On the Recurrence Coefficients for Generalized q-Laguerre Polynomials

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In this paper we consider a semi-classical variation of the weight related to the q-Laguerre polynomials and study their recurrence coefficients. In particular, we obtain a second degree second order discrete equation which in particular cases can be reduced to either the qP_V or the qP_{III} equation.

Keywords: Orthogonal polynomials; discrete Painlevé equations.

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1. Introduction

1.1. *Orthogonal polynomials*

One of the most important properties of orthogonal polynomials is the three-term recurrence relation [6, 13]. For a sequence $(P_n)_{n\geq 0}$ of monic polynomials (of degree n in x) orthogonal with respect to a positive measure μ with support on the real line

$$\int P_n(x)P_m(x)d\mu(x) = \zeta_n\delta_{n,m}, \quad \zeta_n > 0, \quad n,m = 0,1,2,\ldots, \tag{1.1}$$

where $\delta_{n,m}$ is the Kronecker delta, this relation takes the following form:

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x) \tag{1.2}$$

with the recurrence coefficients given by the following integrals

$$\alpha_n = \frac{1}{\zeta_n}\int xP_n^2(x)\,d\mu(x), \qquad \beta_n = \frac{1}{\zeta_{n-1}}\int xP_n(x)P_{n-1}(x)\,d\mu(x). \tag{1.3}$$

The initial conditions are taken to be $\beta_0 P_{-1} = 0$ and $P_0 = 1$. In this paper we use the following expression for monic polynomials:

$$P_n(x) = x^n + \gamma_n x^{n-1} + \ldots. \tag{1.4}$$

In this paper we will work with a weight function w on the positive half line, so that the orthogonality conditions can be written as

$$\int_0^\infty P_n(x)P_m(x)w(x)dx = \zeta_n\delta_{n,m}. \tag{1.5}$$

The recurrence coefficients can be expressed in terms of determinants containing the moments of the orthogonality measure [6]. For classical orthogonal polynomials (Hermite, Laguerre, Jacobi) one knows these recurrence coefficients explicitly in contrast to non-classical weights. The recurrence coefficients of semi-classical weights obey nonlinear recurrence relations, which, in many cases, can be identified as discrete Painlevé equations; see [4] and the references therein. The Painlevé equations (both discrete and continuous) have many important applications in mathematics and mathematical physics; cf. [7, 12, 16]. Relations between the semi-classical orthogonal polynomials and the continuous Painlevé equations can be found in, for instance, [2, 9, 11, 15]. Relations between the semi-classical orthogonal polynomials and the discrete Painlevé equations can be found in, for instance, [1, 4, 14, 17] and the references therein.

In this paper we consider the recurrence coefficients of the generalized q-Laguerre polynomials for the weight function

$$w(x) = \frac{x^\alpha(-p_1/x;q)_\infty(-p_2/x;q)_\infty}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty}, \tag{1.6}$$

where $x \in (0,+\infty)$, $|q| < 1$, $p_1 > 0$, $p_2 > 0$, $p_1p_2 < q^{2-\alpha}$, $\alpha \geq 0$. Here

$$(a;q)_\infty = \prod_{k=0}^\infty(1-aq^k).$$

The case $p_1 + p_2 = 0$, $p_1p_2 = -p$ was considered in Section 7.2 [1]. It was shown that the recurrence coefficients are related to the q-discrete Painlevé equation qP_V. A particular case when $p = q^2$ was considered in [4]. The proof was based on the compatibility of the ladder operators for orthogonal polynomials [5].

1.2. *Ladder operators*

In the case of continuous q-orthogonal polynomials on the positive half line, the ladder operators were first considered in [5]. We repeat the main statements which we use later on to be self-contained following [5] and Section 1.3 [1].

The q-difference operator is given by

$$(D_qf)(x) = \begin{cases} \frac{f(x)-f(qx)}{x(1-q)}, & x \neq 0, \\ f'(0) & x = 0. \end{cases}$$

Let us define the function u, called the potential, by the following formula:

$$u(x) = -\frac{D_{q^{-1}}w(x)}{w(x)}. \tag{1.7}$$

Then the polynomials satisfy the following lowering equation:

$$D_qP_n(x) = A_n(x)P_{n-1}(x) - B_n(x)P_n(x),$$

where the functions $A_n(x)$ and $B_n(x)$ are given by

$$A_n(x) = \frac{1}{\zeta_n} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_n(y/q) w(y) dy, \tag{1.8}$$

$$B_n(x) = \frac{1}{\zeta_{n-1}} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q) w(y) dy. \tag{1.9}$$

Furthermore, the following relations (compatibility conditions) hold:

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n) A_n(x) + x(q-1) \sum_{j=0}^n A_j(x) - u(qx), \tag{1.10}$$

$$1 + (x - \alpha_n) B_{n+1}(x) - (qx - \alpha_n) B_n(x) = \beta_{n+1} A_{n+1}(x) - \beta_n A_{n-1}(x). \tag{1.11}$$

In addition, we have the following identities:

$$\int_0^\infty u(y) P_n(y) P_n(y/q) w(y) dy = 0$$

and

$$\int_0^\infty u(y) P_{n+1}(y) P_n(y/q) w(y) dy = \frac{1 - q^{n+1}}{1 - q} q \zeta_n.$$

They are due to orthogonality and the integration by parts formula

$$\int_0^\infty f(x) D_q g(x) dx = -\frac{1}{q} \int_0^\infty g(x) D_{q^{-1}} f(x) dx,$$

where the good behaviour of the weight near zero and infinity guarantees that this formula can be applied.

2. Main Result

In this section we prove the main result for the recurrence coefficients of monic orthogonal polynomials with respect to the weight (1.6).

Theorem 2.1. *The recurrence coefficients α_n and β_n in the three-term recurrence relation for monic polynomials*

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

for the weight

$$w(x) = \frac{x^\alpha (-p_1/x; q)_\infty (-p_2/x; q)_\infty}{(-x^2; q^2)_\infty (-q^2/x^2; q^2)_\infty}$$

with $x \in (0, +\infty)$, $|q| < 1$, $p_1 > 0$, $p_2 > 0$, $p_1 p_2 < q^{2-\alpha}$, $\alpha \geq 0$, can be expressed in terms of the function y_n, which satisfies the second order second degree discrete equation

$$(c_n^2 - b_n b_{n-1})^2 - a_n a_{n-1} c_n (c_n^2 + b_n b_{n-1}) - c_n^2 (a_n^2 b_{n-1} + a_{n-1}^2 b_n) = 0,$$

with

$$a_n = q^{n-1}(p_1 + p_2),$$
$$b_n = q^{2n+\alpha}(y_{n+1}y_n - p_1 p_2 q^{-\alpha-2}),$$
$$c_n = q^{n-1}\frac{(y_n + q^{-\alpha})(y_n + p_1 p_2 q^{-2})}{y_n + q^{-n-\alpha}},$$

where

$$\beta_n = q^{1-n}(y_n + q^{-n-\alpha})$$

and

$$q^{2\alpha+2n+2}(q^{\alpha+2}y_n y_{n+1} - p_1 p_2)\alpha_n^2 + (p_1 + p_2)q^{\alpha+n+1}(q^2 + q^\alpha(p_1 p_2 + q^2(y_n + y_{n+1})))\alpha_n$$
$$+ (q^2 + q^\alpha(p_1 p_2 + q^2(y_n + y_{n+1})))^2 = 0.$$

Proof. The method applied in the proof is based on the method used in Th. 5.3.2 [1] and earlier in [4].

Using the definition of the weight function, it is easy to find the following relation:

$$w(x/q) = \frac{q^{2-\alpha}}{(x+p_1)(x+p_2)}w(x).$$

The potential (1.7) is given by

$$u(x) = \frac{q}{1-q}\frac{1}{x} - \frac{q^{3-\alpha}}{1-q}\frac{1}{x(x+p_1)(x+p_2)}.$$

Next one can compute

$$\frac{u(qx)-u(y)}{qx-y} = -\frac{1}{qx}u(y) + \frac{q^{2-\alpha}}{1-q}\frac{1}{(y+p_1)(y+p_2)}\left(\frac{q}{(qx+p_1)(qx+p_2)} + \frac{p_1+p_2+y}{x(qx+p_1)(qx+p_2)}\right).$$

The functions $A_n(x)$ and $B_n(x)$ in, respectively, Eq. (1.8) and Eq. (1.9) can be found as follows. We have

$$A_n(x) = \frac{q^2}{1-q}\frac{T_n}{x(qx+p_1)(qx+p_2)} + \frac{q^{n+2}}{1-q}\frac{1}{(qx+p_1)(qx+p_2)},$$

where

$$T_n = q^{n-1}(p_1 + p_2 + \gamma_n - q\gamma_{n+1}) \tag{2.1}$$

since $P_n(qu)u = q^n P_{n+1}(u) + q^{n-1}(\gamma_n - q\gamma_{n+1})P_n(u) +$ lower order terms. Thus,

$$\sum_{j=0}^{n} T_j = \frac{(p_1 + p_2)}{q}\frac{1-q^{n+1}}{1-q} - q^n \gamma_{n+1}. \tag{2.2}$$

Similarly,

$$B_n(x) = -\frac{1}{x}\frac{1-q^n}{1-q} + \frac{q^2}{1-q}\frac{r_n}{(qx+p_1)(qx+p_2)} + \frac{q^2}{1-q}\frac{t_n}{x(qx+p_1)(qx+p_2)}$$

with

$$r_n = \frac{1}{\zeta_{n-1}} \int_0^\infty P_n(qu)P_{n-1}(u)w(u)du,$$

$$t_n = \frac{p_1 + p_2}{q\zeta_{n-1}} \int_0^\infty P_n(qu)P_{n-1}(u)w(u)du + \frac{1}{\zeta_{n-1}} \int_0^\infty uP_n(qu)P_{n-1}(u)w(u)du.$$

Clearly, $r_0 = t_0 = 0$. One can further calculate r_n. Indeed, since $P_n(qu) = q^n P_n(u) + (1 - q)\gamma_n q^{n-1} P_{n-1}(u) +$ lower order terms, and using orthogonality we have

$$r_n = (1 - q)\gamma_n q^{n-1}. \tag{2.3}$$

The compatibility conditions (1.10) and (1.11) give the following relations:

$$t_{n+1} + t_n + p_1 p_2 q^{-2}(q^{n+1} + q^n - 1) = q^{-\alpha} - \alpha_n T_n, \tag{2.4}$$

$$r_{n+1} + r_n = -\alpha_n q^n + T_n - (1 - q)\sum_{j=0}^n T_j - (q^n + q^{n-1} - q^{-1})(p_1 + p_2), \tag{2.5}$$

$$r_{n+1} - qr_n + q^n(1 - q)\alpha_n = 0, \tag{2.6}$$

$$\beta_{n+1} T_{n+1} - \beta_n T_{n-1} = -\alpha_n(t_{n+1} - t_n) + \alpha_n q^n (1 - q)p_1 p_2 q^{-2}, \tag{2.7}$$

$$t_{n+1} - qt_n - \alpha_n r_{n+1} + \alpha_n r^n = \beta_{n+1} q^{n+1} - \beta_n q^{n-1} - q^{n-1}(1 - q)(p_1 + p_2)\alpha_n. \tag{2.8}$$

Using Eq. (2.1) and the fact that $\alpha_n = \gamma_n - \gamma_{n+1}$ which can be easily seen from the three-term recurrence relation comparing the coefficients at x^n, Eq. (2.6) is trivial (i.e., the equation is automatically satisfied). Similarly, using Eq. (2.1), Eq. (2.3) and Eq. (2.2), Eq. (2.5) is also trivial. Further, using Eq. (2.1) and Eq. (2.3) we have

$$r_{n+1} - r_n = (1 - q)(\gamma_{n+1} q^n - \gamma_n q^{n-1}) = (1 - q)((p_1 + p_2)q^{n-1} - T_n).$$

Inserting this into Eq. (2.8) we obtain

$$t_{n+1} - qt_n + (1 - q)\alpha_n T_n = \beta_{n+1} q^{n+1} - \beta_n q^{n-1}. \tag{2.9}$$

Using Eq. (2.4) for replacement of $\alpha_n T_n$ in the last expression and multiplying both sides by q^n we obtain

$$q^{n-\alpha}(1 - q) + q^{n+1}t_{n+1} - q^n t_n + (1 - q)q^{-2}p_1 p_2(q^n - q^{2n}(1 + q)) = \beta_{n+1} q^{2n+1} - \beta_n q^{2n-1}.$$

Taking a telescopic sum (i.e., summing the last expression from 0 to $n - 1$), we get

$$\beta_n q^{2n-1} = (1 - q^n)(q^{-\alpha} + p_1 p_2 q^{-2}) - (1 - q^{2n})p_1 p_2 q^{-2} + q^n t_n. \tag{2.10}$$

Thus, β_n is expressed in terms of t_n.

Next, we multiply Eq. (2.7) by T_n. The expression $\alpha_n T_n$ is replaced by using Eq. (2.4) for the term $t_{n+1} - t_n$ and by using Eq. (2.9) in another case. We also use Eq. (2.10) on the right hand side to replace β_n. As a result we have

$$\beta_{n+1} T_{n+1} T_n - \beta_n T_n T_{n-1} = t_{n+1}^2 - t_n^2 - (t_{n+1} - t_n)(q^{-\alpha} + p_1 p_2 q^{-2}) + 2 p_1 p_2 q^{n-1} t_{n+1}$$
$$-2 p_1 p_2 q^{n-2} t_n + q^{n-2}(1-q)(q^{-\alpha} + p_1 p_2 q^{-2}) p_1 p_2 - q^{2n-4}(1-q^2) p_1^2 p_2^2.$$

Taking the telescopic sum and taking into account that $t_0 = \beta_0 = 0$, we get

$$\beta_n T_n T_{n-1} = t_n^2 - t_n(q^{-\alpha} + p_1 p_2 q^{-2}) + 2 p_1 p_2 q^{n-2} t_n \tag{2.11}$$
$$+ (1-q^n)(q^{-\alpha} + p_1 p_2 q^{-2}) p_1 p_2 q^{-2} - (1-q^{2n}) p_1^2 p_2^2 q^{-4}.$$

Replacing t_n and t_{n+1} in Eq. (2.7) using Eq. (2.10) yields

$$\beta_{n+1}(T_{n+1} + q^n \alpha_n) - \beta_n(T_{n-1} + q^{n-1} \alpha_n) = \alpha_n(1-q) q^{-n-\alpha-1}.$$

Replacing $\alpha_n = \gamma_n - \gamma_{n+1}$ and T_n by Eq. (2.1) on the left hand side, we get

$$\beta_{n+1}(p_1 + p_2 + \gamma_n - q\gamma_{n+2}) q^{2n+1}$$
$$-\beta_n(p_1 + p_2 + \gamma_{n-1} - q\gamma_{n+1}) q^{2n-1} = \alpha_n(1-q) q^{-\alpha}.$$

Taking a telescopic sum, we obtain

$$q^{2n+\alpha-1} \beta_n(p_1 + p_2 + \gamma_{n-1} - q\gamma_{n+1}) = (q-1)\gamma_n. \tag{2.12}$$

Using Eq. (2.1) and replacing $\gamma_{n+1} = \gamma_n - \alpha_n$ we get

$$q^n \alpha_n = T_n - q^{n-1}(p_1 + p_2 + (1-q)\gamma_n).$$

Next we replace γ_n using Eq. (2.12) and again use $\gamma_{n+1} = \gamma_n - \alpha_n$. We also express γ_n in terms of T_{n-1} and γ_{n-1} from Eq. (2.1). Finally we get

$$\alpha_n q^n(1 - q^{2n+\alpha-1} \beta_n) = T_n + q^{2n+\alpha} \beta_n T_{n-1} - q^{n-1}(p_1 + p_2). \tag{2.13}$$

Next we multiply Eq. (2.13) by T_n and substitute $\alpha_n T_n$ into Eq. (2.4). As a result we get

$$q^n(1 - q^{2n+\alpha-1} \beta_n)(q^{-\alpha} - t_{n+1} - t_n - (q^{n+1} + q^n - 1) p_1 p_2 q^{-2}) =$$
$$T_n^2 + q^{2n+\alpha} \beta_n T_n T_{n-1} - q^{n-1}(p_1 + p_2) T_n.$$

In the last expression we use Eq. (2.11) to replace $\beta_n T_n T_{n-1}$ and Eq. (2.10) to replace β_n. We also introduce a new variable

$$y_n = t_n - q^{-\alpha} - (1-q^n) p_1 p_2 q^{-2}.$$

Finally, we get

$$T_n(T_n - q^{n-1}(p_1 + p_2)) = q^{2n+\alpha}(y_{n+1} y_n - p_1 p_2 q^{-\alpha-2}). \tag{2.14}$$

Eq. (2.11) can be written in new variables as

$$q^{1-n}(y_n + q^{-n-\alpha}) T_n T_{n-1} = (y_n + q^{-\alpha})(y_n + p_1 p_2 q^{-2}) \tag{2.15}$$

since from Eq. (2.10) we have

$$\beta_n = q^{1-n}(y_n + q^{-n-\alpha}). \tag{2.16}$$

Eq. (2.14) and Eq. (2.15) can be used to eliminate T_n and T_{n-1} to get a second order second degree nonlinear discrete equation for y_n (here we mean that the equation involves y_{n-1}, y_n, y_{n+1} and is of second degree in y_{n+1}) as follows: to eliminate T_n one can compute the resultant for Eq. (2.14) and Eq. (2.15). If we first rewrite these two equations as

$$T_n^2 - a_n T_n - b_n = 0, \tag{2.17}$$
$$T_{n-1}T_n - c_n = 0, \tag{2.18}$$

with

$$a_n = q^{n-1}(p_1 + p_2),$$
$$b_n = q^{2n+\alpha}(y_{n+1}y_n - p_1 p_2 q^{-\alpha-2}),$$
$$c_n = q^{n-1}\frac{(y_n + q^{-\alpha})(y_n + p_1 p_2 q^{-2})}{y_n + q^{-n-\alpha}},$$

this leads to

$$b_n T_{n-1}^2 + a_n c_n T_{n-1} - c_n^2 = 0. \tag{2.19}$$

Next, using this expression and Eq. (2.17) with $n-1$, one eliminates T_{n-1}, again by computing the resultant. This gives

$$(c_n^2 - b_n b_{n-1})^2 - a_n a_{n-1} c_n (c_n^2 + b_n b_{n-1}) - c_n^2 (a_n^2 b_{n-1} + a_{n-1}^2 b_n) = 0. \tag{2.20}$$

Substituting the expression of β_n in terms of y_n into Eq. (2.13) and replacing T_{n-1} from Eq. (2.15), we get an expression for α_n, y_n and T_n. Eliminating T_n between this expression and Eq. (2.14), we get a quadratic expression for α_n in terms of y_n and y_{n+1}:

$$q^{2\alpha+2n+2}(q^{\alpha+2}y_n y_{n+1} - p_1 p_2)\alpha_n^2 + (p_1 + p_2)q^{\alpha+n+1}(q^2 + q^{\alpha}(p_1 p_2 + q^2(y_n + y_{n+1})))\alpha_n$$
$$+ (q^2 + q^{\alpha}(p_1 p_2 + q^2(y_n + y_{n+1})))^2 = 0. \tag{2.21}$$

This completes the proof of the theorem.

$\square$

In particular, formally if $p_1 + p_2 = 0$ and $p = -p_1^2$, all a_i are zero and we obtain $c_n^2 = b_n b_{n-1}$, or in terms of y_n:

$$(y_n y_{n-1} - pq^{-\alpha-2})(y_n y_{n+1} - pq^{-\alpha-2}) = \frac{(y_n + q^{-\alpha})^2(y_n + pq^{-2})^2}{(q^{\alpha+n}y_n + 1)^2}.$$

This case was considered in Section 7.2 [1] and was shown to be a particular case of qP_V after some change of variables. If we take $p_1 = p_2 = 0$, i.e. a special case of the previous one, we get the equation

$$y_{n-1}y_{n+1} = \frac{(y_n + q^{-\alpha})^2}{(q^{n+\alpha}y_n + 1)^2}.$$

This is the *q*-discrete Painlevé equation qP_{III}, obtained in Section 7.1 [1].

The initial conditions for this recurrence relation can be expressed in terms of the moments of the weight. With

$$\mu_k = \int_0^\infty x^k w(x)\,dx,$$

we have from Eq. (2.16) and $\beta_0 = 0$ that $y_0 = -q^{-\alpha}$, and from Eq. (2.21) and $\alpha_0 = \frac{\mu_1}{\mu_0}$ we deduce that

$$y_1 = -\frac{\mu_1^2}{\mu_0^2} - \frac{p_1+p_2}{q}\frac{\mu_1}{\mu_0} - \frac{p_1 p_2}{q^2}.$$

However, it should be noted that the recurrence relation (2.20) is quadratic in y_{n+1}, so when one tries to find y_{n+1} out of y_{n-1} and y_n, one still has a sign choice to make. It is clear that $\beta_n > 0$ for $n > 0$, but this additional information does not seem to allow one to pick the right sign. This is a complication that was not present in [1]. A similar remark holds when one tries to find α_n from y_n, using Eq. (2.21). Moreover, the method of using a Painlevé equation to calculate recurrence coefficients is usually an unstable one, see e.g. Sections 3.3 and 6.5 in [1], or [3].

Remark 2.1. [a] Eq. (2.20) can be further simplified and written as follows:

$$(c_n^2 - b_n b_{n-1})^2 = c_n(a_{n-1}c_n + a_n b_{n-1})(a_n c_n + a_{n-1}b_n).$$

By setting

$$a = p_1 + p_2, \quad B_n = q^{n+\alpha}(y_{n+1}y_n - p_1 p_2 q^{-\alpha-2}), \quad C_n = \frac{(y_n + q^{-\alpha})(y_n + p_1 p_2 q^{-2})}{y_n + q^{-n-\alpha}}, \qquad (2.22)$$

Eq. (2.20) simplifies further to

$$(C_n^2 - q B_n B_{n-1})^2 = a^2 q^{n-2} C_n(C_n + q B_{n-1})(C_n + B_n). \qquad (2.23)$$

However, the expressions of α_n and β_n in terms of B_n and C_n are cumbersome.

3. Discussions

There are a lot of examples which show a connection between the recurrence coefficients of the orthogonal polynomials and the Painlevé equations, both discrete and continuous. A natural question arises what happens if one modifies further such weights by introducing more parameters. In this paper we studied the case of weight (1.6) and show that the recurrence coefficients satisfy a complicated second order second degree discrete equation, which in particular cases can be reduced to qP_V or qP_{III}. In a recent preprint [8] we showed that one can simultaneously consider a large class of weights giving potential (1.7) of a certain form and derive second order second degree discrete equations for the recurrence coefficients.

A lot of other instances of orthogonal polynomials giving rise to discrete or q-discrete Painlevé equations are known. We give a few examples of the weight, the corresponding Painlevé equation and a reference.

- $w(x) = |x|^\rho e^{-x^4}$ on $\mathbb{R}$ is related to dP_I [10, 11, 14];

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[a]The authors are grateful to the referee for suggesting the expressions in this remark.

- $w(k) = \frac{a^k}{(k!)^2}$ on $\mathbb{N}$ is related to dP_{II} [18];
- $w(x) = x^\alpha e^{-x^2}$ on $\mathbb{R}^+$ is related to dP_{IV} [4];
- $w(x) = (q^4 x^4; q^4)_\infty$ on $\{\pm q^k | k \in \mathbb{N}\}$ is related to qP_I [17];
- $w(x) = |x|^\alpha (q^2 x^2; q^2)_\infty (cq^2 x^2; q^2)_\infty$ on $\{\pm q^k | k \in \mathbb{N}\}$ is related to $\alpha q P_V$ [1].

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