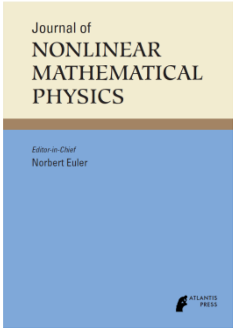

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SOLUTIONS AND LAX PAIRS BASED ON BILINEAR BÄCKLUND TRANSFORMATIONS OF SOME SUPERSYMMETRIC EQUATIONS

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The paper investigates solutions and Lax pairs through bilinear Bäcklund transformations for some supersymmetric equations. We derive variety of solutions from the known bilinear Bäcklund transformations. Besides, using the gauge invariance of (super) Hirota bilinear derivatives we may get deformed bilinear Bäcklund transformations and consequently deformed Lax pairs with fermionic parameters. Examples are the $\mathcal{N}=1$ supersymmetric KdV equation and the $\mathcal{N}=2$ supersymmetric KdV_1 equation.

Keywords: Supersymmetric equations; bilinear Bäcklund transformations; multiple-pole solutions; Lax pairs.

Mathematics Subject Classification: 35Q51, 35Q58, 35Q53

1. Introduction

Supersymmetry is a symmetry that makes particles of different spins and statistical properties link together, or it is a symmetry that makes fermions and bosons transform to each other, moreover, it is a new type of symmetry that makes space-time symmetry and internal symmetry combine together. In 1971, the term of supersymmetry [10] was first used in Gol'fand and Linkman's work. In recent decades, supersymmetry was widely applied in various fields, such as: relativity, non-relativity, nuclear physics, quantum field theory, superstring and so on. Mathematicians and theoretical physicists have aroused great interest in supersymmetry. Supersymmetry reflected in mathematics is based on the introduction of Grassmann variables. Meanwhile, in the field theory it is not only a Grassmann odd field but also an equation of motion.

For the integrability of supersymmetric (SUSY) systems, in 1984 Kupershmidt [17] introduced an integrable super KdV equation through the fermionic extension of the KdV equation. Later, Manin and Radul [26] obtained the SUSY KdV equation. From then on many standard integrable equations and their integrable characteristics have been extended to their SUSY counterparts, see, for example, [20, 21, 24, 25, 28].

With regard to solutions, in 1978 Chaichian and Kulish [5] considered the inverse scattering problem and Bäcklund transformations (BTs) of the SUSY Liouville equation and the SUSY sine-Gordon equation. One of powerful solving tool is bilinear method. In 1993 McArthur and Yung [27] first introduced a super Hirota bilinear derivative and gave explicit bilinear forms of equations in the SUSY KdV hierarchy. In 2000 under the gauge-invariance principle [12] Carstea redefine the super Hirota derivatives and obtained bilinear forms for more SUSY equations, from which one can derived multi-super-soliton solutions [3, 4, 16]. In 2005 Liu and Hu [22] constructed the bilinear BT for the SUSY KdV equation from which they got a nonlinear superposition formula and a new Lax representation for the SUSY KdV equation. There are other interesting progress related bilinear method, such as quasi-periodic solutions [9] of SUSY equations and applications of super Bell polynomials [7, 8], and so on.

Using bilinear equations one can construct bilinear BTs [14]. Usually there is an arbitrary parameter in a bilinear BT and multi-soliton solutions can be derived step by step by taking different values for the parameter in each step. For the classical integrable systems, their bilinear BTs can be deformed. This is based on gauge invariance of Hirota bilinear derivatives. The deformation brings more freedom for choosing parameter values. As a result, many "new" solutions (which are different from solitons and in fact limit-solutions or multiple-pole solutions (cf. [31, 32])) can be derived [1, 6, 30].

In the paper, we will consider bilinear BTs and the related solutions and Lax pairs for some SUSY equations. Since the super Hirota bilinear derivatives also admit gauge invariance, one can deform bilinear BTs for equations. Surprisingly, different from what we mentioned previously for ordinary integrable systems, one can get variety of solutions but deformations of BTs are not necessary. However, the deformations do bring something new. A suitable deformation will bring a bilinear BT with a fermionic type parameter, which leads to a Lax pair with a fermionic type "spectral" parameter. We will also investigate bilinear BTs for the $\mathcal{N}=2$ SUSY KdV_1 equation.

The paper is organized as follows. In Sec. 2, we introduce some notations and properties on the ordinary and super Hirota bilinear derivatives. This section also includes the deformed BT and related solutions for the KdV equation. Then, in Sec. 3, we give detail investigation on the bilinear BT of $\mathcal{N}=1$ SUSY KdV equation and the related solutions, deformations and Lax pairs. Section 4 is for the $\mathcal{N}=2$ SUSY KdV_1 equation.

2. Preliminary

2.1. *Notations and properties*

For two sufficiently differential ordinary functions $a=a(x,t)$ and $b=b(x,t)$, their Hirota bilinear derivative is defined as [15]:

$$D_x^m D_t^n a\cdot b=(\partial_x-\partial_{x'})^m(\partial_t-\partial_{t'})^n a(x,t)b(x',t')|_{x=x',t=t'}. \tag{2.1}$$

It can be proved that

$$\mathcal{F}(D_x,D_t)(e^{\zeta_1'}a)\cdot(e^{\zeta_2'}b)=e^{\zeta_1'+\zeta_2'}\mathcal{F}(D_x+(p_1'-p_2'),D_t+(q_1'-q_2'))a\cdot b, \tag{2.2}$$

where $\mathcal{F}(D_x, D_t)$ is a polynomial of the operators D_x and D_t, and ζ_i' are linear functions defined as

$$\zeta_i' = p_i' x + q_i' t + \zeta_i^{(0)'}, \quad i = 1, 2. \tag{2.3}$$

When $\zeta_1' = \zeta_2'$, the equality (2.2) coincides with the gauge invariance of the Hirota bilinear equations [12].

For the Grassman-valued functions

$$f \doteq f(x, t, \theta), \quad g \doteq g(x, t, \theta),$$

where θ is a Grassmann variable denoting fermionic counterpart of the spatial variable x, one can define a super Hirota derivative as [3]:

$$S_x^N f \cdot g = \sum_{i=0}^{N} (-1)^{i|f| + \frac{i(i+1)}{2}} \begin{bmatrix} N \\ i \end{bmatrix} (\mathcal{D}^{N-i} f(x, t, \theta))(\mathcal{D}^i g(x, t, \theta)), \tag{2.4}$$

where $\mathcal{D}$ is the super covariant derivative

$$\mathcal{D} \doteq \mathcal{D}_\theta = \partial_\theta + \theta \partial_x,$$

$|f|$ is the Grassmann parity of the function f defined as

$$|f| = \begin{cases} 1, & f \text{ is odd (fermionic)}, \\ 0, & f \text{ is even (bosonic)}, \end{cases}$$

$\left[\begin{smallmatrix} N \\ i \end{smallmatrix}\right]$ is the superbinomial coefficient [26] defined as

$$\begin{bmatrix} N \\ i \end{bmatrix} = \begin{cases} 0, & (N, i) \equiv (0, 1) \mod 2, \\ C_{[N/2]}^{[i/2]} = \dfrac{[N/2]!}{([N/2] - [i/2])! [i/2]!}, & \text{others}, \end{cases}$$

and $[k]$ is the integer part of the real number k, i.e., $([k] \leq k \leq [k] + 1)$. Since $\mathcal{D}^2 = \partial_x$ and consequently

$$S_x^{2N} f \cdot g = D_x^N f \cdot g,$$

one only needs to consider the first order super Hirota derivative which is as the following [3]

$$S_x f \cdot g = (\mathcal{D} f) g - (-1)^{|f|} f (\mathcal{D} g). \tag{2.5}$$

One can further finds that

$$S_x D_x^m D_t^n f \cdot g = (\mathcal{D}_\theta - \mathcal{D}_{\theta'})(\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t, \theta) g(t', x', \theta') \mid_{x=x', t=t', \theta=\theta'}, \tag{2.6}$$

where f, g are Grassmann even functions.

With regards to the properties of the operator S_x, if

$$\zeta_i = p_i x + q_i t + \theta \varrho_i + \zeta_i^{(0)} \tag{2.7}$$

with bosonic constants $(p_i, q_i, \xi_i^{(0)})$ and fermionic constant ϱ_i, it then follows that

$$S_x^{2N+1} e^{\zeta_1} \cdot e^{\zeta_2} = S_x D_x^N e^{\zeta_1} \cdot e^{\zeta_2} = [\varrho_1 - \varrho_2 + \theta(p_1 - p_2)](p_1 - p_2)^N e^{\zeta_1 + \zeta_2}, \tag{2.8a}$$

and

$$\begin{aligned} &S_x D_x^m D_t^n e^{\zeta_1} f \cdot e^{\zeta_2} g \\ &\quad = e^{\zeta_1+\zeta_2}[S_x + (\varrho_1 - \varrho_2) + \theta(p_1 - p_2][D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g. \end{aligned} \tag{2.8b}$$

Further we can find

$$\begin{aligned} \mathcal{F}(S_x, D_x, D_t)(e^{\zeta_1} f) \cdot (e^{\zeta_2} g) = e^{\zeta_1+\zeta_2} \mathcal{F}(S_x + (\varrho_1 - \varrho_2) + \theta(p_1 - p_2), D_x \\ + (p_1 - p_2), D_t + (q_1 - q_2)) f \cdot g, \end{aligned} \tag{2.8c}$$

where $\mathcal{F}(S_x, D_x, D_t)$ is a polynomial of S_x, D_x, D_t with bosonic coefficients. Equation (2.8c) is an analogue of the formula (2.2), and exhibits the gauge invariance property of super Hirota derivative [3]. This formula will play an important role in getting the deformed bilinear BTs. The proof for the formula is given below,

$$\begin{aligned} &S_x D_x^m D_t^n (e^{\zeta_1} f) \cdot (e^{\zeta_2} g) \\ &\quad = (\mathcal{D}_\theta - \mathcal{D}_{\theta'})\{e^{\zeta_1(\theta)+\zeta_2(\theta')}[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f(\theta) \cdot g(\theta')\}|_{\theta'=\theta} \\ &\quad = e^{\xi+\eta}(\mathcal{D}(\xi - \eta))[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g] \\ &\qquad + e^{\xi+\eta}(\mathcal{D}_\theta - \mathcal{D}_{\theta'})\{[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f(\theta) \cdot g(\theta')\}|_{\theta'=\theta} \\ &\quad = e^{\xi+\eta}(\mathcal{D}(\xi - \eta))[D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g] \\ &\qquad + e^{\xi+\eta} S_x [D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g] \\ &\quad = e^{\xi+\eta}[S_x + (\varrho_1 - \varrho_2) + \theta(p_1 - p_2)][D_x + (p_1 - p_2)]^m [D_t + (q_1 - q_2)]^n f \cdot g. \end{aligned}$$

2.2. *BTs of the KdV equation*

The KdV equation always plays a paradigmatic role in classical integrable systems. To have a comparison with SUSY systems and also to understand the main ideas of the paper, let us recall the BT and related topics of the KdV equation. The KdV equation reads

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.9}$$

By the transformation

$$u = 2(\ln f)_{xx}, \tag{2.10}$$

it is bilinearized to [13],

$$(D_x D_t + D_x^4) f \cdot f = 0, \tag{2.11}$$

which generates the bilinear BT [14]

$$(D_x^2 - \lambda) f \cdot g = 0, \tag{2.12a}$$

$$(D_t + D_x^3 + 3\lambda D_x) f \cdot g = 0, \tag{2.12b}$$

where λ is a parameter and D is the well-known operator defined in (2.1). If f solves (2.11), so does g provided (f, g) satisfy the above BT, and

$$u = 2(\ln g)_{xx} \tag{2.13}$$

gives a new solution to the KdV equation. Besides, the BT (2.12) is related to Lax pair of the KdV equation, which is

$$\phi_{xx} = (\lambda - u)\phi, \tag{2.14a}$$

$$\phi_t = -4\phi_{xxx} - 3u_x\phi - 6u\phi_x, \tag{2.14b}$$

through [14]

$$\phi = \frac{g}{f}, \quad u = 2(\ln f)_{xx}. \tag{2.15}$$

Usually, with respect to generating multi-soliton solutions, the perturbation expansion technique (expanding f and g as (2.17) for the bilinear BT (2.12)) is not as effective as from the bilinear equation (2.11). In fact, at the first glance, when $f = 1$, g cannot be in the form of $g = 1 + \varepsilon g_1$ due to the term $\lambda f g$ in (2.12a) unless $\lambda = 0$. In practice, one may first take $f = 1$ and solve (2.12) with $\lambda = p_1'$ to get $g = g(p_1')$ which gives 1-soliton solution via (2.13). Then, in a new turn let $f = g(p_1')$ and solve (2.12) with $\mu = p_2' \neq p_1'$ for a new $g(p_1', p_2')$, which yields a 2-soliton solution via (2.13). We also note that in the second turn the restriction $p_2' \neq p_1'$ is necessary. Otherwise, g does not create new solutions. Repeating the procedure by taking f to be g which is obtained in the previous round and solving (2.12) with λ being a number which is different from those p_j''s previously used, the obtained g will provide a new multi-soliton solutions.

In order to use the perturbation expansion technique and also get more solutions than solitons, Chen *et al.* [6] deformed the bilinear BT (2.12) by replacing (f, g) by $(e^{\zeta_1'} f, e^{\zeta_2'} g)$ in (2.12), where ζ_i' are defined as (2.3). They got which by using (2.2) yielded

$$(D_x^2 + \lambda' D_x) f \cdot g = 0, \tag{2.16a}$$

$$(D_t + D_x^3) f \cdot g = 0, \tag{2.16b}$$

where the parametrization is

$$\lambda' = 2(p_1' - p_2'), \quad \lambda = (p_1' - p_2')^2, \quad q_1' - q_2' = 4(p_1' - p_2')^3, \quad \frac{\lambda'^2}{4} = \lambda.$$

Obviously, guaranteed by the gauge invariance of Hirota bilinear equations [12], if f solves (2.11) and (2.16) (together with g), then g satisfies (2.11). This means the deformed BT (2.16) does play a role of a bilinear BT for the KdV equation. Moreover, (2.16) admits more freedom than (2.12).