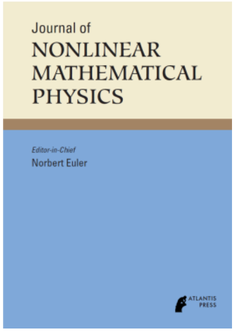

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Point classification of second order ODEs and its application to Painlevé equations

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The first part of this work is a review of the point classification of second order ODEs done by Ruslan Sharipov. His works were published in 1997-1998 in the Electronic Archive at LANL. The second part is an application of this classification to Painlevé equations. In particular, it allows us to solve the equivalence problem for Painlevé equations in an algorithmic form.

Keywords: Invariant; Problem of equivalence; Point transformation; Painlevé equation

2000 Mathematics Subject Classification: 53A55, 34A26, 34A34, 34C14, 34C20, 34C41

1. Introduction

It is a well-known fact that the following class of the second order ODEs

$$y'' = P(x,y) + 3\,Q(x,y)y' + 3\,R(x,y)y'^2 + S(x,y)y'^3 \tag{1.1}$$

is closed under the generic point transformations

$$\tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y). \tag{1.2}$$

It means that the transformed equation is again given by (1.1) but with some other coefficients:

$$\tilde{y}'' = \tilde{P}(\tilde{x},\tilde{y}) + 3\,\tilde{Q}(\tilde{x},\tilde{y})\tilde{y}' + 3\,\tilde{R}(\tilde{x},\tilde{y})\tilde{y}'^2 + \tilde{S}(\tilde{x},\tilde{y})\tilde{y}'^3. \tag{1.3}$$

Suppose we are given two arbitrary equations (1.1) and (1.3). The problem of existence of the change of variables (1.2) that transforms equations (1.1) and (1.3) one into the other is called *the Equivalence Problem*. If we apply transformation (1.2) for equation (1.1), we get the explicit formulas for the coefficients $\tilde{P}(\tilde{x}(x,y),\tilde{y}(x,y))$, $\tilde{Q}(\tilde{x}(x,y),\tilde{y}(x,y))$, $\tilde{R}(\tilde{x}(x,y),\tilde{y}(x,y))$, and $\tilde{S}(\tilde{x}(x,y),\tilde{y}(x,y))$ in terms of $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ and the partial derivations of the unknown functions $\tilde{x}(x,y)$ and $\tilde{y}(x,y)$ on x and y up to the third order. These formulas are rather complicated, and in general situation the equivalence problem can not be solved explicitly.

The main approach usually employed is to find invariants of equations (1.1). *Invariant* is a function that is preserved by transformations (1.2), i.e., $I(x,y) = I(\tilde{x}(x,y),\tilde{y}(x,y))$. Invariants Theory of equations (1.1) was initiated in the works of R.Liouville [16], S.Lie [15], A.Tresse [19, 20], E.Cartan [4, 18] (Late 19th- and Early 20th-Century) and continued in the Late 20th-Century in works [1, 3, 7, 8, 11] and others. Background is described in papers of L. Bordag [1, 2].

However, only advanced computer software for symbolic calculations gave an opportunity to make a substantial progress. In the series of papers [6, 21, 22] Ruslan Sharipov succeeded to construct the system of (pseudo)invariants which he calculated explicitly in the terms of the coefficients of equations (1.1). On the basis of this system he classified equations (1.1). This classification is more general than all previous ones. The relation between the (pseudo)invariants from works [21, 22] and the semiinvariants from works [4, 16] (as they were presented in [2]) was shown in paper [13] and here in Section 7. Moreover, in all possible cases the set of the invariants can be broadened. By employing this technique, in [12], [13] and [14] the equivalence problem for some equations was solved.

The first part of the present paper is a survey of [6, 21, 22]. We also add additional subcases (see Subsection 5.8) not mentioned in the cited works. The second part is an application of this classification for studying Painlevé equations.

2. Classification

Pseudoinvariant of weight m is a function transformed under transformations (1.2) with the factor $\det T$ (the Jacobi determinant) in the power m,

$$J(x,y) = (\det T)^m \cdot J(\tilde{x}(x,y),\tilde{y}(x,y)), \quad T = \begin{pmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial y \\ \partial\tilde{y}/\partial x & \partial\tilde{y}/\partial y \end{pmatrix}.$$

Pseudotensorial field of weight m and valence (r,s) is an indexed set transformed under change of variables (1.2) by the rule

$$F^{i_1\ldots i_r}_{j_1\ldots j_s} = (\det T)^m \sum_{p_1\ldots p_r} \sum_{q_1\ldots q_s} S^{i_1}_{p_1} \ldots S^{i_r}_{p_r} T^{q_1}_{j_1} \ldots T^{q_s}_{j_s} \tilde{F}^{p_1\ldots p_r}_{q_1\ldots q_s}, \quad \text{where} \quad S = T^{-1}.$$

Given the coefficients P, Q, R, and S of equation (1.1), we introduce a three-dimensional array by the rule

$$\Theta_{111} = P, \qquad \Theta_{121} = \Theta_{211} = \Theta_{112} = Q, \qquad \Theta_{122} = \Theta_{212} = \Theta_{221} = R, \qquad \Theta_{222} = S.$$

As "Gramian matrices" we take the following two,

$$d^{ij} = d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad d^{ij} \text{ is a pseudotensorial field of weight } 1, \quad d_{ij} \text{ is of weight -1.}$$

We raise the first index

$$\Theta^k_{ij} = \sum_{r=1}^{2} d^{kr}\Theta_{rij}. \tag{2.1}$$

Under the change of variables (1.2) the quantities Θ^k_{ij} are transformed "almost" as an affine connection (for transformation rule see [6]).

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Using Θ^k_{ij} as the affine connection, we construct the "curvature tensor"

$$\Omega^k_{rij} = \frac{\partial \Theta^k_{jr}}{\partial u^i} - \frac{\partial \Theta^k_{ir}}{\partial u^j} + \sum_{q=1}^{2} \Theta^k_{iq}\Theta^q_{jr} - \sum_{q=1}^{2} \Theta^k_{jq}\Theta^q_{ir}, \quad \text{here } u^1 = x,\, u^2 = y,$$

and the "Ricci tensor" $\Omega_{rj} = \sum_{k=1}^{2} \Omega^k_{rkj}$. Both these objects are not tensors. On the contrary, the three-dimensional array

$$W_{ijk} = \nabla_i \Omega_{jk} - \nabla_j \Omega_{ik}$$

is a tensor. Here we employ Θ^k_{ij} in covariant differentiation instead of the affine connection.

Using the tensor W_{ijk}, we introduce two extra pseudocovectorial fields,

$$\alpha_k = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} W_{ijk} d^{ij} \qquad \text{is a pseudocovectorial field of weight 1,}$$

$$\beta_i = 3\nabla_i \alpha_k d^{kr} \alpha_r + \nabla_r \alpha_k d^{kr} \alpha_i \quad \text{is a pseudocovectorial field of weight 3.}$$

The pseudovectorial fields are $\alpha^j = d^{jk}\alpha_k$ of weight 2 and $\beta^j = d^{ji}\beta_i$ of weight 4.

There are only three possible cases:

1. *Maximal degeneration case*, in which α=0.
2. *Intermediate degeneration case*, in which $3F^5 = \alpha^i \beta_i = 0$, i.e. the fields α and β are collinear.
3. *General case*, in which $3F^5 = \alpha^i \beta_i \neq 0$, i.e. the fields α and β are non-collinear.

3. Maximal degeneration case

The coordinates of the pseudovectorial field α are $\alpha^1 = B$, $\alpha^2 = -A$, where

$$\begin{aligned} A &= P_{0.2} - 2Q_{1.1} + R_{2.0} + 2PS_{1.0} + SP_{1.0} - 3PR_{0.1} - 3RP_{0.1} - 3QR_{1.0} + 6QQ_{0.1}, \\ B &= S_{2.0} - 2R_{1.1} + Q_{0.2} - 2SP_{0.1} - PS_{0.1} + 3SQ_{1.0} + 3QS_{1.0} + 3RQ_{0.1} - 6RR_{1.0}. \end{aligned} \tag{3.1}$$

Hereinafter symbol $K_{i.j}$ denotes the partial differentiation: $K_{i.j} = \partial^{i+j}K/\partial x^i \partial y^j$. In this case both the conditions $A = 0$ and $B = 0$ hold.

Theorem 3.1. ***(Lie)***. *All equations (1.1) with $A = 0$ and $B = 0$, where A and B given by (3.1), are equivalent to $\tilde{y}'' = 0$ by point transformation (1.2). They have an 8-dimension point symmetries algebra.*

For the details see papers [16], [7], and others.

Example. For equations No. 6.113, 6.134, 6.169 in the handbook by E.Kamke [10] we have

$$\begin{array}{ll} 6.113 & yy'' - y'^2 - y^2 \ln y = 0, \quad dim(\mathscr{L}) = 8, \\ 6.134 & (y-x)y'' - 2y'(y'+1) = 0, \quad dim(\mathscr{L}) = 8, \\ 6.169 & xyy'' + xy'^2 - yy' = 0 \quad dim(\mathscr{L}) = 8. \end{array}$$

4. General case

The pseudovectorial fields $\alpha = (B, -A)$ and $\beta = (G, H)$ are non-collinear, so their scalar product is non-zero. The pseudoinvariant F of weight 5 is defined as

$$\begin{aligned} 3F^5 &= AG + BH, \qquad \text{where A and B are from (3.1),} \\ G &= -BB_{1.0} - 3AB_{0.1} + 4BA_{0.1} + 3SA^2 - 6RBA + 3QB^2, \\ H &= -AA_{0.1} - 3BA_{1.0} + 4AB_{1.0} - 3PB^2 + 6QAB - 3RA^2. \end{aligned} \tag{4.1}$$

Since $F \neq 0$, the functions $\varphi_1 = -\partial \ln F / \partial x$ and $\varphi_2 = -\partial \ln F / \partial y$ are well-defined. Employing Θ_{ij}^k from (2.1), we construct an affine connection Γ_{ij}^k and two non-collinear vectorial fields X and Y

$$\Gamma_{ij}^k = \Theta_{ij}^k - \frac{\varphi_k \delta_j^k + \varphi_k \delta_i^k}{3}, \qquad X = \frac{\alpha}{F^2}, \qquad Y = \frac{\beta}{F^4}.$$

Their covariant derivatives are linear combinations of the basis fields X and Y,

$$\begin{aligned} \nabla_X X &= \hat{\Gamma}_{11}^1 X + \hat{\Gamma}_{11}^2 Y, \qquad \nabla_X Y = \hat{\Gamma}_{12}^1 X + \hat{\Gamma}_{12}^2 Y, \\ \nabla_Y X &= \hat{\Gamma}_{21}^1 X + \hat{\Gamma}_{21}^2 Y, \qquad \nabla_Y Y = \hat{\Gamma}_{22}^1 X + \hat{\Gamma}_{22}^2 Y. \end{aligned}$$

Here $\hat{\Gamma}_{ij}^k$ are scalar invariants of equation (1.1). In paper [21] they were denoted by

$$I_3 = \hat{\Gamma}_{12}^1, \quad I_6 = \hat{\Gamma}_{21}^2, \quad I_7 = \hat{\Gamma}_{22}^1, \quad I_8 = \hat{\Gamma}_{22}^2.$$

Differentiating these invariants along vector fields X and Y produces more invariants

$$XI_k = I_{k+8}, \qquad YI_k = I_{k+16}.$$

Repeating the procedure of differentiation along X and Y, we can construct an infinite sequence of invariants. The explicit formulas for the basic four invariants read as

$$\begin{aligned} I_3 &= \frac{B(HG_{1.0} - GH_{1.0})}{3F^9} - \frac{A(HG_{0.1} - GH_{0.1})}{3F^9} + \frac{HF_{0.1} + GF_{1.0}}{3F^5} + \\ &+ \frac{BG^2P}{3F^9} - \frac{(AG^2 - 2HBG)Q}{3F^9} + \frac{(BH^2 - 2HAG)R}{3F^9} - \frac{AH^2S}{3F^9}, \\ I_6 &= \frac{H(AB_{0.1} - BA_{0.1})}{3F^7} + \frac{G(AB_{1.0} - BA_{1.0})}{4F^7} - \frac{(AF_{0.1} - BF_{1.0})}{3F^3} - \\ &- \frac{GB^2P}{3F^7} - \frac{(HB^2 - 2GBA)Q}{3F^7} - \frac{(GA^2 - 2HBA)R}{3F^7} - \frac{HA^2S}{3F^7}, \\ I_7 &= \frac{GHG_{1.0} - G^2H_{1.0} + H^2G_{0.1} - HGH_{0.1} + G^3P + 3G^2HQ + 3GH^2R + H^3S}{3F^{11}}, \\ I_8 &= \frac{G(AG_{1.0} + BH_{1.0})}{3F^9} + \frac{H(AG_{0.1} + BH_{0.1})}{3F^9} - \frac{10(HF_{0.1} + GF_{1.0})}{3F^5} - \\ &- \frac{BG^2P}{3F^9} + \frac{(AG^2 - 2HBG)Q}{3F^9} - \frac{(BH^2 - 2HAG)R}{3F^9} + \frac{AH^2S}{3F^9}. \end{aligned}$$

The case of general position splits into three subcases:

1. In the infinite sequence of invariants I_k there exist two functionally independent ones; in this case the dimension of the point symmetries algebra is $dim(\mathscr{L}) = 0$.

2. Invariants I_k are functionally dependent, but not all of them are constants; in this case $dim(\mathscr{X}) = 1$.
3. All invarians in the sequence I_k are constants; here $dim(\mathscr{X}) = 2$.

Example. For equation 6.54 and 6.109 in the handbook by E. Kamke [10] we have

$$6.54 \qquad y'' = y^2 + 4yy' + y^2y'^2, \qquad dim(\mathscr{X}) = 1,$$
$$6.109 \qquad y'' = \frac{y'}{y} - \frac{y'^2}{y}, \qquad dim(\mathscr{X}) = 2.$$

5. Intermediate degeneration case

In this case $F = 0$, but $A \neq 0$ or $B \neq 0$, and the pseudovectorial fields α and β are collinear.

In the case $A \neq 0$ by φ_1 and φ_2 we redenote the functions

$$\varphi_1 = -3\frac{BP + A_{1.0}}{5A} + \frac{3}{5}Q, \;\; \varphi_2 = 3B\frac{BP + A_{1.0}}{5A^2} - 3\frac{B_{1.0} + A_{0.1} + 3BQ}{5A} + \frac{6}{5}R, \tag{5.1}$$

and in the case $B \neq 0$ we let

$$\varphi_1 = -3A\frac{AS - B_{0.1}}{5B^2} - 3\frac{A_{0.1} + B_{1.0} - 3AR}{5B} - \frac{6}{5}Q, \quad \varphi_2 = 3\frac{AS - B_{0.1}}{5B} - \frac{3}{5}R. \tag{5.2}$$

Employing the introduced functions and Θ^k_{ij} from (2.1), we construct the affine connection Γ^k_{ij} and a pseudoinvariant Ω of weight 1,

$$\Gamma^k_{ij} = \Theta^k_{ij} - \frac{\varphi_k\delta^k_j + \varphi_k\delta^k_i}{3}, \qquad \Omega = \frac{5}{3}\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right). \tag{5.3}$$

As $A \neq 0$, the explicit formula for the pseudoinvariant Ω reads as

$$\begin{aligned}\Omega = &\frac{2BA_{1.0}(BP + A_{1.0})}{A^3} - \frac{(2B_{1.0} + 3BQ)A_{1.0}}{A^2} + \frac{(A_{0.1} - 2B_{1.0})BP}{A^2} - \\ &- \frac{BA_{2.0} + B^2P_{1.0}}{A^2} + \frac{B_{2.0}}{A} + \frac{3B_{1.0}Q + 3BQ_{1.0} - B_{0.1}P - BP_{0.1}}{A} + Q_{0.1} - 2R_{1.0}.\end{aligned} \tag{5.4}$$

And in the case $B \neq 0$ the similar formula is

$$\begin{aligned}\Omega = &\frac{2AB_{0.1}(AS - B_{0.1})}{B^3} - \frac{(2A_{0.1} - 3AR)B_{0.1}}{B^2} + \frac{(B_{1.0} - 2A_{0.1})AS}{B^2} + \\ &+ \frac{AB_{0.2} - A^2S_{0.1}}{B^2} - \frac{A_{0.2}}{B} + \frac{3A_{0.1}R + 3AR_{0.1} - A_{1.0}S - AS_{1.0}}{B} + R_{1.0} - 2Q_{0.1}.\end{aligned} \tag{5.5}$$

The rule of covariant differentiation of the pseudotensorial field was given in [6],

$$\nabla_k F^{i_1\ldots i_r}_{j_1\ldots j_s} = \frac{\partial F^{i_1\ldots i_r}_{j_1\ldots j_s}}{\partial u^k} + \sum_{n=1}^{r}\sum_{v_n=1}^{2} \Gamma^{i_n}_{kv_n} F^{i_1\ldots v_n\ldots i_r}_{j_1\ldots j_s} - \sum_{n=1}^{s}\sum_{w_n=1}^{2} \Gamma^{w_n}_{kj_n} F^{i_1\ldots i_r}_{j_1\ldots w_n\ldots j_s} + m\varphi_k F^{i_1\ldots i_r}_{j_1\ldots j_s}.$$

If a pseudotensorial field F has valence (r,s) and weight m, the pseudotensorial field ∇F has valence $(r,s+1)$ and weight m.

The pseudovectorial fields α and β are collinear, hence there exists a coefficient N such that $\beta = 3N\alpha$. N is a pseudoinvariant of weight 2.

We let

$$\xi^i = d^{ij}\nabla_j N, \qquad M = -\alpha_i \xi^i, \qquad \gamma = -\xi - 2\Omega\alpha, \tag{5.6}$$

Here ξ is a pseudovectorial field of weight 3, M is a pseudoinvariant of weight 4, γ is a pseudovectorial field of weight 3.

In the cases $A \neq 0$ and $B \neq 0$ the pseudoinvariant N is given by the formulas

$$N = -\frac{H}{3A}, \qquad N = \frac{G}{3B}, \tag{5.7}$$

respectively. The pseudoinvariant M in the case $A \neq 0$ reads as

$$M = -\frac{12BN(BP + A_{1.0})}{5A} + BN_{1.0} + \frac{24}{5}BNQ + \frac{6}{5}NB_{1.0} + \frac{6}{5}NA_{0.1} - AN_{0.1} - \frac{12}{5}ANR, \tag{5.8}$$

and in the case $B \neq 0$ it is given by the formula

$$M = -\frac{12AN(AS - B_{0.1})}{5B} - AN_{0.1} + \frac{24}{5}ANR - \frac{6}{5}NA_{0.1} - \frac{6}{5}NB_{1.0} + BN_{1.0} - \frac{12}{5}BNQ. \tag{5.9}$$

In the case $A \neq 0$ the field γ is

$$\begin{aligned} \gamma^1 &= -\frac{6BN(BP + A_{1.0})}{5A^2} + \frac{18NBQ}{5A} + \frac{6N(B_{1.0} + A_{0.1})}{5A} - N_{0.1} - \frac{12}{5}NR - 2\Omega B, \\ \gamma^2 &= -\frac{6N(BP + A_{1.0})}{5A} + N_{1.0} + \frac{6}{5}NQ + 2\Omega A. \end{aligned} \tag{5.10}$$

In the case $B \neq 0$ the field γ is

$$\begin{aligned} \gamma^1 &= -\frac{6N(AN - B_{0.1})}{5B} - N_{0.1} + \frac{6}{5}NR - 2\Omega B, \\ \gamma^2 &= -\frac{6AN(AS - B_{0.1})}{5B^2} + \frac{18NAR}{5B} - \frac{6N(A_{0.1} + B_{1.0})}{5B} + N_{1.0} - \frac{12}{5}NQ + 2\Omega A. \end{aligned} \tag{5.11}$$

5.1. *First case of intermediate degeneration:* $M \neq 0$

If in (5.8), (5.9) $M \neq 0$, the pseudovectorial fields α in (3.1) and γ in (5.10), (5.11) are non-collinear. Moreover, it means that $N \neq 0$ in (5.7). Consider the expansion $\nabla_\gamma \gamma = \hat{\Gamma}^1_{22}\alpha + \hat{\Gamma}^2_{22}\gamma$. The basic invariants are the following ones,

$$I_1 = \frac{M}{N^2}, \qquad I_2 = \frac{\Omega^2}{N}, \qquad I_3 = \frac{\hat{\Gamma}^1_{22}}{M}. \tag{5.12}$$

Here$M$, N, and Ω are from (5.8), (5.9), (5.7), (5.4), (5.5). The explicit formula for $\hat{\Gamma}^1_{22}$ is

$$\hat{\Gamma}^1_{22} = \frac{\gamma^1\gamma^2(\gamma^1_{1.0} - \gamma^2_{0.1})}{M} + \frac{(\gamma^2)^2\gamma^1_{0.1} - (\gamma^1)^2\gamma^2_{1.0}}{M} + \frac{P(\gamma^1)^3 + 3Q(\gamma^1)^2\gamma^2 + 3R\gamma^1(\gamma^2)^2 + S(\gamma^2)^3}{M}.$$

$\hat{\Gamma}^1_{22}$ is pseudoinvariant of the weight 4. By differentiating the invariants I_1, I_2 and I_3 along fields α and γ we get new invariants

$$I_{k+3} = \frac{\nabla_\alpha I_k}{N}, \qquad I_{k+6} = \frac{(\nabla_\gamma I_k)^2}{N^3}. \tag{5.13}$$

The first case of intermediate degeneration splits into three subcases

1. In the infinite sequence of invariants I_k there exist two functionally independent ones; in this case the dimension of the point symmetries algebra is $dim(\mathscr{L}) = 0$.
2. Invariants I_k are functionally dependent but not all of them are constants; here we have $dim(\mathscr{L}) = 1$.
3. All invariants in the sequence I_k are constants; here $dim(\mathscr{L}) = 2$.

Example. For equations 6.45, 6.174 in the handbook by E. Kamke [10] we have

$$6.45 \qquad y'' = ay'^2 + by, \quad a, b = const, \quad ab \neq 0, \qquad dim(\mathscr{L}) = 1,$$
$$6.174 \qquad xyy'' - 2xy'^2 + (y+1)y' = 0, \qquad dim(\mathscr{L}) = 2.$$

5.2. *Second case of intermediate degeneration*

If in (5.8), (5.9) $M = 0$, the pseudovectorial fields α in (3.1) and γ in (5.10), (5.11) are collinear. Hence, there exists a coefficient Λ such that $\gamma = \Lambda\alpha$. Here Λ is a pseudoinvariant of weight 1, in the cases $A \neq 0$ and $B \neq 0$ being respectively

$$\Lambda = -\frac{\gamma^2}{A}, \quad A \neq 0, \qquad \text{or} \qquad \Lambda = \frac{\gamma^1}{B}, \quad B \neq 0.$$

The explicit formulas for Λ are

$$\Lambda = -\frac{6N(AS - B_{0.1})}{5B^2} - \frac{N_{0.1}}{B} + \frac{6NR}{5B} - 2\Omega. \tag{5.14}$$

$$\Lambda = \frac{6N(BP + B_{1.0})}{5A^2} - \frac{N_{1.0}}{A} - \frac{6NQ}{5A} - 2\Omega. \tag{5.15}$$

Let us calculate the curvature tensor using the connections (5.3):

$$R^k_{qij} = \frac{\partial \Gamma^k_{jk}}{\partial u^i} - \frac{\partial \Gamma^k_{iq}}{\partial u^j} + \sum_{s=1}^{2} \Gamma^k_{is}\Gamma^s_{jq} - \sum_{s=1}^{2} \Gamma^k_{js}\Gamma^s_{iq}, \quad u^1 = x, u^2 = y,$$

and the pseudotensorial field of the weight 1:

$$R^k_q = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} R^i_{qij} d^{ij},$$

where λ_1 and λ_2 are its eigenvalues. Now we construct the pseudocovectorial field of the weight -1. If $A \neq 0$, we let

$$\omega_1 = -\frac{R^2_1}{A}, \qquad \omega_2 = \frac{\lambda_2 - R^2_2}{A}, \tag{5.16}$$

where

$$\omega_1 = \frac{12PR}{5A} - \frac{54}{25}\frac{Q^2}{A} - \frac{P_{0.1}}{A} + \frac{6Q_{1.0}}{5A} - \frac{PA_{0.1} + BP_{1.0} + A_{2.0}}{5A^2} - \\ - \frac{2B_{1.0}P}{5A^2} + \frac{3QA_{1.0} - 12PBQ}{25A^2} + \frac{6B^2P^2 + 12A_{1.0}BP + 6A^2_{1.0}}{25A^3},$$
$$\omega_2 = \frac{6\Lambda + 3\Omega}{5A} + \frac{-5BP_{0.1} + 6BQ_{1.0} + 12RBP}{5A^2} - \frac{54}{25}\frac{BQ^2}{A^2} - \frac{12B^2PQ}{25A^3} + \frac{3BQA_{1.0}}{25A^3} - \\ - \frac{2BB_{1.0}P + BA_{0.1}P + B^2P_{1.0} + BA_{2.0}}{5A^3} + \frac{6BA^2_{1.0} + 6B^3P^2 + 12B^2A_{1.0}P}{25A^4}.$$

And if $B \neq 0$,

$$\omega_1 = \frac{R_1^1 - \lambda_2}{B}, \qquad \omega_2 = \frac{R_2^1}{B}, \qquad \text{and} \tag{5.17}$$

$$\omega_1 = -\frac{6\Lambda + 3\Omega}{5B} + \frac{5AS_{1.0} - 6AR_{0.1} + 12QAS}{5B^2} - \frac{54}{25}\frac{AR^2}{B^2} - \frac{12A^2SR}{25B^3} + \frac{3ARB_{0.1}}{25B^3} + \\ + \frac{2AA_{0.1}S + AB_{1.0}S + A^2S_{0.1} - AB_{0.2}}{5B^3} + \frac{6AB_{0.1}^2 + 6A^3S^2 - 12A^2B_{0.1}S}{25B^4},$$
$$\omega_2 = \frac{12SQ}{5B} - \frac{54}{25}\frac{R^2}{B} + \frac{S_{1.0}}{B} - \frac{6R_{0.1}}{5B} + \frac{SB_{1.0} + AS_{0.1} - B_{0.2}}{5B^2} + \\ + \frac{2A_{0.1}S}{5B^2} - \frac{3RB_{0.1} + 12SAR}{25B^2} + \frac{6A^2S^2 - 12B_{0.1}AS + 6B_{0.1}^2}{25B^3}.$$

We introduce one more pseudocovectorial field of weight 1, $w = N\omega + \nabla\Lambda + \nabla\Omega/3$. It is collinear to the pseudocovectorial field α in (3.1), and thus there exists a coefficient K such that $w = K\alpha$,

$$K = \frac{\Lambda_{1.0} + \Lambda\varphi_1}{A} + \frac{\Omega_{1.0} + \Omega\varphi_1}{3A} + \frac{N\omega_1}{A}, \qquad A \neq 0. \tag{5.18}$$

$$K = \frac{\Lambda_{0.1} + \Lambda\varphi_2}{B} + \frac{\Omega_{0.1} + \Omega\varphi_2}{3B} + \frac{N\omega_2}{B}, \qquad B \neq 0. \tag{5.19}$$

K is a scalar field of the weight 0.

By ε we denote a pseudocovectorial field of weight 1, $\varepsilon = N\omega + \nabla\Lambda$. Raising indices by matrix d^{ij}, we get the pseudovectorial field ε of weight 2,

$$\varepsilon^1 = N\omega_2 + \Lambda_{0.1} + \varphi_2\Lambda, \qquad \varepsilon^2 = -N\omega_1 - \Lambda_{1.0} - \varphi_1\Lambda. \tag{5.20}$$

The fields ε in (5.20) and α in (3.1) are non-collinear, and we can write

$$\nabla_\varepsilon\varepsilon = \hat{\Gamma}_{22}^1\alpha + \hat{\Gamma}_{22}^2\varepsilon,$$

$$\hat{\Gamma}_{22}^1 = \frac{5\varepsilon^1\varepsilon^2(\varepsilon_{1.0}^1 - \varepsilon_{0.1}^2)}{3N\Omega} + \frac{5(\varepsilon^2)^2\varepsilon_{0.1}^1 - 5(\varepsilon^1)^2\varepsilon_{1.0}^2}{3N\Omega} + \frac{5P(\varepsilon^1)^3 + 15Q(\varepsilon^1)^2\varepsilon^2 + 15R\varepsilon^1(\varepsilon^2)^2 + 5S(\varepsilon^2)^3}{3N\Omega}.$$

$\hat{\Gamma}_{22}^1$ is pseudoinvariant of the weight 2. We introduce pseudoscalar fields L of the weight 2 and E of the weight 2

$$L = KN + \frac{5}{9}N + 3\Lambda\Omega + \frac{7}{9}\Omega^2 + 2\Lambda^2. \tag{5.21}$$

$$E = \hat{\Gamma}_{22}^1 - \frac{\nabla_\varepsilon L}{N} + \frac{4\Lambda\nabla_\varepsilon\Lambda}{N} + \frac{17\Omega\nabla_\varepsilon\Lambda}{6N} + \frac{12L^2}{5N} - \frac{53L\Lambda\Omega}{5N} - \frac{48L\Lambda^2}{5N} - \\ - \frac{62L\Omega^2}{15N} - \frac{8L}{3} + \frac{48\Lambda^4}{5N} + \frac{106\Lambda^3\Omega}{5N} + \frac{16\Lambda^2}{3} + \\ + \frac{1163\Lambda^2\Omega^2}{60N} + \frac{137\Lambda\Omega^3}{18N} + \frac{50\Lambda\Omega}{9} + \frac{203\Omega^2}{108} - \frac{77\Omega^4}{135N} + \frac{20N}{27}. \tag{5.22}$$

Employing the above objects, we can define invariants

$$I_1 = \frac{\Lambda^{12}}{\Omega^8 N^2}, \qquad I_2 = \frac{L^4}{N^2\Omega^4}, \qquad I_3 = \frac{E^6 N^4}{\Omega^{20}}.$$

Here Λ is from (5.14), (5.15), Ω is from (5.4), (5.5), N is from (5.7), L is from (5.21), E is from (5.22).

In the second case of intermediate degeneration the algebra of the point symmetries of equation (1.1) is 1-dimensional if and only if all invariants I_1, I_2, I_3 are identically constant. In other cases it is trivial.

Example. For the following equation one has

$$y'' = 3yy' - y^3 + \frac{a}{(b+y)^3}, \quad a, b = const, \quad a \neq 0, \qquad dim(\mathscr{L}) = 1.$$

5.3. *Third case of intermediate degeneration*

In this case $N \neq 0$ in (5.7), $M = 0$ in (5.8), (5.9), $\Omega = 0$ in (5.4), (5.5), $\Lambda \neq 0$ in (5.14), (5.15). Consider again the pseudocovectorial field ω of the weight -1 from (5.16), (5.17). Raising indices by the matrix d^{ij}, we get the vector field ω, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$. Since $\Lambda \neq 0$, ω, and α are non-collinear, we obtain the following relation

$$\nabla_\omega \omega = \hat{\Gamma}^1_{22}\alpha + \hat{\Gamma}^2_{22}\omega,$$

$$\hat{\Gamma}^1_{22} = \frac{5\omega^1\omega^2(\omega^1_{1.0} - \omega^2_{0.1})}{6\Lambda} + \frac{5(\omega^2)^2\omega^1_{0.1} - 5(\omega^1)^2\omega^2_{1.0}}{6\Lambda} + \\ + \frac{5P(\omega^1)^3 + 15Q(\omega^1)^2\omega^2 + 15R\omega^1(\omega^2)^2 + 5S(\omega^2)^3}{6\Lambda}.$$

$\hat{\Gamma}^1_{22}$ has weight -2. In this case we define new fields L of the weight 0 and E of the weight -2,

$$\begin{aligned} L &= K + \frac{5}{9} + \frac{2\Lambda^2}{N}, \quad \text{with } K \text{ from (5.18), (5.19)} \\ E &= \hat{\Gamma}^1_{22} - \frac{\nabla_\omega L}{N} + \frac{9L^2}{5N} - \frac{2L}{N} - \frac{12L\Lambda^2}{5N^2} + \frac{7\Lambda^2}{3N^2} + \frac{5}{9N} + \frac{63\Lambda^4}{20N^3}. \end{aligned} \tag{5.23}$$

Let us construct the invariants:

$$I_1 = \frac{L^8 N^6}{\Lambda^{12}}, \qquad I_2 = \frac{EN^3}{\Lambda^4}.$$

Here L, E are from (5.23), N is from (5.7), Λ is from (5.14), (5.15).

In the third case of intermediate degeneration the algebra of the point symmetries of equation (1.1) is 1-dimensional if and only if both the invariants I_1, I_2 are identically constant. In other cases it is trivial.

Example. For Emden-Fowler equation 6.11 with $n = -3$ in the handbook by E. Kamke [10]

$$6.11 \qquad y'' = -\frac{ax^m}{y^3}, \quad a = const \neq 0, \quad m \neq 0, \quad \text{one has} \quad dim(\mathscr{L}) = 1.$$

5.4. *Fourth case of intermediate degeneration*

In this case $N \neq 0$ in (5.7), $M = 0$ in (5.8), (5.9), $\Omega = 0$ in (5.4), (5.5), $\Lambda = 0$ in (5.14), (5.15), $K \neq -5/9$ in (5.18), (5.19). Consider again the vectorial field ω, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$, from (5.16), (5.17). Since $\Lambda = 0$, ω, and α are collinear, we can define a new scalar field Θ by the relationship $\omega = \Theta\alpha$, field Θ has weight -2,

$$\Theta = \frac{\omega_1}{A}, \quad A \neq 0, \qquad \Theta = \frac{\omega_2}{B}, \quad B \neq 0. \tag{5.24}$$

The covariant differential $\theta = \nabla\Theta$ is a pseudocovectorial field of weight -2,

$$\theta_1 = \Theta_{1.0} - 2\varphi_1\Theta, \qquad \theta_2 = \Theta_{0.1} - 2\varphi_2\Theta. \tag{5.25}$$

The corresponding pseudovectorial field of weight -1 is $\theta^1 = \theta_2$, $\theta^2 = -\theta_1$. Let us calculate its convolution with α from (3.1),

$$L = -\frac{5}{9}\sum_{i=1}^{2}\alpha_i\theta^i = \frac{5}{9}\sum_{i=1}^{2}\theta_i\alpha^i. \quad \text{Note that} \quad L = K + \frac{5}{9}, \tag{5.26}$$

where K is from (5.18), (5.19). Field L has weight 0. Since $L \neq 0$, fields θ (5.25) and α (3.1) are non-collinear,

$$\nabla_\theta\theta = \hat{\Gamma}^1_{22}\alpha + \hat{\Gamma}^2_{22}\theta,$$

$$\hat{\Gamma}^1_{22} = -\frac{5\theta^1\theta^2(\theta^1_{1.0} - \theta^2_{0.1})}{9L} - \frac{5(\theta^2)^2\theta^1_{0.1} - 5(\theta^1)^2\theta^2_{1.0}}{9L} - \\ - \frac{5P(\theta^1)^3 + 15Q(\theta^1)^2\theta^2 + 15R\theta^1(\theta^2)^2 + 5S(\theta^2)^3}{9L}.$$

$\hat{\Gamma}^1_{22}$ has weight -4. We introduce one more pseudoscalar field E of the weight -4,

$$E = \hat{\Gamma}^1_{22} + \frac{27N}{5}\left(\Theta + \frac{5}{9N}\right)^3 - \frac{3}{4}\left(\Theta + \frac{5}{9N}\right)^2 \tag{5.27}$$

that gives rise to the invariant

$$I_1 = \frac{E^6N^{12}}{L^{20}}$$

where E is from (5.27), N is from (5.7), L is from (5.26).

In the fourth case of intermediate degeneration the algebra of the point symmetries of equation (1.1) is 1-dimensional if and only if the invariant I_1 is identically constant. Otherwise it is trivial.

Example. For the following equation one has

$$y'' = \frac{5y'^2}{4y} + y^{\frac{5}{4}} + ay^2, \quad a = const \neq 0, \qquad dim(\mathscr{L}) = 1.$$

5.5. *Fifth case of intermediate degeneration*

In this case $N \neq 0$ in (5.7), $M = 0$ in (5.8), (5.9), $\Omega = 0$ in (5.4), (5.5), $\Lambda = 0$ in (5.14), (5.15), $K = -5/9$ in (5.18), (5.19). All equations (1.1) are equivalent to

$$y'' = \frac{1}{y^3}, \quad \text{or another form} \quad y'' = -\frac{5}{4x}y' + \frac{4}{3}x^2 y'^3.$$

The algebra of point symmetries is 3-dimensional, see also [17].

Example. For equations 6.81, 6.138 in the handbook by E. Kamke [10] we have

$$\begin{array}{ll} 6.81 & 2xy'' + y'^3 + y' = 0, \qquad dim(\mathscr{L}) = 3, \\ 6.138 & 2yy'' - y'^2 + a = 0, \quad a = const, \qquad dim(\mathscr{L}) = 3. \end{array}$$

5.6. *Sixth case of intermediate degeneration*

In this case $N = 0$ in (5.7), $\Omega \neq 0$ in (5.4), (5.5). The pseudovectorial fields ω, $\omega^1 = \omega_2$, $\omega^2 = -\omega_1$ from (5.16), (5.17) and α from (3.1) are non-collinear,

$$\nabla_\omega \omega = \hat{\Gamma}^1_{22}\alpha + \hat{\Gamma}^2_{22}\omega,$$

$$\hat{\Gamma}^1_{22} = -\frac{5\omega^1\omega^2(\omega^1_{1.0} - \omega^2_{0.1})}{9\Omega} - \frac{5(\omega^2)^2\omega^1_{0.1} - 5(\omega^1)^2\omega^2_{1.0}}{9\Omega} - \\ - \frac{5P(\omega^1)^3 + 15Q(\omega^1)^2\omega^2 + 15R\omega^1(\omega^2)^2 + 5S(\omega)^3}{9\Omega}.$$

$\hat{\Gamma}^1_{22}$ has weight -2. We introduce a new scalar field L of the weight 0

$$L = \nabla_\omega K - \frac{21}{25}K^2 - K. \tag{5.28}$$

Then the corresponding invariants are

$$I_1 = L,$$
$$I_2 = \Omega^2\hat{\Gamma}^1_{22} - \nabla_\omega L - \frac{72}{625}K^3 + \frac{63}{50}K^2 + \frac{12}{25}KL - K - L.$$

where K is from (5.18), (5.19), Ω is from (5.4), (5.5), ω is from (5.16), (5.17), L is from (5.28).

In the sixth case of intermediate degeneration the algebra of point symmetries of equation (1.1) is 1-dimensional if and only if both invariants I_1, I_2 are identically constant. In other cases it is trivial.

Example. For the following equation one has

$$y'' = 3yy' - y^3 + \frac{y^2}{2} + ay + b, \quad a, b = const, \qquad dim(\mathscr{L}) = 1.$$

5.7. *Seventh case of intermediate degeneration*

In this case $N = 0$ in (5.7), $\Omega = 0$ in (5.4), (5.5). The pseudovectorial fields θ from (5.25) and α from (3.1) are non-collinear,

$$\nabla_\theta \theta = \hat{\Gamma}^1_{22}\alpha + \hat{\Gamma}^2_{22}\theta,$$

$$\hat{\Gamma}^1_{22} = \theta^1\theta^2(\theta^1_{1.0} - \theta^2_{0.1}) - (\theta^2)^2\theta^1_{0.1} + (\theta^1)^2\theta^2_{1.0} - P(\theta^1)^3 - 3Q(\theta^1)^2\theta^2 - 3R\theta^1(\theta^2)^2 - S(\theta)^3.$$

$\hat{\Gamma}^1_{22}$ has weight -4. We define the pseudoscalar fields L of the weight -4 and L_1 of the weight -5

$$L = \hat{\Gamma}^1_{22} - \frac{1}{2}\Theta^2, \quad L_1 = \nabla_\theta L = L_{1.0}\theta^1 + L_{0.1}\theta^2 - 4L(\varphi_1\theta^1 + \varphi_2\theta^2). \tag{5.29}$$

Then the invariants are

$$I_1 = \frac{L_1^4}{L^5}, \qquad I_2 = \frac{\Theta^2}{L}, \tag{5.30}$$

where we have employed (5.29), (5.24) and (5.1), (5.2).

The seventh case of intermediate degeneration splits into three subcases:

1. $L = 0$ from (5.29), then $dim(\mathscr{L}) = 2$.
2. $L \neq 0$ from (5.29) and I_1 from (5.30) is identically constant; here we have $dim(\mathscr{L}) = 1$.
3. In other cases $dim(\mathscr{L}) = 0$.

Example. For equations 6.2 and 6.5 in the handbook by E. Kamke [10] one has

6.2 $\quad y'' = 6y^2, \qquad dim(\mathscr{L}) = 2,$

6.5 $\quad y'' = ay^2 + bx + c, \quad a \neq 0 \qquad dim(\mathscr{L}) = 2$ if $b^2 = ac$ otherwise $dim(\mathscr{L}) = 1.$

5.8. *Additional subcases of intermediate degeneration*

We define a new pseudocovectorial field $\eta = \nabla M$. Let Z is a convolution η with the field ξ from (5.6). Z is a pseudoinvariant of weight 7,

$$Z = \eta_i \xi^i. \tag{5.31}$$

Then the first case of the intermediate degeneration splits into four subcases.

Case 1.1. $M \neq 0$, $\Omega \neq 0$, $Z \neq 0$.

Case 1.2. $M \neq 0$, $\Omega \neq 0$, $Z = 0$.

Case 1.3. $M \neq 0$, $\Omega = 0$, $Z \neq 0$.

Case 1.4. $M \neq 0$, $\Omega = 0$, $Z = 0$.

Subject to the pseudoinvariant Θ from (5.24), the seventh case of the intermediate degeneration splits into the two subcases.

Case 7.1. $N = 0$, $\Omega = 0$, $\Theta \neq 0$.

Case 7.2. $N = 0$, $\Omega = 0$, $\Theta = 0$.

5.9. *Tree of intermediate degeneration cases*

The following diagram illustrates the cases of the intermediate degeneration.

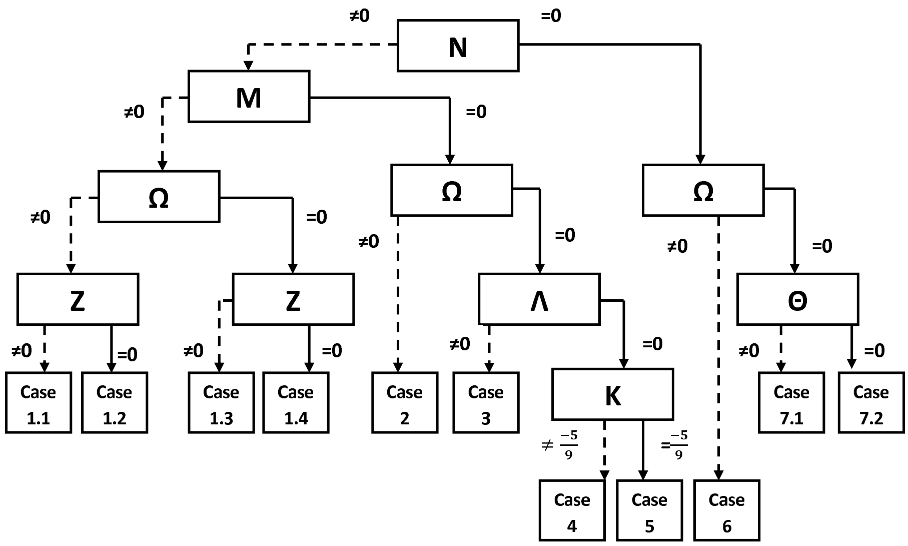

Fig. 1. Tree of intermediate degeneration cases.

6. Classification of Painlevé equations

Let us show how Painlevé equations are included into the proposed classification scheme.

1. Equation Painlevé I is in Case 7.1 of intermediate degeneration.
2. Equations Painlevé III-VI (except special cases!) are in Case 1.3 of intermediate degeneration.
3. Special cases.
   (a) Equation Painlevé II is in Case 1.4 of intermediate degeneration.
   (b) Equation Painlevé III with 3 zero parameters is in Case 1.4 of intermediate degeneration.
   (c) Equation Painlevé III $(0,b,0,d)$ or $(a,0,c,0)$ (they are equivalent) is in Case 1.4 of intermediate degeneration.
   (d) Equation Painlevé V $(a,b,0,0)$ is in Case 1.4 of intermediate degeneration.
   (e) Equation Painlevé III $(0,0,0,0)$ is in Case of maximal degeneration.
   (f) Equation Painlevé V $(0,0,0,0)$ is in Case of maximal degeneration.
   (g) Equation Painlevé VI $(0,0,0,1/2)$ is in Case of maximal degeneration.

7. Relation between the semiinvariants

In work of E. Cartan [4] the notations $P=-a_4$, $Q=-a_3$, $R=-a_2$, $S=-a_1$, $A=-L_1$, $B=-L_2$ were adopted, where L_1 and L_2 are the components of the projective curvature tensor. In the work of R.Liouville [16] there were provided the semiinvariants v_5, w_1, i_2, and R_1 (see review [2]). Relations between them and the pseudoinvariants F, Ω, N and the component H are as follows,

$$F^5=v_5,\quad H=L_1(L_2)_x-L_2(L_1)_x+3R_1,\quad \Omega=-w_1-\frac{v_5a_4}{L_1^3}-4\frac{(L_1)_xR_1}{L_1^3},\quad N=\frac{i_2}{3}.$$

Other pseudovectorial fields and pseudoinvariants appeared firstly in $[6,21,22]$.

8. Solution of equivalence problem for some Painlevé equations

8.1. *Painlevé I equation*

The equivalence problem for this equation was effectively solved in paper [12].

Theorem 8.1. *Equation (1.1) is equivalent to Painlevé I equation*

$$\tilde{y}'' = 6\tilde{y}^2 + \tilde{x}$$

under the point transformations (1.2) if and only if the following conditions hold: $F = 0$ in (4.1), $A \neq 0$ or $B \neq 0$ in (3.1), $\Omega = 0$ in (5.4), (5.5), $N = 0$ in (5.7), $W = 0$ in (8.1), $V = 0$ in (8.2), $\Theta \neq 0$ in (5.24), $L_1 \neq 0$ in (5.29). Invariants I_1 and I_2 in (5.30) are functionally independent. The point transformation is $\tilde{x} = 1/\sqrt[5]{12 I_1}$, $\tilde{y} = \pm\sqrt{I_2}(\sqrt[5]{12^3}\,\sqrt[10]{I_1})$.

Here the pseudoinvariant W is introduced by (5.29), (5.24) and (5.1), (5.2),

$$W = \nabla_\theta L_1 = (L_1)_{1.0}\theta^1 + (L_1)_{0.1}\theta^2 - 5L_1(\varphi_1\theta^1 + \varphi_2\theta^2), \tag{8.1}$$

and the pseudoinvariant V is introduced by (5.29), (3.1) and (5.1), (5.2),

$$V = \nabla_\alpha L_1 = (L_1)_{1.0}B - (L_1)_{0.1}A - 5L_1(B\varphi_1 - A\varphi_2). \tag{8.2}$$

Example. The equation

$$y'' = -\sin^3 y(6x\cos^2 y + \sin y) + \frac{1}{x}(-18x^3\cos^3 y \sin^2 y - 3x^2\sin^3 y\cos y - 2)y' -$$
$$- (18x^3\cos^4 y\sin y + 3x^2\sin^2 y\cos^2 y)y'^2 - (6x^4\cos^5 y + x^3\sin y\cos^3 y + x)y'^3$$

is equivalent to Painlevé I equation. The corresponding invariants and the change of variables are

$$I_1 = \frac{1}{12}\frac{1}{x^5\sin^5 y}, \quad I_2 = \frac{12x\cos^2 y}{\sin y}, \quad \tilde{y} = x\cos y, \quad \tilde{x} = x\sin y.$$

Example. The equation

$$y'' = 6y^2 + f(x)$$

is equivalent to Painlevé I equation if and only if $f(x) = mx + n$, where m, n are the constants, $m \neq 0$. Let us check the conditions of Theorem 8.1:

$$A = 12, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad N = 0, \quad W = \frac{f''(x)}{248832}, \quad V = 0,$$

$$\Theta = -\frac{y}{12}, \quad L_1 = -\frac{f'(x)}{20736}, \quad I_1 = \frac{f'^4(x)}{12f^5(x)}, \quad I_2 = \frac{12y^2}{f(x)}.$$

Example. Only these equations in the handbook by E. Kamke [10]

6.3	$y'' = 6y^2 + x$,	Painlevé I
6.5	$y'' = -ay^2 - bx - c$,	$a, b = const$, $a, b \neq 0$

are equivalent to Painlevé I equation.

8.2. ***Painlevé II equation***

The equivalence problem for this equation was effectively solved in papers [12] and [13].

Theorem 8.2. *Equation (1.1) is equivalent to Painlevé II equation*

$$\tilde{y}'' = 2\tilde{y}^3 + \tilde{x}\tilde{y} + \tilde{a}$$

with the parameter $\tilde{a} = \pm J$ *with* $J = (4 + 10I_6 - 60I_3)/(50\sqrt{I_9})$ *if and only if the following conditions hold:* $F = 0$ *in (4.1),* $A \neq 0$ *or* $B \neq 0$ *in (3.1),* $\Omega = 0$ *in (5.4), (5.5),* $M \neq 0$ *in (5.8), (5.9),* $I_1 = 18/5$ *in (5.12),* $I_9 \neq 0$ *in (5.13), invariant* J *is a constant. Among the invariants* I_3*,* I_6*, and* I_9 *from (5.12), (5.13,* $k = 3$*) one can find two functionally independent. The point transformation is* $\tilde{y} = 1/\sqrt[6]{2500I_9}$*,* $\tilde{x} = 5I_6/\sqrt[6]{2500I_9} - 3J\sqrt[6]{2500I_9}/2$.

Example. The equation

$$y'' = (-2x^3 - xy + a)y'^3 \quad a = const,$$

is equivalent to Painlevé II equation with $\tilde{a} = \pm a$. Let us check the conditions of Theorem 8.2,

$$A = 0,\ B = -12x,\ F = 0,\ \Omega = 0,\ M = \frac{288}{5},\ I_1 = \frac{18}{5},\ J = \pm a,$$

$$I_3 = \frac{2x^3 + xy - a}{30x^3},\ I_6 = \frac{2xy - 3a}{10x^3},\ I_9 = \frac{1}{2500x^6}, \quad \tilde{y} = x,\ \tilde{x} = y.$$

Example. The equation

$$y'' = y^3 + f(x)y + g(x)$$

is equivalent to Painlevé II equation if and only if $g(x) = c = const,\ f(x) = mx + n$, where m, n are the constants, $m \neq 0$. Let us check the conditions of Theorem 8.2,

$$A = 6y,\ B = 0,\ F = 0,\ \Omega = 0,\ M = \frac{72}{5},\ I_1 = \frac{18}{5},\ J = \frac{g(x)y}{\sqrt{2}(f'(x)y + g'(x))},$$

$$I_3 = \frac{y^3 + f(x)y + g(x)}{15y^3}, \quad I_6 = \frac{2f(x)y + 3g(x)}{5y^3}, \quad I_9 = \frac{2(f'(x)y + g'(x))^2}{625y^8}.$$

Example. Only these equations in the handbook by E. Kamke [10]

6.6 $\quad y'' = 2y^3 + xy + a, \quad a = const,$ Painlevé II,

6.8 $\quad y'' = 2a^2y^3 - 2abxy + b, \quad a, b = const, \quad a, b \neq 0,$ where $\tilde{a} = \pm\frac{1}{2}$,

6.9 $\quad y'' = -ay^3 - bxy - cy - d, \quad a, b, c, d = const, \quad a, b \neq 0,$ where $\tilde{a} = \pm\frac{d\sqrt{a}}{b\sqrt{-2}}$,

6.142 $\quad y'' = \frac{y'^2}{2y} + 4y^2 + 2xy,$ where $\tilde{a} = 0$,

6.145 $\quad y'' = \frac{y'^2}{2y} - \frac{ay^2}{2} - \frac{bxy}{2}, \quad a, b = const, \quad a, b \neq 0,$ where $\tilde{a} = 0$,

6.27 $\quad y'' = -ay' - bx^my^n, \quad a, b = const, \quad n = 3,\ m, b \neq 0,$ where $\tilde{a} = 0$.

are equivalent to Painleve II equation.

8.3. *Painlevé III equation with 3 zero parameters*

A general form of the Painlevé equations III reads as

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{ay^2+b}{x} + cy^3 + \frac{d}{y}.$$

It is a 4-parameter family of equations, which we denote by $PIII(a,b,c,d)$. If three of four parameters vanish, all these equations are equivalent one to another. Referring to work [8], we write the change of variables: 1), 3): $x = \tilde{x}, y = 1/\tilde{y}$, 2): $x = \tilde{x}^2/2, y = \tilde{y}^2$.

$$PIII(0,b,0,0) \overset{1)}{\to} PIII(-b,0,0,0) \overset{2)}{\to} PIII(0,0,-b,0) \overset{3)}{\to} PIII(0,0,0,b),$$

The equivalence problem for this equation was effectively solved in paper [13].

Theorem 8.3. *Equation (1.1) is equivalent to Painlevé III equation with 3 zero parameters if and only if the following conditions hold:* $F = 0$ *in (4.1),* $A \neq 0$ *or* $B \neq 0$ *in (3.1),* $\Omega = 0$ *in (5.4), (5.5),* $M \neq 0$ *in (5.8), (5.9),* $I_1 = 3/5$, $I_3 = 1/15$ *from (5.12).*

Example. The equation 6.75 in the handbook by E. Kamke [10]

$$6.75 \qquad y'' = -\frac{2}{x}y' - e^y$$

is not equivalent to Painlevé III equation with 3 zero parameters, in that

$$A = -e^y, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad M = \frac{e^{2y}}{15}, \quad I_1 = \frac{3}{5}, \quad I_3 = \frac{1}{15} - \frac{4}{15x^2e^y}.$$

Example. The following equation

$$y'' = f(x)y' - e^y$$

is equivalent to Painlevé III equation with 3 zero parameters if and only if function $f(x)$ is the solution of equation $f^2(x) - f'(x) = 0$. Hence $f(x) = 1/(c-x)$, where c is the constant. Let us check the conditions of Theorem 8.3,

$$A = -e^y, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad M = \frac{e^{2y}}{15}, \quad I_1 = \frac{3}{5}, \quad I_3 = \frac{1}{15} - \frac{2(f^2(x) - f'(x))}{15e^y}.$$

Example. Only these equations in the handbook by E. Kamke [10]

6.14 $\quad y'' = e^y,$

6.28 $\quad y'' = -ay' - be^y + 2a, \quad b = const \neq 0, \quad a = 0$ or $a = -1,$

6.76 $\quad y'' = -\frac{a}{x}y' - be^y, \quad b = const \neq 0, \quad a = 0$ or $a = 1,$

6.77 $\quad y'' = \frac{a}{x}y' - bx^{4-2a}e^y, \quad b = const \neq 0, \quad a = 1,$

6.83 $\quad y'' = -\frac{a(e^y-1)}{x^2}, \quad a = -2,$

6.110(111) $\quad y'' = \frac{y'^2}{y} \pm \frac{1}{y},$

6.118 $\quad y'' = \frac{y'^2}{y} - ay' - by^2 + 2ay, \quad b = const \neq 0, \quad a = 0$ or $a = -1,$

6.127 $\quad y'' = \frac{y'^2}{y} - by^2, \quad b = const \neq 0,$

6.172 $\quad y'' = \frac{y'^2}{y} - a\frac{y'}{x} - by^2, \quad b = const \neq 0, \qquad a = 0 \quad \text{or} \quad a = 1$

are equivalent to to Painlevé III equation with 3 zero parameters.

8.4. *Equations PIII(0,0,0,0), PV(0,0,0,0), PVI(0,0,0,1/2)*

For the equations $PIII(0,0,0,0)$, $PV(0,0,0,0)$, $PVI(0,0,0,1/2)$ the conditions $A = 0$ and $B = 0$ from (3.1) hold then according to Theorem 3.1 (Lie) they are equivalent to $y'' = 0$.

$$PIII(0,0,0,0): y'' = \frac{y'^2}{y} - \frac{y'}{x}, \quad PV(0,0,0,0): y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x},$$
$$PVI(0,0,0,\frac{1}{2}): y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\frac{y'^2}{2} - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)}{2x(x-1)(y-x)}.$$

8.5. *Algorithm for Painlevé III-VI equations*

Equations Painlevé III-VI are 4-parameter families, they depend on parameters a, b, c, and d. As they are in the first case of intermediate degeneration, we should use formulas (5.12), (5.13) to calculate the invariants. It is a well-known fact that $I_2 = 0$ for all Painlevé equations, and therefore only two sequences generated by I_1 and I_3 are nontrivial.

The next useful fact is that all invariants are rational functions of the variables “x” and “y”. Let us take the invariant I_1 (its formula is the simplest). First we regard the symbol I_1 as a parameter in order to convert rational function into polynomial. This polynomial depend on variables “x” and “y”, parameters a, b, c, d (they are parameters of the equation), and the new parameter I_1. For example, invariant I_1 for the Painlevé III equation is

$$I_1 = \frac{3}{5}\frac{64(c^2y^8 - 22cy^4d + d^2)x^2 - 4(cy^6a + db + 49(cy^4b + day^2))yx + a^2y^6 - 22ay^4b + b^2y^2}{(8cy^4x - 8dx - yb + ay^3)^2}$$

The associated polynomial is the following one,

$$\begin{aligned} P_1(x,y;a,b,c,d,I_1) =& 5I_1(8cy^4x - 8dx - yb + ay^3)^2 - 3\big(64(c^2y^8 - 22cy^4d + d^2)x^2 - \\ & 4(cy^6a + db + 49(cy^4b + day^2))yx + a^2y^6 - 22ay^4b + b^2y^2\big) = 0. \end{aligned}$$

We shall call “y” a “higher” variable. In the same way we construct a polynomial $P_4(x,y;a,b,c,d,I_4)$ from the formula for invariant I_4. Using Buchberger’s algorithm (see [5]), we reduce polynomials $P_1(x,y;a,b,c,d,I_1)$ and $P_4(x,y;a,b,c,d,I_4)$ with respect to ”higher” variable “y”. Aa a result we get a polynomial $Q_1(x;a,b,c,d,I_1,I_4)$ and a formula for variable $y = R_1(x;a,b,c,d,I_1,I_4)$, where R_1 is a rational function. In the same way we construct a polynomial $P_7(x,y;a,b,c,d,I_7)$ by the invariant I_7. Then we reduce it together with the polynomial $P_1(x,y;a,b,c,d,I_1)$ and get a new polynomial $Q_2(x;a,b,c,d,I_1,I_7)$.

Now we reduce polynomials $Q_1(x;a,b,c,d,I_1,I_4)$ and $Q_2(x;a,b,c,d,I_1,I_7)$ with respect to the variable “x”. We get a quantity $K(a,b,c,d,I_1,I_4,I_7)$ and a formula for variable $x = R_2(a,b,c,d,I_1,I_4,I_7)$, where R_2 is also a rational function.

Repeating this procedure as many times as necessary, we obtain a relation between the invariants $K(I_1, I_4, \dots) = 0$ that is a *necessary condition* of the equivalence as well as the formulas for the parameters a, b, c, and d and for the variables x and y via invariants. These formulas form the *sufficient conditions* and complete the solution.

The main difficulty of this method is a bulky form of these polynomials, and this is why at present the equivalence problem is successfully solved only for Painlevé IV equation, see [14]. But the final formulas are too complicated, so here we present only the necessary conditions of the equivalence.

8.5.1. *Necessary conditions for Painlevé IV equation*

Equation Painlevé IV depends on two parameters a and b,

$$PIV(a,b): \quad y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - a)y + \frac{b}{y}.$$

We introduce additional invariants using formulas (5.12), (5.13),

$$J_1 = \frac{5I_1}{72}, \qquad J_4 = \frac{I_4}{2160}, \qquad J_{10} = \frac{I_{10}}{12960}, \qquad I_{10} = \frac{\nabla_\alpha I_4}{N} = \frac{(I_4)'_x B - (I_4)'_y A}{N}.$$

Theorem 8.4. *If equation (1.1) is equivalent to Painlevé IV equation under the transformations (1.2), then the following necessary conditions hold:* $F = 0$ *in (4.1),* $A \neq 0$ *or* $B \neq 0$ *in (3.1),* $\Omega = 0$ *in (5.4), (5.5),* $M \neq 0$ *in (5.8), (5.9),* $Z \neq 0$ *in (5.31), a)* $K_0 = 0$ *in (8.3) for PIV(a,0) equation; b)* $K_n = 0$ *in (8.4) for PIV(a,b) equation,* $b \neq 0$.

$$\begin{aligned} K_0 = &4608J_1^4 - 3248J_1^3 + 808J_1^2 + 48000J_4J_1^2 - \\ &- 16500J_4J_1 - 83J_1 + 1125J_4 + 125000J_4^2 + 3, \end{aligned} \tag{8.3}$$

$$\begin{aligned} K_n = &2^{22}3^9J_1^9 - 2^{18}3^4 7229J_1^8 + 2^{14}3^2(20412 \cdot 10^3 J_4 + 795377)J_1^7 + \\ &+ 2^{10} \cdot 3 \cdot 5(11664000J_{10} - 293875200J_4 - 3170041)J_1^6 + \\ &+ 2^9 \cdot 3 \cdot 5(47628 \cdot 10^5 J_4^2 + 347502500J_4 - 33816 \cdot 10^3 J_{10} + 1574799)J_1^5 + \\ &+ 2^8(550148750J_{10} + 1701 \cdot 10^7 J_{10}J_4 - 15275925 \cdot 10^4 J_4^2 - 31879206254 - \\ &- 7217838)J_1^4 + 2^5(5312667 + 437746 \cdot 10^4 J_4 + 405 \cdot 10^7 J_{10}^2 + 46305 \cdot 10^8 J_4^3 + \\ &+ 479194 \cdot 10^6 J_4^2 - 1168733750J_{10} - 129705 \cdot 10^6 J_{10}4)J_1^3 + 2^2(-2157057 - \\ &- 337746700J_4 + 12948575 \cdot 10^2 J_{10} + 6615 \cdot 10^9 J_{10}J_4^2 + 33184 \cdot 10^7 J_{10}J_4 - \\ &- 697457 \cdot 10^6 J_4^2 - 219765 \cdot 10^8 J_4^3 - 23075 \cdot 10^6 J_{10}^2)J_1^2 + 2^2 \cdot 5(9675 \cdot 10^5 J_{10}^2 + \\ &- 17852625J_{10} + 33823650J_4 - 847425 \cdot 10^4 J_{10}J_4 - 615125 \cdot 10^6 J_{10}J_4^2 + \\ &+ 8080625 \cdot 10^5 J_4^3 + 11864525 \cdot 10^3 J_4^2 + 9261 + 7875 \cdot 10^7 J_{10}^2 J_4)J_1 + \\ &+ 5^2(15435J_{10} - 21609J_4 - 12027400J_4^2 - 16033 \cdot 10^5 J_4^3 + 11606 \cdot 10^3 J_{10}J_4 + \\ &+ 5 \cdot 10^7 J_{10}^3 + 343 \cdot 10^8 J_4^4 + 20875 \cdot 10^5 J_{10}J_4^2 - 58 \cdot 10^7 J_{10}^2 J_4 - 175 \cdot 10^4 J_{10}^2). \end{aligned} \tag{8.4}$$

Example. The equation No. 34 from the book [9] named ”Painlevé 34” equation

$$XXXIV. \qquad y'' = \frac{y'^2}{2y} + 4ay^2 - xy - \frac{1}{2y}, \quad a = const \neq 0$$

is not equivalent to Painlevé IV equation, although $K_n = 0$ from (8.4). Let us check the conditions of Theorem 8.4,

$$A = 6a - \frac{3}{2y^3}, \quad B = 0, \quad F = 0, \quad \Omega = 0, \quad M = \frac{9a(35+4ay^3)}{10y^5}, \quad Z = 0, \quad K_n = 0.$$

9. Cases of Painlevé equation with non-trivial algebra of point symmetries

In a general situation Painlevé equations have the trivial algebra of point symmetries. But in some special cases the dimension of the point symmetries algebra is 8, 2, or 1.

1. For equations $PIII(0,0,0,0)$, $PV(0,0,0,0)$, $PVI(0,0,0,1/2)$ we have $dim(\mathscr{L}) = 8$.
2. For equation $PIII$ with 3 zero parameters we have $dim(\mathscr{L}) = 2$, and the operators are

$$PIII(0,b,0,0): \; y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{b}{x}, \quad X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \; X_2 = x\ln x\frac{\partial}{\partial x} + y(\ln x + 2)\frac{\partial}{\partial y}.$$

3. For equation $PIII(0,b,0,d)$ or $PIII(-b,0,-d,0)$ (they are equivalent under the transformations $x = \tilde{x}$, $y = 1/\tilde{y}$, see [8]) and equation PV(a,b,0,0) we have $dim(\mathscr{L}) = 1$.

$$PIII(0,b,0,d): \quad y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{b}{x} + \frac{d}{y}, \qquad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$$PV(a,b,0,0): \qquad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(ay + \frac{b}{y}\right), \quad X = x\frac{\partial}{\partial x}.$$

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References

[1] M.V.Babich and L.A.Bordag, Projective Differential Geometrical Structure ot the Painlevé Equations, *Journal of Diff. Equations* **157**(2) (1999), 452–485.

[2] C.Bandle and L.A.Bordag, Equivalence classes for Emden equations, *Nonlinear Analysis* **50** (2002), 523–540.

[3] L.A.Bordag and V.S.Dryuma, Investigation of dynamical systems using tools of the theory of invariants and projective geometry, *NTZ-Preprnt 24/95*, (Leipzig, 1995).

[4] E.Cartan, Sur les varietes a connection projective, *Bull. de Soc. Math. de Fr.* **52** (1924), 205-241.

[5] D.Cox, J.Little, and D.O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra* (Springer, 1997).

[6] V.V.Dmitrieva, R.A.Sharipov, On the point transformations for the second order differential equations, *Electronic archive at LANL* (solv-int #9703003, 1997), 1–14.

[7] C.Grissom, G.Thompson and G.Wilkens, Linearisation of Second Order Ordinary Differential Equations via Cartan's Equivalence Method, *Diff. Equations* **77** (1989), 1–15.

[8] J.Hietarinta and V.Dryuma, Is my ODE is Painlevé equation in disguise? *Journal of Nonlin. Math. Phys.* **9**(1) (2002), 67–74.
[9] E.L.Ince, *Ordinary differential equations* (Courier Dover Publications, 1956).
[10] E.Kamke, *Handbook of ordinary differential equations* (Moscow, Nauka, 1976).
[11] N.Kamran, K.G.Lamb & W.F.Shadwick, The local equivalence problem for $d^2y/dx^2 = F(x,y,dy/dx)$ and the Painlevé transcendents, *J. Diff. geometry* **22** (1985), 139–150.
[12] V.V.Kartak, Explicit solution of the equivalence problem for certain Painlevé equations, *Ufimskii Math. Journal* **1**(3) (2009), 46–56 (in Russian).
[13] V.V.Kartak, Equivalence classes of the second order ODEs with the constant Cartan invariant, *Journal of Nonlin. Math. Phys.* **18**(4) (2011), 613–640.
[14] V.V.Kartak, Solution of the equivalence problem for the Painlevé IV equation, *Theor. and Math. Phys.*, **173**(2) (2012), 1541-1564.
[15] S.Lie *Theorie der Transformationsgruppen III* (Teubner Verlag, Leipzig, 1930).
[16] R.Liouville, Sur les invariants de certaines equations differentielles et sur leurs applications, *J. de L'Ecole Polytechnique* **59** (1889), 7–76.
[17] Yu.R.Romanovskii, Computation of local symmetries of second-order ordinary differential equations by the Cartan equivalence method, *Math. Notes* **60**(1), 56-67.
[18] G.Thomsen, Über die topologischen Invarianten der Differentialgleichung $y'' = f(x,y)y'^3 + g(x,y)y'^2 + h(x,y)y' + k(x,y)$, *Abhandlungen aus dem math. Seminar der Hamburgischen Univ.* **7** (1930), 301–328.
[19] A.Tresse, Sur les invariants differenties des groupes continus de transformations, *Acta Math.* **18** (1894), 1–88.
[20] A.Tresse, Determination des Invariants ponctuels de l"Equation differentielle ordinaire de second ordre: $y'' = w(x,y,y')$, *Preisschriften der fr̈stlichen Jablonowski'schen Gesellschaft XXXII* (S.Hirzel, Leipzig, 1896).
[21] R.A.Sharipov, On the point transformations for the equation $y'' = P + 3Qy' + 3Ry'^2 + Sy'^3$, *Electronic archive at LANL* (solv-int #9706003, 1997), 1–35.
[22] R.A.Sharipov, Effective procedure of point classification for the equations $y'' = P + 3Qy' + 3Ry'^2 + Sy'^3$, *Electronic archive at LANL* (MathḊG #9802027, 1998), 1–35.