpha}; i, k) T_{\underline{\alpha}} T_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k} = \epsilon_{ijk} T_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_j} T_{\underline{\alpha} + \underline{\delta}_j - \underline{\delta}_k}, \quad (1.6)$$

for distinct  $i, j, k$  with  $1 \leq j \leq 3$  and  $1 \leq i, k \leq 6$ . Here

$$\partial_j = b_j(t) \frac{d}{dt}, \quad \text{and} \quad b_1(t) = t(t-1), \quad b_2(t) = t, \quad b_3(t) = -t^2$$

and  $n_1(\underline{\alpha}; i, k)$  is a certain constant, which is given in (3.4). From this we deduce the following Bäcklund equation for the Jimbo-Miwa-Okamoto  $\sigma$ -function for distinct  $i, j, k$  with  $1 \leq j \leq 3$  and  $1 \leq i, k \leq 6$ :

$$\begin{aligned} & \sigma_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_j}(t) + \sigma_{\underline{\alpha} + \underline{\delta}_j - \underline{\delta}_k}(t) - \sigma_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k}(t) - \sigma_{\underline{\alpha}}(t) = \\ & = G_{ijk}(\underline{\alpha}; t) + t(t-1) \frac{d}{dt} \log (\sigma_{\underline{\alpha}}(t) - \sigma_{\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k}(t) + H_{ijk}(\underline{\alpha}; t)). \end{aligned} \quad (1.7)$$

Here  $G_{ijk}(\underline{\alpha}; t)$  and  $H_{ijk}(\underline{\alpha}; t)$  are certain first order polynomials that can be determined explicitly, see (3.13).

## 2. The polynomial Grassmannian and the (modified) 3-component KP-hierarchy

The geometry behind the rational Painlevé VI solutions of [1] is the infinite polynomial (3-component) Grassmannian. Let  $H = \{\sum_i c_i \lambda^i \mid c_i \in \mathbb{C}^3, c_i = 0 \text{ for } i < 0\}$  and  $H_+ = \{\sum_i c_i \lambda^i \mid c_i \in \mathbb{C}^3, c_i = 0 \text{ for } i < 0\}$ . On  $H$  we have a natural bilinear form given by

$$\left( \sum_i c_i \lambda^i \mid \sum_j d_j \lambda^j \right) = \sum_i (c_i, d_{-1-i}), \quad (2.1)$$

where  $(\cdot, \cdot)$  is the standard bilinear form on  $\mathbb{C}^3$  given by

$$(w, v) = w_1 v_1 + w_2 v_2 + w_3 v_3. \quad (2.2)$$

The Grassmannian consists of linear subspaces  $W \subset H$ , that satisfy certain conditions. Here we will consider only very special linear subspaces  $W$  of  $H$ , viz. the ones that satisfy the following conditions:

- There exist positive integers  $m$  and  $n$  such that  $\lambda^n H_+ \subset W \subset \lambda^{-m} H_+$ ,
- $W$  satisfy the condition  $\lambda W \subset W$ ,
- $W$  has a basis of elements  $v(\lambda)$  that are homogeneous in  $\lambda$ , i.e.  $\lambda \frac{dv(\lambda)}{d\lambda} = dv(\lambda)$  with  $d \in \mathbb{Z}$ .

All this gives that such a  $W$  can be described as follows, see [1] for more details. Choose 3 linearly independent vectors in  $\mathbb{C}^3$

$$w^{(i)} = \left( w_1^{(i)}, w_2^{(i)}, w_3^{(i)} \right), \quad i = 1, 2, 3,$$

and let

$$w_{(i)} = \left( w_{(i)}^1, w_{(i)}^2, w_{(i)}^3 \right)$$

be the dual basis with respect to the bilinear form (2.2).

Let

$$\underline{\mu} = (\mu_1, \mu_2, \mu_3) = \mu_1 \underline{\varepsilon}_1 + \mu_2 \underline{\varepsilon}_2 + \mu_3 \underline{\varepsilon}_3 \in \mathbb{Z}^3, \quad (2.3)$$

where  $\underline{\varepsilon}_j$  is a basis vector in  $\mathbb{Z}^3$ , so  $\underline{\varepsilon}_j = (\delta_{j1}, \delta_{j2}, \delta_{j3})$ . Then such a  $W$  is equal to  $W(\underline{\mu})$ , where

$$W(\underline{\mu}) = \sum_{i \geq \mu_1} \mathbb{C} \lambda^i w^{(1)} + \sum_{j \geq \mu_2} \mathbb{C} \lambda^j w^{(2)} + \sum_{k \geq \mu_3} \mathbb{C} \lambda^k w^{(3)}.$$

Let  $e_a$ ,  $a = 1, 2, 3$ , be the standard basis of  $\mathbb{C}^3$ , then

$$W(\underline{\mu}) = \sum_{i=\mu_1}^{\max \mu_\ell - 1} \mathbb{C} \lambda^i w^{(1)} + \sum_{j=\mu_2}^{\max \mu_\ell - 1} \mathbb{C} \lambda^j w^{(2)} + \sum_{k=\mu_3}^{\max \mu_\ell - 1} \mathbb{C} \lambda^k w^{(3)} + \sum_{a=1}^3 \sum_{m \geq \max \mu_\ell} \mathbb{C} \lambda^m e_a.$$

Note that

$$W(\underline{0}) = \sum_{a=1}^3 \sum_{m \geq 0} \mathbb{C} \lambda^m e_a = H_+.$$

With respect to the bilinear form (2.1) on  $H$  we can find the maximal orthocomplement  $W^\perp(\underline{\mu})$ . This space is given by

$$W^\perp(\underline{\mu}) = \sum_{i \geq -\mu_1} \mathbb{C} \lambda^i w_{(1)} + \sum_{j \geq -\mu_2} \mathbb{C} \lambda^j w_{(2)} + \sum_{k \geq -\mu_3} \mathbb{C} \lambda^k w_{(3)}.$$

Note that

$$W^\perp(\underline{\mu}) = \sum_{i=-\mu_1}^{\max -\mu_\ell - 1} \mathbb{C} \lambda^i w_{(1)} + \sum_{j=-\mu_2}^{\max -\mu_\ell - 1} \mathbb{C} \lambda^j w_{(2)} + \sum_{k=-\mu_3}^{\max -\mu_\ell - 1} \mathbb{C} \lambda^k w_{(3)} + \sum_{a=1}^3 \sum_{m \geq \max -\mu_\ell} \mathbb{C} \lambda^m e_a.$$

If we define the following ordering on  $\mathbb{Z}^3$

$$\underline{\mu} \leq \underline{\lambda} \quad \text{if } \mu_i \leq \lambda_i \quad \text{for all } i = 1, 2, 3,$$

then

$$W(\underline{\lambda}) \subset W(\underline{\mu}) \quad \text{and } W^\perp(\underline{\mu}) \subset W^\perp(\underline{\lambda}) \quad \text{iff } \underline{\mu} \leq \underline{\lambda}.$$

Next, we associate to  $W(\underline{\mu})$  the following vector in a semi-infinite wedge space:

$$\begin{aligned} |W(\underline{\mu})\rangle = & \lambda^{\mu_1} w^{(1)} \wedge \lambda^{\mu_1+1} w^{(1)} \wedge \dots \wedge \lambda^{\max \mu_\ell - 1} w^{(1)} \wedge \lambda^{\mu_2} w^{(2)} \wedge \lambda^{\mu_2+1} w^{(2)} \wedge \dots \\ & \dots \wedge \lambda^{\max \mu_\ell - 1} w^{(2)} \wedge \lambda^{\mu_3} w^{(3)} \wedge \lambda^{\mu_3+1} w^{(3)} \wedge \dots \wedge \lambda^{\max \mu_\ell - 1} w^{(3)} \wedge \\ & \lambda^{\max \mu_\ell} e_1 \wedge \lambda^{\max \mu_\ell} e_2 \wedge \lambda^{\max \mu_\ell} e_3 \wedge \lambda^{\max \mu_\ell + 1} e_1 \wedge \lambda^{\max \mu_\ell + 1} e_2 \wedge \dots . \end{aligned} \quad (2.4)$$

If we define the grading

$$\deg(|W(\underline{0})\rangle) = 0 \quad \text{and} \quad \deg(\lambda^k w^{(j)}) = \frac{1}{2} - k,$$

then

$$\deg(|W(\underline{\mu})\rangle) = \frac{1}{2} (\mu_1^2 + \mu_2^2 + \mu_3^2). \quad (2.5)$$

For any  $v \in (\mathbb{C}[\lambda, \lambda^{-1}])^3$  we can define creation and annihilation operators, see e.g. [6] for more details. Let  $v_0 \wedge v_1 \wedge v_2 \wedge \dots$  be an element in the semi-infinite wedge space, then we define

$$\psi^+(v) v_0 \wedge v_1 \wedge v_2 \wedge \dots = v \wedge v_0 \wedge v_1 \wedge v_2 \wedge \dots$$

and

$$\psi^-(v) v_0 \wedge v_1 \wedge v_2 \wedge \dots = \sum_{i=0}^{\infty} (-)^i (v|v_i) v_0 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots .$$

These elements form a Clifford algebra, they satisfy the anti-commutation relations

$$\begin{aligned} \psi^+(v) \psi^+(w) + \psi^+(w) \psi^+(v) &= 0, & \psi^-(v) \psi^-(w) + \psi^-(w) \psi^-(v) &= 0, \\ \psi^+(v) \psi^-(w) + \psi^-(w) \psi^+(v) &= (v|w). \end{aligned}$$

Note that

$$\psi^+(v) |W(\underline{\mu})\rangle = 0 \quad \text{for } v \in W(\underline{\mu}), \quad \psi^-(v) |W(\underline{\mu})\rangle = 0 \quad \text{for } v \in W^\perp(\underline{\mu}).$$

Let  $V_0 = v_0 \wedge v_1 \wedge v_2 \wedge \dots$  and  $V_k = v_{-k} \wedge v_{-k+1} \wedge v_{-k+2} \wedge \dots$  for  $k \geq 0$ , then, since

$$v = \sum_{a=1}^3 \sum_{j \in \mathbb{Z}} (\lambda^{-j-1} e^a | v) \lambda^j e_a,$$

we find that

$$\begin{aligned}
 & \sum_{a=1}^3 \sum_{j \in \mathbb{Z}} \psi^+(\lambda^j e_a) V_k \otimes \psi^-(\lambda^{-j-1} e_a) V_0 = \\
 & = \sum_{a=1}^3 \sum_{j \in \mathbb{Z}} \lambda^j e_a \wedge V_k \otimes \left( \sum_{i=0}^{\infty} (-)^i (\lambda^{-j-1} e^a | v_i) v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \right) \\
 & = \sum_{a=1}^3 \sum_{j \in \mathbb{Z}} \sum_{i=0}^{\infty} (-)^i (\lambda^{-j-1} e^a | v_i) \lambda^j e_a \wedge V_k \otimes v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \\
 & = \sum_{i=0}^{\infty} (-)^i v_i \wedge V_k \otimes v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \\
 & = \sum_{i=0}^{\infty} \psi^+(v_i) V_k \otimes \psi^-(v_i^*) V_0 = 0.
 \end{aligned}$$

Here  $v_i^*$  is the dual vector of  $v_i$  with respect to the bilinear form (2.1). So in particular for  $W(\underline{v}) \subset W(\underline{\mu})$  one has

$$\sum_{a=1}^3 \sum_{k \in \mathbb{Z}} \psi^+(\lambda^k e_a) |W(\underline{\mu})\rangle \otimes \psi^-(\lambda^{-k-1} e_a) |W(\underline{v})\rangle = 0 \quad \text{for } \underline{\mu} \leq \underline{v}. \quad (2.6)$$

In a similar way we see that for  $i \neq j$ :

$$\sum_{a=1}^3 \sum_{k \in \mathbb{Z}} \psi^+(\lambda^k e_a) |W(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j)\rangle \otimes \psi^-(\lambda^{-k-1} e_a) |W(\underline{\mu})\rangle = \varepsilon_{ij} |W(\underline{\mu} - \underline{\varepsilon}_j)\rangle \otimes |W(\underline{\mu} + \underline{\varepsilon}_i)\rangle. \quad (2.7)$$

Let  $\delta(z - \lambda) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{\lambda}\right)^n$  and introduce the fields

$$\psi^\pm(\delta(z - \lambda) e_a) = \sum_{n \in \mathbb{Z}} \psi^\pm(\lambda^n e_a) z^{-n-1}.$$

Then (2.6) is equivalent to

$$\text{Res}_z \sum_{a=1}^3 \psi^+(\delta(z - \lambda) e_a) |W(\underline{\mu})\rangle \otimes \psi^-(\delta(z - \lambda) e_a) |W(\underline{v})\rangle = 0 \quad \text{for } \underline{\mu} \leq \underline{v}. \quad (2.8)$$

Using the boson-fermion correspondence we can express every such semi-infinite wedge  $|W(\underline{\mu})\rangle$  as a function in  $F = \mathbb{C}[q_a, q_a^{-1}, x_i^{(a)}; a = 1, 2, 3, i = 1, 2, 3, \dots]$ . We identify  $|W(\underline{0})\rangle$  with  $1 \in F$ . Let  $\sigma$  be the corresponding isomorphism, then

$$\sigma \psi^\pm(\delta(z - \lambda) e_a) \sigma^{-1} = q_a^{\pm 1} z^{\pm q_a \frac{\partial}{\partial q_a}} \exp \left( \pm \sum_{i=1}^{\infty} x_i^{(a)} z^i \right) \exp \left( \mp \sum_{i=1}^{\infty} \frac{\partial}{\partial x_i^{(a)}} \frac{z^{-i}}{i} \right). \quad (2.9)$$

The fact that  $q_a$  and  $q_b$  for  $a \neq b$  anticommute requires that they have to be ordered. We assume that

$$\sigma(|W(\underline{0})\rangle) = 1 \quad \text{and} \quad \sigma(|W(\underline{\mu})\rangle) = \sum_{\alpha \in \mathbb{Z}^3} \tau_{\alpha}(\underline{\mu}; x) q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3}. \quad (2.10)$$

Such  $\tau_{\underline{\alpha}}(\underline{\mu}; x)$  is equal, up to a sign, to the coefficient of

$$\lambda^{-\alpha_1} e_1 \wedge \lambda^{1-\alpha_1} e_1 \wedge \dots \wedge \lambda^{\max(-\alpha_\ell)-1} e_1 \wedge \lambda^{-\alpha_2} e_2 \wedge \lambda^{1-\alpha_2} e_2 \wedge \dots \wedge \lambda^{\max(-\alpha_\ell)-1} e_2 \wedge \lambda^{-\alpha_3} e_3 \wedge \dots \\ \dots \wedge \lambda^{\max(-\alpha_\ell)-1} e_3 \wedge \lambda^{\max(-\alpha_\ell)} e_1 \wedge \lambda^{\max(-\alpha_\ell)} e_2 \wedge \lambda^{\max(-\alpha_\ell)} e_3 \wedge \lambda^{\max(-\alpha_\ell)+1} e_1 \wedge \dots$$

of

$$e^{\sum_{a=1}^3 \sum_{i>0} x_i^{(a)} E_{aa}} \lambda^i |W(\underline{\mu})\rangle = \\ w_{\mu_1}^{(1)}(x) \wedge w_{\mu_1+1}^{(1)}(x) \wedge \dots \wedge w_{\max \mu_\ell-1}^{(1)}(x) \wedge w_{\mu_2}^{(2)}(x) \wedge w_{\mu_2+1}^{(2)}(x) \wedge \dots \\ \dots \wedge w_{\max \mu_\ell-1}^{(2)}(x) \wedge w_{\mu_3}^{(3)}(x) \wedge w_{\mu_3+1}^{(3)}(x) \wedge \dots \wedge w_{\max \mu_\ell-1}^{(3)}(x) \wedge \\ \lambda^{\max \mu_\ell} e_1 \wedge \lambda^{\max \mu_\ell} e_2 \wedge \lambda^{\max \mu_\ell} e_3 \wedge \lambda^{\max \mu_\ell+1} e_1 \wedge \lambda^{\max \mu_\ell+1} e_2 \wedge \dots,$$

where  $w_k^{(a)}(x)$  is the  $x$ -dependent vector

$$w_k^{(a)}(x) = \sum_{b=1}^3 \sum_{i=0}^{\max \mu_\ell-k-1} w_b^{(a)} S_i(x^{(b)}) \lambda^{k+i} e_b.$$

The  $S_i(x)$  appearing in the vector are the elementary Schur functions defined by  $e^{\sum_{i>0} x_i z^i} = \sum_{j \in \mathbb{Z}} S_j(x) z^j$ . Note that  $S_j(x) = 0$  if  $j < 0$ . A tedious but straightforward calculation (see e.g. [1] for more details) then gives that  $\tau_{\underline{\alpha}}(\underline{\mu}; x) = \pm \det A(\underline{\alpha}, \underline{\mu}; x)$ , where  $A(\underline{\alpha}, \underline{\mu}; x)$  is the  $(3p + \alpha_1 + \alpha_2 + \alpha_3) \times (3p + \alpha_1 + \alpha_2 + \alpha_3)$ -matrix

$$A(\underline{\alpha}, \underline{\mu}; x) = \sum_{a,b=1}^3 \sum_{i=1}^{p+\alpha_a \max(\mu_\ell)-\mu_b} \sum_{j=1}^{\max(\mu_\ell)-\mu_b} w_a^{(b)} S_{i-\alpha_a-\mu_b-j}(x^{(a)}) E_{\beta_a+i, \gamma_b+j} + \\ + \sum_{a=1}^3 \sum_{i=1}^{p-\max(\mu_\ell)} E_{\max(\mu_\ell)+\alpha_a+\beta_a+i, (4-a)\max(\mu_\ell)+(a-1)p+\alpha_1+\alpha_2+\alpha_3+i}$$

and  $p = \max(\mu_\ell, -\alpha_\ell; \ell = 1, 2, 3)$ ,  $\beta_1 = \gamma_1 = 0$ ,  $\beta_2 = p + \alpha_1$ ,  $\beta_3 = 2p + \alpha_1 + \alpha_2$ ,  $\gamma_2 = \max(\mu_\ell) - \mu_1$  and  $\gamma_3 = 2\max(\mu_\ell) - \mu_1 - \mu_2$ . It is straightforward to check that

$$\tau_{\underline{\alpha}}(\underline{\mu}; x) = 0 \quad \text{for } \mu_1 + \mu_2 + \mu_3 + \alpha_1 + \alpha_2 + \alpha_3 \neq 0 \quad (2.11)$$

and using (2.5) that

$$R(\underline{\mu}, \underline{\alpha}) := \deg \left( \tau_{\underline{\alpha}}(\underline{\mu}; x) \right) = \frac{1}{2} (\mu_1^2 + \mu_2^2 + \mu_3^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2). \quad (2.12)$$

Having this in mind, it will be useful to introduce the following subset of  $\mathbb{Z}^3$ :

$$L_{\underline{\mu}} = \{ \underline{\alpha} \in \mathbb{Z}^3 \mid \mu_1 + \mu_2 + \mu_3 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \}.$$

Note that since the form of the vectors  $|W(\underline{\mu})\rangle$  and  $|W(\underline{\mu} - (\underline{\varepsilon}_1 + \underline{\varepsilon}_2 + \underline{\varepsilon}_3))\rangle$  (2.4) are similar, one finds that

$$\tau_{\underline{\alpha}}(\underline{\mu}; x) = (-1)^{\alpha_2} \tau_{\underline{\alpha} + (\underline{\varepsilon}_1 + \underline{\varepsilon}_2 + \underline{\varepsilon}_3)}(\underline{\mu} - (\underline{\varepsilon}_1 + \underline{\varepsilon}_2 + \underline{\varepsilon}_3); x). \quad (2.13)$$

We can use (2.9) and (2.10) to rewrite (2.8) as a generating series of Hirota bilinear equations. We forget the tensor symbol and write  $x'$  for its first component and  $x''$  for its second component.

Define

$$q^\alpha = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3}$$

and let

$$\varepsilon(\underline{\varepsilon}_j, \underline{\alpha}) = \begin{cases} 1 & \text{for } j = 1, \\ (-1)^{\alpha_1} & \text{for } j = 2, \\ (-1)^{\alpha_1 + \alpha_2} & \text{for } j = 3, \end{cases}$$

then (2.8) is equivalent to

$$\begin{aligned} & \text{Res}_z \left( \sum_{a=1}^3 \sum_{\underline{\alpha} \in L_{\underline{\mu}}, \underline{\beta} \in L_{\underline{\nu}}} \varepsilon(\underline{\varepsilon}_a, \underline{\alpha} - \underline{\beta}) z^{\alpha_a - \beta_a} \exp \left( \sum_{k=1}^{\infty} (x_k^{(a)'} - x_k^{(a)'') } z^k \right) \right. \\ & \left. \exp \left( - \sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_k^{(a)'}} - \frac{\partial}{\partial x_k^{(a)'}} \right) \frac{z^{-k}}{k} \right) \tau_{\underline{\alpha}}(\underline{\mu}; x') (q^{\underline{\alpha} + \underline{\varepsilon}_a})' \tau_{\underline{\beta}}(\underline{\nu}; x'') (q^{\underline{\beta} - \underline{\varepsilon}_a})'' \right) = 0, \quad \underline{\mu} \leq \underline{\nu} \end{aligned} \quad (2.14)$$

and (2.7) is equivalent to (  $\varepsilon_{ij} = \varepsilon(\underline{\varepsilon}_i, \underline{\varepsilon}_j)$ ):

$$\begin{aligned} & \text{Res}_z \left( \sum_{a=1}^3 \sum_{\underline{\alpha} \in L_{\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j}, \underline{\beta} \in L_{\underline{\mu}}} \varepsilon(\underline{\varepsilon}_a, \underline{\alpha} - \underline{\beta}) z^{\alpha_a - \beta_a} \exp \left( \sum_{k=1}^{\infty} (x_k^{(a)'} - x_a^{(a)'') } z^k \right) \right. \\ & \left. \exp \left( - \sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_k^{(a)'}} - \frac{\partial}{\partial x_k^{(a)'}} \right) \frac{z^{-k}}{k} \right) \tau_{\underline{\alpha}}(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j; x') (q^{\underline{\alpha} + \underline{\varepsilon}_a})' \tau_{\underline{\beta}}(\underline{\mu}; x'') (q^{\underline{\beta} - \underline{\varepsilon}_a})'' \right) \\ & = \varepsilon_{ij} \sum_{\underline{\gamma} \in L_{\underline{\mu} - \underline{\varepsilon}_j}, \underline{\delta} \in L_{\underline{\mu} + \underline{\varepsilon}_i}} \tau_{\underline{\gamma}}(\underline{\mu} - \underline{\varepsilon}_j; x') (q^{\underline{\gamma}})' \tau_{\underline{\delta}}(\underline{\mu} + \underline{\varepsilon}_i; x'') (q^{\underline{\delta}})''. \end{aligned} \quad (2.15)$$

Taking the coefficient of  $(q^\alpha)'(q^\beta)''$  in (2.14) for  $\underline{\alpha} \in L_{\underline{\mu} - \underline{\varepsilon}_i}$  and  $\underline{\beta} \in L_{\underline{\nu} + \underline{\varepsilon}_i}$  we obtain:

$$\begin{aligned} & \text{Res}_z \left( \sum_{a=1}^3 \varepsilon(\underline{\varepsilon}_a, \underline{\alpha} - \underline{\beta}) z^{\alpha_a - \beta_a - 2} \exp \left( \sum_{k=1}^{\infty} (x_k^{(a)'} - x_k^{(a)'') } z^k \right) \right. \\ & \left. \exp \left( - \sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_k^{(a)'}} - \frac{\partial}{\partial x_k^{(a)'}} \right) \frac{z^{-k}}{k} \right) \tau_{\underline{\alpha} - \underline{\varepsilon}_a}(\underline{\mu}; x') \tau_{\underline{\beta} + \underline{\varepsilon}_a}(\underline{\nu}; x'') \right) = 0, \quad \underline{\mu} \leq \underline{\nu} \end{aligned} \quad (2.16)$$

and in a similar way (2.15) gives:

$$\begin{aligned} & \text{Res}_z \left( \sum_{a=1}^3 \varepsilon(\underline{\varepsilon}_a, \underline{\alpha} - \underline{\beta}) z^{\alpha_a - \beta_a - 2} \exp \left( \sum_{k=1}^{\infty} (x_k^{(a)'} - x_k^{(a)'') } z^k \right) \right. \\ & \left. \exp \left( - \sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_k^{(a)'}} - \frac{\partial}{\partial x_k^{(a)'}} \right) \frac{z^{-k}}{k} \right) \tau_{\underline{\alpha} - \underline{\varepsilon}_a}(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j; x') \tau_{\underline{\beta} + \underline{\varepsilon}_a}(\underline{\mu}; x'') \right) \\ & = \varepsilon_{ij} \tau_{\underline{\alpha}}(\underline{\mu} - \underline{\varepsilon}_j; x') \tau_{\underline{\beta}}(\underline{\mu} + \underline{\varepsilon}_i; x''). \end{aligned} \quad (2.17)$$

Now making the change of variables  $x_k^{(j)} = \frac{1}{2}(u_k^{(j)'} + u_k^{(j)'')}$ ,  $y_k^{(j)} = \frac{1}{2}(u_k^{(j)'} - u_k^{(j)'')}$ , one gets for (2.16) for  $\underline{\mu} \leq \underline{\nu}$ :

$$\begin{aligned} & \text{Res}_z \left( \sum_{j=1}^3 \varepsilon(\underline{\varepsilon}_j, \underline{\alpha} - \underline{\beta}) z^{\alpha_j - \beta_j - 2} \right. \\ & \left. \times \exp \left( \sum_{k=1}^{\infty} 2y_k^{(j)} z^k \right) \exp \left( - \sum_{k=1}^{\infty} \frac{\partial}{\partial y_k^{(j)}} \frac{z^{-k}}{k} \right) \tau_{\underline{\alpha} - \underline{\varepsilon}_j}(\underline{\mu}; x + y) \tau_{\underline{\beta} + \underline{\varepsilon}_j}(\underline{\nu}; x - y) \right) = 0. \end{aligned} \quad (2.18)$$



Using elementary Schur functions we rewrite this again as

$$\sum_{j=1}^3 \varepsilon(\underline{\varepsilon}_j, \underline{\alpha} - \underline{\beta}) \sum_{k=0}^{\infty} S_k(2y^{(j)}) S_{k-1+\alpha_j-\beta_j} \left( -\frac{\tilde{\partial}}{\partial y^{(j)}} \right) \tau_{\underline{\alpha}-\underline{\varepsilon}_j}(\underline{\mu}; x+y) \tau_{\underline{\beta}+\underline{\varepsilon}_j}(\underline{\nu}; x-y) = 0. \quad (2.19)$$

Here and further we use the notation  $\frac{\tilde{\partial}}{\partial y} = (\frac{\partial}{\partial y_1}, \frac{1}{2} \frac{\partial}{\partial y_2}, \frac{1}{3} \frac{\partial}{\partial y_3}, \dots)$ . Using Taylor's formula we can rewrite this once more:

$$\begin{aligned} & \sum_{j=1}^3 \varepsilon(\underline{\varepsilon}_j, \underline{\alpha} - \underline{\beta}) \sum_{k=0}^{\infty} S_k(2y^{(j)}) S_{k-1+\alpha_j-\beta_j} \left( -\frac{\tilde{\partial}}{\partial t^{(j)}} \right) \\ & \times e^{\sum_{j=1}^3 \sum_{r=1}^{\infty} y_r^{(j)} \frac{\partial}{\partial t_r^{(j)}}} \tau_{\underline{\alpha}-\underline{\varepsilon}_j}(\underline{\mu}; x+t) \tau_{\underline{\beta}+\underline{\varepsilon}_j}(\underline{\nu}; x-t)|_{t=0} = 0. \end{aligned} \quad (2.20)$$

This last equation can be written as the following generating series of Hirota bilinear equations:

$$\sum_{j=1}^3 \varepsilon(\underline{\varepsilon}_j, \underline{\alpha} - \underline{\beta}) \sum_{k=0}^{\infty} S_k(2y^{(j)}) S_{k-1+\alpha_j-\beta_j} (-\widetilde{D^{(j)}}) e^{\sum_{j=1}^3 \sum_{r=1}^{\infty} y_r^{(j)} D_r^{(j)}} \tau_{\underline{\alpha}-\underline{\varepsilon}_j}(\underline{\mu}) \cdot \tau_{\underline{\beta}+\underline{\varepsilon}_j}(\underline{\nu}) = 0, \quad (2.21)$$

for all  $\underline{\alpha} \in L_{\underline{\mu}-\underline{\varepsilon}_i}$ ,  $\underline{\beta} \in L_{\underline{\nu}+\underline{\varepsilon}_i}$  and  $\underline{\mu} \leq \underline{\nu}$ , see [6] for more details.

Now take  $\underline{\mu} = \underline{\nu}$ , then for  $\underline{\alpha} \in L_{\underline{\mu}}$  and  $1 \leq i, j \leq 3$  distinct indices  $i$  and  $j$  one finds the following equation:

$$D_1^{(i)} D_1^{(j)} \tau_{\underline{\alpha}}(\underline{\mu}) \cdot \tau_{\underline{\alpha}}(\underline{\mu}) = 2 \tau_{\underline{\alpha}+\underline{\varepsilon}_i-\underline{\varepsilon}_j}(\underline{\mu}) \tau_{\underline{\alpha}+\underline{\varepsilon}_j-\underline{\varepsilon}_i}(\underline{\mu}) \quad (2.22)$$

and for each triple of distinct indices  $i, j, k$ :

$$D_1^{(j)} \tau_{\underline{\alpha}}(\underline{\mu}) \cdot \tau_{\underline{\alpha}+\underline{\varepsilon}_i-\underline{\varepsilon}_k}(\underline{\mu}) = \varepsilon_{ijk} \tau_{\underline{\alpha}+\underline{\varepsilon}_i-\underline{\varepsilon}_j}(\underline{\mu}) \tau_{\underline{\alpha}+\underline{\varepsilon}_j-\underline{\varepsilon}_k}(\underline{\mu}). \quad (2.23)$$

If  $\underline{\mu} = \underline{\nu} - \underline{\varepsilon}_\ell$ , choose first  $\underline{\alpha}$  and  $\underline{\beta}$  such that  $\underline{\alpha} - \underline{\beta} = \underline{\varepsilon}_1 + \underline{\varepsilon}_2 + \underline{\varepsilon}_3$ , then we find the following Hirota-Miwa equation:

$$\tau_{\underline{\beta}+\underline{\varepsilon}_2+\underline{\varepsilon}_3}(\underline{\mu}) \tau_{\underline{\beta}+\underline{\varepsilon}_1}(\underline{\mu} + \underline{\varepsilon}_\ell) - \tau_{\underline{\beta}+\underline{\varepsilon}_1+\underline{\varepsilon}_3}(\underline{\mu}) \tau_{\underline{\beta}+\underline{\varepsilon}_2}(\underline{\mu} + \underline{\varepsilon}_\ell) + \tau_{\underline{\beta}+\underline{\varepsilon}_1+\underline{\varepsilon}_2}(\underline{\mu}) \tau_{\underline{\beta}+\underline{\varepsilon}_3}(\underline{\mu} + \underline{\varepsilon}_\ell) = 0. \quad (2.24)$$

Secondly choose  $\underline{\alpha}$  and  $\underline{\beta}$  such that  $\underline{\alpha} - \underline{\beta} = 2\underline{\varepsilon}_i + \underline{\varepsilon}_j$ , with  $i$  and  $j$  distinct, then we find

$$D_1^{(i)} \tau_{\underline{\gamma}}(\underline{\mu}) \cdot \tau_{\underline{\gamma}-\underline{\varepsilon}_j}(\underline{\mu} + \underline{\varepsilon}_\ell) = \varepsilon_{ij} \tau_{\underline{\gamma}-\underline{\varepsilon}_i}(\underline{\mu} + \underline{\varepsilon}_\ell) \tau_{\underline{\gamma}+\underline{\varepsilon}_i-\underline{\varepsilon}_j}(\underline{\mu}), \quad (2.25)$$

or equivalently

$$D_1^{(i)} \tau_{\underline{\gamma}}(\underline{\mu}) \cdot \tau_{\underline{\gamma}+\underline{\varepsilon}_j}(\underline{\mu} - \underline{\varepsilon}_\ell) = \varepsilon_{ji} \tau_{\underline{\gamma}+\underline{\varepsilon}_j-\underline{\varepsilon}_i}(\underline{\mu}) \tau_{\underline{\gamma}+\underline{\varepsilon}_i}(\underline{\mu} - \underline{\varepsilon}_\ell). \quad (2.26)$$

In a similar way (2.17) can be rewritten as the following generating series of Hirota bilinear equations ( $i \neq j$ ):

$$\begin{aligned} & \sum_{a=1}^3 \varepsilon(\underline{\varepsilon}_a, \underline{\alpha} - \underline{\beta}) \sum_{k=0}^{\infty} S_k(2y^{(a)}) S_{k-1+\alpha_a-\beta_a} (-\widetilde{D^{(a)}}) e^{\sum_{a=1}^3 \sum_{r=1}^{\infty} y_r^{(a)} D_r^{(a)}} \tau_{\underline{\alpha}-\underline{\varepsilon}_a}(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j) \cdot \tau_{\underline{\beta}+\underline{\varepsilon}_a}(\underline{\mu}) \\ & = \varepsilon_{ij} e^{\sum_{a=1}^3 \sum_{r=1}^{\infty} y_r^{(a)} D_r^{(a)}} \tau_{\underline{\alpha}}(\underline{\mu} - \underline{\varepsilon}_j) \cdot \tau_{\underline{\beta}}(\underline{\mu} + \underline{\varepsilon}_i), \end{aligned} \quad (2.27)$$

for all  $\underline{\alpha} \in L_{\underline{\mu}-\underline{\varepsilon}_i}$ ,  $\underline{\beta} \in L_{\underline{\nu}+\underline{\varepsilon}_i}$  and  $\underline{\mu} \leq \underline{\nu}$ . Now taking  $\underline{\alpha} - \underline{\beta} = \underline{\varepsilon}_k + \underline{\varepsilon}_\ell$ , with  $k \neq \ell$ , where  $k$  or  $\ell$  may be equal to  $i$  or  $j$ , we find another version of the Hirota-Miwa equation ( $i \neq j, k \neq \ell$ ):

$$\varepsilon_{k\ell} \tau_{\underline{\beta}+\underline{\varepsilon}_\ell}(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j) \tau_{\underline{\beta}+\underline{\varepsilon}_k}(\underline{\mu}) + \varepsilon_{\ell k} \tau_{\underline{\beta}+\underline{\varepsilon}_k}(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j) \tau_{\underline{\beta}+\underline{\varepsilon}_\ell}(\underline{\mu}) - \varepsilon_{ij} \tau_{\underline{\beta}+\underline{\varepsilon}_k+\underline{\varepsilon}_\ell}(\underline{\mu} - \underline{\varepsilon}_j) \tau_{\underline{\beta}}(\underline{\mu} + \underline{\varepsilon}_i) = 0. \quad (2.28)$$

Next taking  $\underline{\alpha} - \underline{\beta} = 2\underline{\varepsilon}_k$ , where  $k$  may be equal to  $i$  or  $j$ , we find ( $i \neq j$ ):

$$D_1^{(k)} \tau_{\underline{\gamma}}(\underline{\mu}) \cdot \tau_{\underline{\gamma}}(\underline{\mu} + \underline{\varepsilon}_i - \underline{\varepsilon}_j) = \varepsilon_{ji} \tau_{\underline{\gamma} + \underline{\varepsilon}_k}(\underline{\mu} - \underline{\varepsilon}_j) \tau_{\underline{\gamma} - \underline{\varepsilon}_k}(\underline{\mu} + \underline{\varepsilon}_i). \quad (2.29)$$

In the above construction the pair

$$(\underline{\alpha}, \underline{\mu}) = (\alpha_1, \alpha_2, \alpha_3, \mu_1, \mu_2, \mu_3) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$$

can be seen as an element in the root lattice  $Q(A_5)$  of  $sl_6$  (see(1.4)). Note that the tau function corresponding to such a pair  $(\underline{\alpha}, \underline{\mu})$  is 0, whenever this pair is not in  $Q(A_5)$ , see (2.11). A basis of this root lattice is given by the roots  $\underline{\delta}_i - \underline{\delta}_{i+1}$  for  $1 \leq i \leq 5$ . Using the degree of the tau function given in (2.12), we define a similar grading on this root lattice by

$$R(\underline{\alpha}) = R\left(\sum_{i=1}^6 \alpha_i \underline{\delta}_i\right) = \deg\left(\sum_{i=1}^6 \alpha_i \underline{\delta}_i\right) = \frac{1}{2} (\alpha_4^2 + \alpha_5^2 + \alpha_6^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2). \quad (2.30)$$

In this light the equations (2.23), (2.25), (2.29) can be rewritten to one equation. Let  $\underline{\beta}$  be an element in the root lattice of  $sl_6$ , then for distinct  $i, j, k$  with  $1 \leq j \leq 3$  and  $1 \leq i, k \leq 6$  one has:

$$D_1^{(j)} \tau_{\underline{\beta}} \cdot \tau_{\underline{\beta} + \underline{\delta}_i - \underline{\delta}_k} = \varepsilon_{ijk} \tau_{\underline{\beta} + \underline{\delta}_i - \underline{\delta}_j} \tau_{\underline{\beta} + \underline{\delta}_j - \underline{\delta}_k}, \quad j = 1, 2, 3, \quad i, k = 1, 2, \dots, 6. \quad (2.31)$$

Finally we note that (2.13) can be rewritten to

$$\tau_{\underline{\alpha}} = (-1)^{\alpha_2} \tau_{\underline{\alpha} + \underline{\delta}_1 + \underline{\delta}_2 + \underline{\delta}_3 - \underline{\delta}_4 - \underline{\delta}_5 - \underline{\delta}_6}. \quad (2.32)$$

### 3. From KP to the Jimbo-Miwa-Okamoto $\sigma$ -equation

To obtain the Jimbo-Miwa-Okamoto  $\sigma$ -form (1.2) of the Painlevé VI equation from the 3-component KP, the following choice of new variables was used in [1] and a similar choice was made in [5]:

$$t = \frac{x_1^{(2)} - x_1^{(1)}}{x_1^{(3)} - x_1^{(1)}}, \quad h = x_1^{(2)} - x_1^{(1)} \quad (3.1)$$

and

$$\frac{\partial}{\partial x_1^{(1)}} = \frac{t(t-1)}{h} \frac{\partial}{\partial t} - \frac{\partial}{\partial h}, \quad \frac{\partial}{\partial x_1^{(2)}} = \frac{t}{h} \frac{\partial}{\partial t} + \frac{\partial}{\partial h}, \quad \frac{\partial}{\partial x_1^{(3)}} = -\frac{t^2}{h} \frac{\partial}{\partial t}. \quad (3.2)$$

Then for  $\underline{\alpha} \in Q(A_5)$  with  $R(\underline{\alpha}) \geq 0$

$$\frac{\partial \tau_{\underline{\alpha}}(t, h)}{\partial h} = R(\underline{\alpha}) \tau_{\underline{\alpha}}(t, h),$$

thus

$$\tau_{\underline{\alpha}}(t, h) = h^{R(\underline{\alpha})} T_{\underline{\alpha}}(t).$$

Using this and equation (3.2) equation (2.22) turns into (cf [13]),

$$\begin{aligned}
 R(\underline{\alpha})T_{\underline{\alpha}}^2 - (t-1)t^2 \left( \frac{dT_{\underline{\alpha}}}{dt} \right)^2 + t^2 T_{\underline{\alpha}} \left( \frac{dT_{\underline{\alpha}}}{dt} + (t-1) \frac{d^2 T_{\underline{\alpha}}}{dt^2} \right) &= T_{\underline{\alpha}+\underline{\delta}_1-\underline{\delta}_2} T_{\underline{\alpha}+\underline{\delta}_2-\underline{\delta}_1}, \\
 t^2 \left( t(t-1) \left( \frac{dT_{\underline{\alpha}}}{dt} \right)^2 + T_{\underline{\alpha}} \left( (1-2t) \frac{dT_{\underline{\alpha}}}{dt} - t(t-1) \frac{d^2 T_{\underline{\alpha}}}{dt^2} \right) \right) &= T_{\underline{\alpha}+\underline{\delta}_1-\underline{\delta}_3} T_{\underline{\alpha}+\underline{\delta}_3-\underline{\delta}_1}, \\
 t^2 \left( -t \left( \frac{dT_{\underline{\alpha}}}{dt} \right)^2 + T_{\underline{\alpha}} \left( \frac{dT_{\underline{\alpha}}}{dt} + t \frac{d^2 T_{\underline{\alpha}}}{dt^2} \right) \right) &= T_{\underline{\alpha}+\underline{\delta}_2-\underline{\delta}_3} T_{\underline{\alpha}+\underline{\delta}_3-\underline{\delta}_2}.
 \end{aligned} \tag{3.3}$$

This gives 3 series of Toda equations that can be used to calculate neighboring tau-functions. Equation (2.31) turns into (1.6), with

$$n_1(\underline{\alpha}; i, k) = -n_2(\underline{\alpha}; i, k) = R(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k) - R(\underline{\alpha}), \quad n_3(\underline{\alpha}; i, k) = 0 \tag{3.4}$$

and (2.32) into

$$T_{\underline{\alpha}} = (-1)^{\alpha_2} T_{\underline{\alpha}+\underline{\delta}_1+\underline{\delta}_2+\underline{\delta}_3-\underline{\delta}_4-\underline{\delta}_5-\underline{\delta}_6}. \tag{3.5}$$

Finally, we have the two Hirota-Miwa equations (2.24) and (2.28), that give:

$$\begin{aligned}
 T_{\underline{\beta}+\underline{\delta}_2+\underline{\delta}_3} T_{\underline{\beta}+\underline{\delta}_1+\underline{\delta}_\ell} - T_{\underline{\beta}+\underline{\delta}_1+\underline{\delta}_3} T_{\underline{\beta}+\underline{\delta}_2+\underline{\delta}_\ell} + T_{\underline{\beta}+\underline{\delta}_1+\underline{\delta}_2} T_{\underline{\beta}+\underline{\delta}_3+\underline{\delta}_\ell} &= 0, \quad \text{for } \ell > 3, \quad \text{and} \\
 \epsilon_{k\ell} T_{\underline{\beta}+\underline{\delta}_\ell+\underline{\delta}_i} T_{\underline{\beta}+\underline{\delta}_k+\underline{\delta}_j} + \epsilon_{\ell k} T_{\underline{\beta}+\underline{\delta}_k+\underline{\delta}_i} T_{\underline{\beta}+\underline{\delta}_\ell+\underline{\delta}_j} + \epsilon_{j-3, i-3} T_{\underline{\beta}+\underline{\delta}_k+\underline{\delta}_\ell} T_{\underline{\beta}+\underline{\delta}_i+\underline{\delta}_j} &= 0, \quad \text{for}
 \end{aligned} \tag{3.6}$$

$1 \leq k, \ell \leq 3, 4 \leq i, j \leq 6$  with  $i \neq j$  and  $k \neq \ell$ . All these equations give Bäcklund transformations for the tau functions  $T_{\underline{\alpha}}$  of the Painlevé VI equation.

All the above type of equations in the case of the affine Lie algebra of type  $A_n$  were obtained by Noumi and Yamada, see e.g. [10] and [11].

We want to rewrite (1.6) and express it in the corresponding Jimbo-Miwa-Okamoto  $\sigma$ -functions. First, we introduce

$$f_{\underline{\alpha}}(t) = t(t-1) \frac{d \log T_{\underline{\alpha}}}{dt}, \tag{3.7}$$

and take the log of the expression (1.6)

$$\begin{aligned}
 &\log(\text{constant}) + \log(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_j}) + \log(T_{\underline{\alpha}+\underline{\delta}_j-\underline{\delta}_k}) = \\
 &= \log(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k} \partial_j(T_{\underline{\alpha}}) - T_{\underline{\alpha}} \partial_j(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}) + n_j(\underline{\alpha}; i, k) T_{\underline{\alpha}} T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}) \\
 &= \log(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}) + \log(T_{\underline{\alpha}}) + \log \left( \frac{\partial_j(T_{\underline{\alpha}})}{T_{\underline{\alpha}}} - \frac{\partial_j(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k})}{T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}} + n_j(\underline{\alpha}; i, k) \right) \\
 &= \log \left( \frac{b_j(t)}{t(t-1)} \right) + \log(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}) + \log(T_{\underline{\alpha}}) + \\
 &\quad + \log \left( \frac{t(t-1) \frac{d}{dt}(T_{\underline{\alpha}})}{T_{\underline{\alpha}}} - t(t-1) \frac{\frac{d}{dt}(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k})}{T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}} + n_j(\underline{\alpha}; i, k) \frac{t(t-1)}{b_j(t)} \right) \\
 &= \log \left( \frac{b_j(t)}{t(t-1)} \right) + \log(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}) + \log(T_{\underline{\alpha}}) + \\
 &\quad + \log \left( t(t-1) \frac{d \log(T_{\underline{\alpha}})}{dt} - t(t-1) \frac{d \log(T_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k})}{dt} + n_j(\underline{\alpha}; i, k) \frac{t(t-1)}{b_j(t)} \right).
 \end{aligned} \tag{3.8}$$

Now take  $t(t-1)\frac{d}{dt}$  of this expression (3.8), we thus obtain:

$$\begin{aligned} f_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_j}(t) + f_{\underline{\alpha}+\underline{\delta}_j-\underline{\delta}_k}(t) - f_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}(t) - f_{\underline{\alpha}}(t) = \\ = g_j(t) + t(t-1)\frac{d}{dt} \log (f_{\underline{\alpha}}(t) - f_{\underline{\alpha}+\underline{\delta}_i-\underline{\delta}_k}(t) + h_j(t)), \end{aligned} \quad (3.9)$$

where

$$g_j(t) = \begin{cases} 0 & \text{if } j=1, \\ -t & \text{if } j=2, \\ 1 & \text{if } j=3, \end{cases} \quad h_j(t) = \begin{cases} n_j(\underline{\alpha}; i, k) & \text{if } j=1, \\ n_j(\underline{\alpha}; i, k)(t-1) & \text{if } j=2, \\ -n_j(\underline{\alpha}; i, k)\frac{t-1}{t} = 0 & \text{if } j=3. \end{cases} \quad (3.10)$$

Following [1] we introduce

$$\sigma_{\underline{\alpha}} = f_{\underline{\alpha}}(t) + c_5(\underline{\alpha})(t-1) - \frac{1}{2}c_6(\underline{\alpha}), \quad (3.11)$$

where

$$\begin{aligned} c_5(\underline{\alpha}) &= -\frac{1}{4}(\alpha_1 - \alpha_3)^2, \\ c_6(\underline{\alpha}) &= R(\underline{\alpha}) + \frac{1}{2}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3), \end{aligned} \quad (3.12)$$

and thus we obtain (1.7), where

$$\begin{aligned} G_{ijk}(\underline{\alpha}; t) &= (c_5(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_j) + c_5(\underline{\alpha} + \underline{\delta}_j - \underline{\delta}_k) - c_5(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k) - c_5(\underline{\alpha}))(1-t) \\ &\quad + \frac{1}{2}(c_6(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_j) + c_6(\underline{\alpha} + \underline{\delta}_j - \underline{\delta}_k) - c_6(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k) - c_6(\underline{\alpha})) + g_j(t), \\ H_{ijk}(\underline{\alpha}; t) &= (c_5(\underline{\alpha}) - c_5(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k))(1-t) + \frac{1}{2}(c_6(\underline{\alpha}) - c_6(\underline{\alpha} + \underline{\delta}_i - \underline{\delta}_k)) + h_j(t). \end{aligned} \quad (3.13)$$

#### 4. Other Bäcklund transformations

Besides the Bäcklund transformations that come from the 3-component Grassmannian structure, there are some other relevant transformations. A first observation that can be made is that the  $\sigma$  equation (1.2) has a natural  $D_4$  symmetry. One can permute all  $v_i$  together with an even number of sign changes.

Secondly, one can choose an other identification (1.5) between the  $\alpha$ 's and  $v$ 's see e.g. [1], section 2.

Thirdly, one can permute the  $\alpha_i$ 's for  $i = 1, 2, 3$  and also separately the  $\mu_i$ 's. All these transformations rearrange the tau functions on the  $sl_6$  root lattice.

Finally, starting with the underlying 3-component KP model one can choose a different identification (3.1) between the  $x_1^{(i)}$  and  $t$  and  $h$ . For instance interchanging  $x_1^{(1)}$  and  $x_1^{(3)}$ , gives a transformation  $t \mapsto 1-t$ , such a transformation leaves Painlevé VI equation (1.1) invariant for  $y \mapsto 1-y$  and appropriate transformations of coefficients, see e.g. Boalch [3] or [12]. The permutation that interchanges  $x_1^{(1)}$  and  $x_1^{(2)}$  (respectively  $x_1^{(2)}$  and  $x_1^{(3)}$ ), gives a transformation  $t \mapsto \frac{t-y}{t-1}$  (resp.  $t \mapsto \frac{1}{t}$ ), such a transformation induces  $y \mapsto \frac{t-y}{t-1}$  (resp.  $y \mapsto \frac{1}{y}$ ), again see [3] or [12], where it is argued that addition of these transformations extend  $D_4^{(1)}$  symmetry to  $F_4^{(1)}$  symmetry.

## 5. Root lattice of $F_4^{(1)}$

Okamoto showed in his fundamental paper [13] that there is a representation of the affine Weyl group of type  $F_4^{(1)}$  that acts on the solutions of the Painlevé VI equation. An element in this Weyl group is related to a birational canonical transformation. We will now show that the  $sl_6$  root lattice of the previous section is related to the root lattice of the affine Lie algebra of type  $F_4^{(1)}$  on which this affine Weyl group acts.

Let

$$\underline{v} = (v_0, v_1, v_2, v_3, v_4) = v_0 \underline{e}_0 + v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 + v_4 \underline{e}_4 \quad (5.1)$$

be a vector in a 5-dimensional vector space. We assume that

$$(\underline{v}, \underline{w}) = \sum_{i=1}^4 v_i w_i.$$

If we make the following identification (see also (1.5)):

$$v_0 = \alpha_1, \quad v_i = \frac{\alpha_1 + \alpha_3}{2} + \mu_i = \frac{\alpha_1 + \alpha_3}{2} + \alpha_{3+i} \quad (i = 1, 2, 3), \quad v_4 = \frac{\alpha_1 - \alpha_3}{2}, \quad (5.2)$$

then the  $v_1, v_2, v_3, v_4$  correspond to the parameters of the Jimbo-Miwa-Okamoto  $\sigma$ -equation (1.2). Moreover, one has the following correspondence, the element  $\sum_{i=1}^6 \alpha_i \delta_i$  is equal to

$$\alpha_1 \underline{e}_0 + \left( \frac{\alpha_1 + \alpha_3}{2} + \alpha_4 \right) \underline{e}_1 + \left( \frac{\alpha_1 + \alpha_3}{2} + \alpha_5 \right) \underline{e}_2 + \left( \frac{\alpha_1 + \alpha_3}{2} + \alpha_6 \right) \underline{e}_3 + \left( \frac{\alpha_1 - \alpha_3}{2} \right) \underline{e}_4.$$

Note that  $\underline{e}_0 = \underline{\delta}_1 + \underline{\delta}_2 + \underline{\delta}_3 - \underline{\delta}_4 - \underline{\delta}_5 - \underline{\delta}_6$ . In this way one gets all elements of the form (5.1) with  $v_0 \in \mathbb{Z}$  and all  $v_i \in \mathbb{Z}$  for  $i > 0$  or all  $v_i \in \frac{1}{2} + \mathbb{Z}$  for  $i > 0$ . This is the root lattice  $Q(F_4^{(1)})$  of the Lie algebra of type  $F_4^{(1)}$ . In fact the simple roots of this affine Lie algebra are:

$$\begin{aligned} \underline{e}_0 - \underline{e}_1 - \underline{e}_2 &= \underline{\delta}_1 + 3\underline{\delta}_2 + \underline{\delta}_3 - 2\underline{\delta}_4 - 2\underline{\delta}_5 - \underline{\delta}_6, \\ \underline{e}_2 - \underline{e}_3 &= \underline{\delta}_5 - \underline{\delta}_6, \\ \underline{e}_3 - \underline{e}_4 &= 2\underline{\delta}_3 - \underline{\delta}_4 - \underline{\delta}_5, \\ \underline{e}_4 &= -\underline{\delta}_2 - 2\underline{\delta}_3 + \underline{\delta}_4 + \underline{\delta}_5 + \underline{\delta}_6, \\ \frac{1}{2}(\underline{e}_1 - \underline{e}_2 - \underline{e}_3 - \underline{e}_4) &= \underline{\delta}_2 + \underline{\delta}_3 - \underline{\delta}_5 - \underline{\delta}_6. \end{aligned}$$

The  $\pm(\underline{\delta}_i - \underline{\delta}_j)$  with  $1 \leq i \leq 6, 1 \leq j \leq 3$  and  $i \neq j$  that appear in the sigma functions of equation (1.7) form up to possibly a translation with the vector  $\underline{e}_0$  all short roots of  $F_4$ , which are ( $\varepsilon_k = \pm 1$ ):

$$\varepsilon_k \underline{e}_k, \quad (k = 1, 2, 3, 2), \quad \frac{1}{2}(\varepsilon_1 \underline{e}_1 + \varepsilon_2 2\underline{e}_2 + \varepsilon_3 \underline{e}_3 + \varepsilon_4 \underline{e}_4).$$

To be more precise they form the union of the sets  $\pm S_j$ , which are defined by

$$\begin{aligned} S_1 &= \{e_0 + e_4, e_0 + \frac{1}{2}(e_1 + e_2 + e_3 + e_4), -e_0 + \frac{1}{2}(e_1 - e_2 - e_3 + e_4), \\ &\quad e_0 + \frac{1}{2}(e_1 - e_2 + e_3 + e_4), e_0 + \frac{1}{2}(e_1 + e_2 - e_3 + e_4)\}, \\ S_2 &= \{e_0 + \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \frac{1}{2}(e_1 + e_2 + e_3 - e_4), e_1, e_2, e_3\}, \\ S_3 &= \{e_0 + e_4, \frac{1}{2}(-e_1 - e_2 - e_3 + e_4), \frac{1}{2}(e_1 - e_2 - e_3 + e_4), \\ &\quad \frac{1}{2}(-e_1 + e_2 - e_3 + e_4), \frac{1}{2}(-e_1 - e_2 + e_3 + e_4)\}. \end{aligned} \quad (5.3)$$

Then the following holds:

Let  $\underline{\beta}$  be an element in the root lattice of  $F_4^{(1)}$  and assume  $\gamma_1, \gamma_2 \in S_j$  for fixed  $j = 1, 2, 3$ , suppose  $\sigma_{\underline{\beta}}$  and  $\sigma_{\underline{\beta}+\gamma_1-\gamma_2}$  are known, then using equation (1.7) one can calculate  $\sigma_{\underline{\beta}+\gamma_1}$  (resp.  $\sigma_{\underline{\beta}-\gamma_2}$ ), if one knows  $\sigma_{\underline{\beta}-\gamma_2}$  (resp.  $\sigma_{\underline{\beta}+\gamma_1}$ ).

Clearly a similar implication also holds for the corresponding tau functions  $T_{\underline{\beta}}$ . Equation (3.3) implies:

Let  $\underline{\beta}$  be an element in the root lattice of  $F_4^{(1)}$  and assume  $\underline{\gamma} = e_0 + \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ ,  $\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$  or  $e_0 + e_4$ , suppose  $T_{\underline{\beta}}$  and  $T_{\underline{\beta} \pm \underline{\gamma}}$  are known, using equation (3.3) one can calculate  $T_{\underline{\beta} \mp \underline{\gamma}}$ .

Equation (3.5) implies:

Let  $\underline{\beta}$  be an element in the root lattice of  $F_4^{(1)}$  then up to a sign  $T_{\underline{\beta}}$  is equal to  $T_{\underline{\beta}+e_0}$

Finally the Hirota-Miwa equation (3.6) also gives a connection between six tau functions in the  $F_4^{(1)}$  root lattice. However it is not so easy to describe this explicitly in this  $F_4^{(1)}$  setting.

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