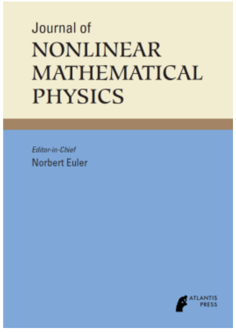

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FOCUSING mKdV BREATHER SOLUTIONS WITH NONVANISHING BOUNDARY CONDITION BY THE INVERSE SCATTERING METHOD

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Using the Inverse Scattering Method with a nonvanishing boundary condition, we obtain an *explicit* breather solution with nonzero vacuum parameter b of the *focusing modified* Korteweg–de Vries (mKdV) equation. Moreover, taking the limiting case of zero frequency, we obtain a generalization of the *double pole* solution introduced by M. Wadati *et al.*

Keywords: mKdV equation; breather; inverse scattering; nonvanishing condition.

Mathematics Subject Classification: 37K15, 35Q53, 35Q51, 37K10

1. Introduction and Main Results

This paper is concerned with the integrability of the *focusing modified* Korteweg–de Vries equation (mKdV)

$$k_t + k_{sss} + 2(k^3)_s = 0, \quad (s,t) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where $k \equiv k(s,t)$ is a real valued function. Specifically, we use here the Inverse Scattering Method (ISM) under a nonvanishing boundary condition (NVBC), as it was devised by Kawata and Inoue [10], to obtain *explicit* breather solutions of mKdV (1.1) which at infinity behave as a nontrivial constant parameter, e.g. $b \in \mathbb{R}$. We also consider here the limiting case of zero frequency of these breather solutions, obtaining a nonzero mean generalization of a special solution (the *double pole* solution) introduced by Wadati *et al.* [17].

The mKdV (1.1) is a nonlinear dispersive and integrable equation with infinitely many conservation laws and well known Lax pair (see [1]). It is characterized to contain a nonlinear part given by the cubic term $(k^3)_s$. The competition between this nonlinear term together with the linear dispersive term k_{sss} allows the existence of well-known soliton as well as exact real breather solutions (see [15, 20] and references therein). These breathers are nonlinear oscillatory modes, defined by means of a periodical-in-time and spatially localized real function (indeed, exponentially decaying in space). Breather solutions of mKdV (1.1) in the line were found by Wadati [20] (see also [15]). Those breather solutions are defined in all the real line, vanish exponentially at infinity and, qualitatively describe traveling wave packets. They were used by Kenig, Ponce and Vega in the proof of the discontinuity of the flowmaps associated to mKdV equation in the Sobolev spaces H^σ, constituted by functions with σ derivatives in $L^2(\mathbb{R})$ (see [12]).

Explicitly, real breather solutions of mKdV in the line are determined by four real parameters, two of them given by the amplitude (β) of the envelope and the frequency (α) of the carried wave and the other two given by time and spatial translations (represented by y_{10}, y_{20} below). Taking $\alpha, \beta \in \mathbb{R}\backslash\{0\}$, it is well known that such breather solutions can be written as follows

$$k(s,t) = -i\frac{\partial}{\partial s}\log\left(\frac{f(y_1)+ig(y_2)}{f(y_1)-ig(y_2)}\right) = 2\frac{\partial}{\partial s}\arctan\left(\frac{g(y_2)}{f(y_1)}\right), \tag{1.2}$$

where $y_1 = 2\beta s + \gamma t + y_{10}$, $y_2 = 2\alpha s + \delta t + y_{20}$, $\gamma = 8\beta(-\beta^2+3\alpha^2)$, $\delta = 8\alpha(\alpha^2-3\beta^2)$, and[a]

$$g(y_2) = \frac{\beta}{\alpha}\sin\left(y_2 - \arctan\left(\frac{\beta}{\alpha}\right)\right), \tag{1.3}$$

$$f(y_1) = \cosh(y_1). \tag{1.4}$$

Note that $\delta \neq \gamma$, for all values of α and β different from zero. This means that variables $2\alpha s + \delta t$ and $2\beta s + \gamma t$ are always independent. Indeed, if $\delta = \gamma$, one has from (1.2)

$$4(\alpha^2+\beta^2) = 0,$$

which means $\alpha = \beta = 0$, a contradiction. Moreover, note that for each fixed time, the breather of the mKdV (1.2) is a function in the Schwartz class, with zero mean:

$$\int_{\mathbb{R}} k(s,t)ds = 0.$$

Therefore, in what follows, we may suppose $\alpha, \beta > 0$, without loss of generality. The parameter γ will be for us the *velocity* of the breather solution. In general, calculations involving the real breather of mKdV (1.2) are cumbersome, but in the particular case of (1.2), it is easy to see that selecting α large such that $\beta/\alpha \ll 1$, the breather (1.2) approximates to

$$k(s,t) \approx 2\beta\cos(2\alpha(s+4(\alpha^2-3\beta^2)t))\,\mathrm{sech}(2\beta(s+4(3\alpha^2-\beta^2)t)), \tag{1.5}$$

where now the traveling wave packet nature of the breather (1.2) is apparent.

From a physical point of view the mKdV (1.1) appears to be relevant in a number of different physical systems (e.g. phonons in anharmonic lattices, models of traffic congestion, curve motion and fluid mechanics). From a mathematical point of view, the mKdV (1.1) has been the focus of an extensive study in the past, being necessary to comment the particular issues about well-posedness of the Cauchy problem (see [11], and references therein) and

[a]The *arctan* phase in the argument of g can be dropped with a suitable translation in time and space, but it will not be done for comparison purposes.

about the asymptotic decomposition of solitons and breathers announced by Schuur (see [19, p. 114, Chap. 5]).

The motivation for this work is focused on the obtention of *explicit* breather solutions that can be used as the curvature of closed curves. The interest of this kind of solutions of the mKdV with NVBC comes from the problem of the evolution of closed planar curves under the mKdV flow. By the work of Goldstein and Petrich [8], the mKdV equation (1.1) is considered as the evolution equation of the curvature of a curve. Therefore, a closed curve can be considered as those whose associated curvature satisfies that its mean is nonzero (indeed, it is propotional to 2π). Moreover, those curvatures with nonzero mean can be obtained from a solution of the focusing mKdV constructed by the *nonlinear* superposition of a constant (e.g. b) plus a traveling wave. In fact, it is possible to find some special breather solutions associated to simple closed curves that, when evolving under the mKdV flow, they create and annihilate self-intersections (see [2]).

In this paper we use the ISM under a NVBC (e.g. [10]) to get *explicit real* breather solutions of mKdV with nonzero mean. In this case, in comparison with the previous works related with the use of this technique (e.g. [5]), some difficulties arise since it is necessary to consider a pair of complex eigenvalues $\lambda_1, \lambda_2 = -\lambda_1^*$. Moreover, since these complex eigenvalues appear inside of squared roots, it is necessary to simplify the complex expressions involved in the calculations and re-define the parameters corresponding to the frequency α and the amplitude β of the breather solution. In this work, we also factorize and simplify the final expression in order to obtain this breather solution with nonzero vacuum parameter b. We also show the structure of the matricial expression that comes from the ISM for this breather solution. Finally, taking the limiting case of zero frequency, we obtain a generalization of the *double pole* solution introduced by Wadati *et al.* in [17].

As far as we know, this is the *first result* in which a breather solution with NVBC of the mKdV (1.1) is obtained by using the ISM, since the previous results of Au-Yeung *et al.* [5] reduce to build, by means of the ISM, the exact squared two-soliton solution with NVBC (see [5]) and the results obtained by Grimshaw, Slunyaev and Pelinovsky (see [9, 18]) for the Gardner equation[b] (further details on this equation in Miura *et al.* [4, 7, 16]) do not contain a detailed description of the ISM. Note that the mKdV breather solutions mentioned above can be obtained alternatively using the Hirota method with a suitable selection of the wavenumbers (e.g. in Chow and Lai [6]). Finally, we believe that breather solutions with NVBC we have built in this work are easily compared with the classical breather solution (1.2) of the mKdV obtained by Wadati [20]. Our main results are the following:

Theorem 1.1 (Breather solution of the mKdV with NVBC). *Let* $\alpha, \beta, b \in \mathbb{R}\backslash\{0\}$ *such that* $\Delta = \alpha^2 + \beta^2 - b^2 > 0$. *Then, up to translations, the real breather solution of the mKdV* (1.1) *with nonvanishing asymptotic constant value b is*

$$k(s,t) = b + 2\frac{\partial}{\partial s}\arctan\left[\frac{\tilde{g}(s,t)}{\tilde{f}(s,t)}\right], \tag{1.6}$$

[b]Note that solutions of the mKdV equation with NVBC are also solutions of the Gardner equation with zero boundary condition and a translation.

where

$$\tilde{g}(s,t) = \frac{\beta}{\alpha}\sqrt{\frac{\Delta+b^2}{\Delta}}\cos(2\alpha s+\delta t) + \frac{b\beta}{\Delta}(\cosh(2\beta s+\gamma t) - \sinh(2\beta s+\gamma t)),$$

$$\tilde{f}(s,t) = \cosh(2\beta s+\gamma t) + \frac{b\beta}{\alpha\sqrt{\Delta}}\sin\left(2\alpha s+\delta t - \arctan\left(\frac{\beta}{\alpha}\right)\right), \tag{1.7}$$

$$\text{and} \quad \gamma = 8\beta(3\alpha^2-\beta^2) - 12b^2\beta, \quad \delta = 8\alpha(\alpha^2-3\beta^2) - 12b^2\alpha.$$

Corollary 1.2 (*Double pole* solution of the mKdV with NVBC). *Let $\beta, b \in \mathbb{R}\backslash\{0\}$ such that $\beta^2 - b^2 > 0$. Then, up to translations, the double pole solution of the mKdV* (1.1) *with nonvanishing asymptotic constant value b is*

$$k(s,t) = b + 2\frac{\partial}{\partial s}\arctan\left[\frac{\tilde{G}(s,t)}{\tilde{F}(s,t)}\right], \tag{1.8}$$

where

$$\tilde{G}(s,t) = \frac{\beta(1-2\beta(s-6(2\beta^2+b^2)t))}{\sqrt{\beta^2-b^2}} + \frac{b\beta(\cosh(y)-\sinh(y))}{\beta^2-b^2},$$

$$\tilde{F}(s,t) = \cosh(y) + \frac{2b\beta(s-6(2\beta^2+b^2)t)}{\sqrt{\beta^2-b^2}}, \quad y = \beta(2s-4(2\beta^2+3b^2)t).$$

Before explaining the main techniques and steps behind the proof of these results, some remarks are in order.

Remarks. (1) Note that it is possible to obtain new solutions of the *defocusing* mKdV equation

$$k_t + k_{sss} - 2(k^3)_s = 0, \quad (s,t) \in \mathbb{R}\times\mathbb{R} \tag{1.9}$$

from the *focusing* mKdV breather solutions (1.6) with a special choice of their free parameters b, β, α. First of all, and having this goal in mind, we apply the translation

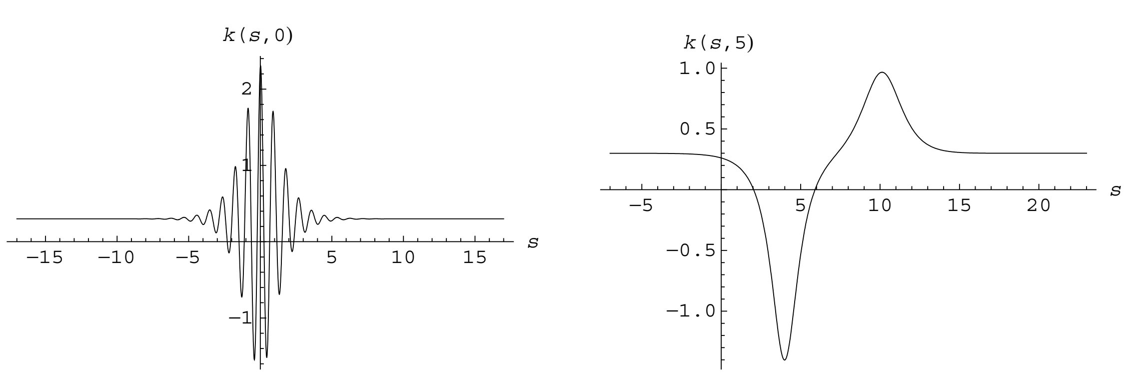

Fig. 1. Left: Breather solution (1.6) with $\alpha = 7, \beta = 1, b = 0.3$ at $t = 0$. Right: *Double pole* solution (1.8) with $\beta = 1, b = 0.3$ at $t = 5$.

$\arctan[\beta/\alpha] + \pi/2$ in the argument of the oscillatory functions of (1.6). Then, we perform the following change of parameters in (2.69): b becomes ib, β becomes $-\beta$ and α becomes $i\alpha$. Finally, we get a new purely complex solution $i\tilde{k}(s,t)$ of the focusing mKdV equation (1.1). Hence $\tilde{k}(s,t)$ is a *real and regular* two soliton solution of the defocusing mKdV equation (1.9) with nonvanishing boundary value b at infinity. Explicitly (with $\alpha, \beta, b \in \mathbb{R}\backslash\{0\}$ such that $\alpha > \beta$ and $\tilde{\Delta} = b^2 - \alpha^2 + \beta^2 > 0$),

$$\tilde{k}(s,t) = b - 2\frac{\partial}{\partial s} \operatorname{arctanh}\left[\frac{\tilde{g}_{def}(s,t)}{\tilde{f}_{def}(s,t)}\right], \tag{1.10}$$

where

$$\begin{aligned} \tilde{g}_{def}(s,t) &= \frac{\beta}{\alpha}\sqrt{\frac{b^2-\tilde{\Delta}}{\tilde{\Delta}}}\sinh(2\alpha s + \delta t) - \frac{b\beta}{\tilde{\Delta}}(\cosh(2\beta s + \gamma t) - \sinh(2\beta s + \gamma t)),\\ \tilde{f}_{def}(s,t) &= \cosh(2\beta s + \gamma t) + \frac{b\beta}{\alpha\sqrt{\tilde{\Delta}}}\cosh\left(2\alpha s + \delta t + \operatorname{arctanh}\left(\frac{\beta}{\alpha}\right)\right), \\ \text{and} \quad \gamma &= 8\beta(-3\alpha^2 - \beta^2) + 12b^2\beta, \quad \delta = 8\alpha(-\alpha^2 - 3\beta^2) + 12b^2\alpha. \end{aligned} \tag{1.11}$$

With other choices of the parameters, b, β, α and with the same procedure, we obtain different complex and regular or singular (depending on the selected parameters) solutions of the Eq. (1.9) with nonvanishing boundary value b at infinity.

(2) We believe that there exist *periodic* breather solutions of the mKdV (1.1) with *nonzero mean*. First results on the existence of *periodic* breathers were obtained by Kevrekidis, Khare and Saxena in [13, 14]. In these works, they showed regular and singular *periodic* breathers for the *focusing* and *defocusing* mKdV by using suitable ansätze with free parameters to be adjusted. Unfortunately for geometrical purposes, these *periodic* breathers have *zero mean* and therefore they cannot be the curvature of a close curve. Henceforth, at least as geometrical motivation, it is necessary to find *periodic* breathers with nonzero mean which play the role of the curvature of closed curves. We base our conjecture about the existence of these *periodic* breathers with nonzero mean on the numerical results obtained by the author in [2]. In this work, some examples of closed curves and their numerical evolution under the geometric mKdV flow were given. The starting point to build those close curves was to use, as initial curvature, the breathers with nonzero mean (1.6) and (1.8) obtained in the present work.

2. Breather Solutions of the Focusing mKdV with Nonvanishing Boundary Condition

In this section we obtain *explicit real* breather solutions of the focusing mKdV (1.1) with a nonvanishing boundary condition by using the ISM for potentials that are not trivial at infinity, as it was developed by Kawata and Inoue in [10]. We also recall the work of Au-Yeung *et al.* [5], in which the same approach was used to obtain one and two soliton solutions with nontrivial values at infinity. First of all, we summarize some basic results from Kawata and Inoue [10] and Au-Yeung *et al.* [5], necessary for our research.

2.1. *Basic results of Kawata and Inoue: General formalism for the mKdV*

Kawata and Inoue considered in [10] a generalized AKNS eigenvalue problem for nonvanishing potentials. The AKNS eigenvalue problem for the mKdV equation is given by the following spatial and time evolution equations:

$$\mathbf{v}_s = D(\lambda)\mathbf{v}, \quad D(\lambda; s,t) = \begin{pmatrix} -i\lambda & q(s,t) \\ r(s,t) & i\lambda \end{pmatrix}, \tag{2.1}$$

$$\mathbf{v}_t = F(\lambda)\mathbf{v}, \quad F(\lambda; s,t) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} A(\lambda) &= -4i\lambda^3 - 2i\lambda qr + rq_s - qr_s, \\ B(\lambda) &= 4\lambda^2 q + 2i\lambda q_s + 2q^2 r - q_{ss}, \\ C(\lambda) &= 4\lambda^2 r - 2i\lambda r + 2qr^2 - r_{ss}, \end{aligned} \tag{2.3}$$

and the potential $q(s,t) = k(s,t)$ and $r(s,t) = -q(s,t)$. Matrices $D(\lambda)$ and $F(\lambda)$ satisfy the well known integrable condition associated to Eqs. (2.1) and (2.2) (i.e. ∂_t (2.1)$=$ ∂_s (2.2)):

$$D_t - F_s + DF - FD = 0. \tag{2.4}$$

They sought a real *potential* solution $q(s,t)$ under the following boundary condition:

$$q(s,t) \to b \quad \text{as } s \to \pm\infty; \quad b^2 = -\lambda_0^2, \tag{2.5}$$

moreover, requiring that $q(s,t)$ is sufficiently smooth and all the s derivatives of q tend to zero as $s \to \pm\infty$. For this purpose, they considered potentials $q(s,t)$ and $r(s,t)$ with the following nonvanishing conditions:

$$\begin{aligned} &q(s,t)(\text{or } r(s,t)) \to q^{\pm}(\text{or } r^{\pm}) \quad \text{as } s \to \pm\infty, \\ &q^+ r^+ = q^- r^- = \lambda_0^2, \end{aligned} \tag{2.6}$$

where $q^{\pm}, r^{\pm}$ and λ_0^2 are constants. Then, the spatial evolution matrix $D(\lambda; s,t)$ can be written as follows:

$$D(\lambda; s,t) = D^{\pm}(\lambda) + \Delta D^{\pm}(s,t), \tag{2.7a}$$

$$D^{\pm}(\lambda) = \lim_{s\to\pm\infty} D(\lambda; s,t) = \begin{pmatrix} -i\lambda & q^{\pm} \\ r^{\pm} & i\lambda \end{pmatrix}, \tag{2.7b}$$

$$\Delta D^{\pm}(s,t) = D(\lambda; s,t) - D^{\pm}(\lambda) = \begin{pmatrix} 0 & q(s,t) - q^{\pm} \\ r(s,t) - r^{\pm} & 0 \end{pmatrix}, \tag{2.7c}$$

and the characteristic roots of $D^{\pm}(\lambda)$ are $\pm i\zeta$, with $\zeta = \sqrt{\lambda^2 - \lambda_0^2}$. Now, they defined

$$T^{\pm}(\lambda, \zeta) = \lim_{s \to \pm\infty} T(\lambda, \zeta; s), \tag{2.8}$$

where

$$T(\lambda, \zeta; s) = \begin{pmatrix} -iq_1(s) & \lambda - \zeta \\ \lambda - \zeta & ir_1(s) \end{pmatrix},$$

and q_1, r_1 are suitable smooth and satisfy that $q_1(s)r_1(s) = \lambda_0^2$ for all $s \in \mathbb{R}$. The matrices $D^{\pm}(\lambda)$ can be diagonalized by $T^{\pm}(\lambda, \zeta)$ as

$$D^{\pm}(\lambda) = -i\zeta T^{\pm}(\lambda, \zeta)\sigma_3[T^{\pm}(\lambda, \zeta)]^{-1}. \tag{2.9}$$

Using (2.9), they could define Jost matrices $\Phi^{\pm}$ as the solutions of (2.1) under conditions

$$\Phi^{\pm}(\lambda, \zeta; s, t) \to T^{\pm}(\lambda, \zeta)J(\zeta s) \quad \text{as } s \to \pm\infty, \tag{2.10}$$

where

$$J(\zeta s) = \begin{pmatrix} e^{-i\zeta s} & 0 \\ 0 & e^{i\zeta s} \end{pmatrix}.$$

Then, a scattering matrix S was defined by

$$\Phi^{-}(\lambda, \zeta; s, t) = \Phi^{+}(\lambda, \zeta; s, t)S(\lambda, \zeta; t), \tag{2.11}$$

and using relations (2.2), (2.9) and (2.11) it was easy to derive the following equation:

$$S_t + SW^{-} - W^{+}S = 0, \quad S|_{t=0} = S_0, \tag{2.12}$$

where S_0 is given by the direct scattering, and

$$W^{\pm} = \zeta\sigma_3 \cdot \sum_{n=1}^{N} a_n \lambda^{n-1} + i\left(\frac{a_0}{\zeta}\right)\begin{pmatrix} -i\lambda & r^{\pm}e^{2i\zeta s} \\ q^{\pm}e^{-2i\zeta s} & i\lambda \end{pmatrix}. \tag{2.13}$$

As they indicated, the solution of (2.12) is easily obtained by integrating directly

$$S(\lambda, \zeta; t) = e^{W^{+}t}S_0(\lambda, \zeta)e^{-W^{-}t}. \tag{2.14}$$

Now, they assumed that zeros of the $S_{11}(\lambda, \zeta; 0)$ matrix element in the region $\mathrm{Im}(\zeta) > 0$ were (λ_j, ζ_j), $\quad j = 1, 2$ where

$$\zeta_j = \sqrt{\lambda_j^2 + b^2}, \quad j = 1, 2, \quad \text{for } y > s. \tag{2.15}$$

Defining the scattering data by $e^{2i\theta(s)} = q_1(s)/r_1(s), \theta^{\pm} = \lim_{s\to\pm\infty}\theta(s)$, it could be seen that the Jost functions satisfy the following equation (see [10, p. 1724] for further details)

$$\phi_1^{+}(\lambda, s)e^{i\zeta s} = T(\lambda, s)[1 + \exp(\theta(\infty) - \theta(s))]\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$- \frac{1}{2\pi i}\int_{\Gamma^{+}} \frac{d\lambda'}{\lambda' - \lambda}\rho_1(\lambda')T(\lambda, s)T^{-1}(\lambda', s)\phi_2^{+}(\lambda', s)e^{i\zeta' s}, \tag{2.16}$$

where $\Phi^{\pm} = (\phi_1^{\pm}, \phi_2^{\pm})$ and $\rho_1(\lambda') = S_{21}(\lambda,\zeta)/S_{11}(\lambda,\zeta)$. Repeating the procedure, it was possible to obtain an equation for the ϕ_2^{+} component,

$$\phi_2^{+}(\lambda,s)e^{-i\zeta s} = T(\lambda,s)[1+\exp(-\theta(\infty)+\theta(s))]\begin{pmatrix}0\\1\end{pmatrix}$$
$$-\frac{1}{2\pi i}\int_{\Gamma^{+}}\frac{d\lambda}{\lambda'-\lambda}\rho_2(\lambda')T(\lambda,s)T^{-1}(\lambda',s)\phi_1^{+}(\lambda',s)e^{-i\zeta' s}, \qquad (2.17)$$
$$\rho_2(\lambda') = S_{12}(\lambda,\zeta)/S_{22}(\lambda,\zeta).$$

Defining the following representation about the Jost matrices

$$\Phi^{\pm}(\lambda,s) = T^{\pm}(\lambda)J(\lambda s) + \int_{-\infty}^{s} K^{\pm}(s,\tau)T^{\pm}(\lambda)J(\lambda\tau)d\tau, \qquad (2.18)$$

and substituting (2.18) into (2.1) and (2.2), the functions $K^{\pm}(s,\cdot)$ satisfy the following equations (for the shake simplicity, we drop the time dependence in $K^{\pm}$ and H_c defined below):

$$\frac{\partial K^{\pm}}{\partial s}(s,y) + \sigma_3\frac{\partial K^{\pm}}{\partial y}(s,y)\sigma_3 + \sigma_3 K^{\pm}(s,y)\sigma_3[D^{\pm}(\lambda)+i\lambda\sigma_3]$$
$$-\begin{pmatrix}0 & q(s,t)\\ -q(s,t) & 0\end{pmatrix}K^{\pm}(s,y) = 0, \qquad (2.19a)$$

$$\sigma_3 K^{\pm}(s,s)\sigma_3 - K^{\pm}(s,s) + \Delta D^{\pm}(\lambda;s,t) = 0, \qquad (2.19b)$$

$$K^{\pm}(s,y)\to 0 \quad \text{as } y\to\pm\infty, \qquad (2.19c)$$

where $D^{\pm}(\lambda)$ and $\Delta D^{\pm}(\lambda;s,t)$ were defined by (2.7) and $\sigma_3 \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Due to the characteristics in (2.19a), given the potentials q and r, there exists a solution of (2.19a) which satisfies the conditions (2.19b) and (2.19c). Therefore, the representation (2.18) is suitable. From (2.18), it was possible to define the following types

$$\phi_1^{+}(\lambda,s) := T^{+}(\lambda)e^{-i\zeta s}\begin{pmatrix}1\\0\end{pmatrix} + \int_{-\infty}^{s} K^{+}(s,\tau)T^{+}(\lambda)e^{-i\zeta\tau}ds\begin{pmatrix}1\\0\end{pmatrix}, \qquad (2.20a)$$

$$\phi_2^{+}(\lambda,s) := T^{+}(\lambda)e^{i\zeta s}\begin{pmatrix}0\\1\end{pmatrix} + \int_{-\infty}^{s} K^{+}(s,\tau)T^{+}(\lambda)e^{i\zeta\tau}ds\begin{pmatrix}0\\1\end{pmatrix}. \qquad (2.20b)$$

Substituting (2.20) in (2.16), they got an equation without Jost functions. Next applying the integrator $\frac{1}{4\pi}\int_{B^{+}}\frac{e^{i(y-s)\zeta}}{\zeta}d\lambda$ $(y>s)$ to both sides of this equation, they finally obtained the Gelfand–Levitan integral equation

$$K^{+}(s,y)\cdot\begin{pmatrix}0\\1\end{pmatrix} + H_1(s+y)\begin{pmatrix}0\\1\end{pmatrix} - \int_{s}^{\infty} K^{+}(s,\tau)H_1(y+\tau)d\tau\begin{pmatrix}0\\1\end{pmatrix} = 0 \quad (y>s), \qquad (2.21)$$

where $H_1(z)$ is given by

$$H_1(z) = \frac{1}{4\pi}\int_{\Gamma^{+}}\frac{e^{i\zeta z}}{\zeta}\rho_1(\lambda,\zeta)T^{+}(\lambda,\zeta)d\lambda. \qquad (2.22)$$

Now we refer to the symmetries arising in Eqs. (2.1) and (2.2)

(1) This eigenvalue problem has a proper symmetry caused by the fact that potentials do not vanish at infinity. The matrices $\Phi^{\pm}(\lambda, -\zeta; s)$ also satisfy Eqs. (2.1) and (2.2), then we can set

$$\Phi^{\pm}(\lambda, \zeta; s) = \Phi^{\pm}(\lambda, -\zeta; s)P^{\pm}(\lambda, \zeta), \tag{2.23}$$

where $P^{\pm}$ are constant matrices given by

$$P^{\pm}(\lambda, \zeta) = \frac{i}{\lambda + \zeta}\begin{pmatrix} 0 & r^{\pm} \\ -q^{\pm} & 0 \end{pmatrix}. \tag{2.24}$$

From Eqs. (2.11) and (2.23) we get

$$S(\lambda, \zeta) = [P^{+}(\lambda, \zeta)]^{-1} S(\lambda, -\zeta)P^{-}(\lambda, \zeta), \tag{2.25}$$

or

$$\begin{aligned} &S_{11}(\lambda, \zeta) = (q^{-}/q^{+})S_{22}(\lambda, -\zeta), \quad S_{11}(\lambda, -\zeta) = (q^{-}/q^{+})S_{22}(\lambda, \zeta), \\ &S_{12}(\lambda, \zeta) = (-r^{-}/q^{+})S_{21}(\lambda, -\zeta), \quad S_{12}(\lambda, -\zeta) = -(r^{-}/q^{+})S_{21}(\lambda, \zeta). \end{aligned} \tag{2.26}$$

(2) The symmetries that arise in the mKdV case $(r(s) = -q(s))$ are

$$\begin{aligned} &S_{11}(\lambda, \zeta) = S_{22}(-\lambda, -\zeta), \quad S_{11}(-\lambda, -\zeta) = S_{22}(\lambda, \zeta), \\ &S_{12}(\lambda, \zeta) = -S_{21}(-\lambda, -\zeta), \quad S_{12}(-\lambda, -\zeta) = -S_{21}(\lambda, \zeta), \end{aligned} \tag{2.27}$$

and also $K^{\pm}(s, y)$ has the additional property

$$K^{\pm}(s, y) = (-\sigma_2)K^{\pm}(s, y)(-\sigma_2), \tag{2.28}$$

where the matrix σ_2 is defined by

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.29}$$

Now, from direct calculations as they indicated, it is possible to rewrite $H_1(z)$ as follows (quantities $H_d(z), H_c(z)$ are called as the discrete and continuous components of H_1, respectively).

$$H_1(z) = H_d(z) - H_c(z), \quad H_{d,c}(y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i(\partial h_{1d,c}/\partial y)(y) + h_{2d,c}(y) \\ ir^{+}h_{1d,c}(y) \end{pmatrix}, \tag{2.30}$$

where

$$\begin{aligned} &h_{1d}(y) = \frac{1}{4\pi}\oint \frac{e^{i\zeta y}}{\zeta}\frac{S_{21}(\lambda, \zeta)}{S_{11}(\lambda, \zeta)}d\lambda, \quad h_{1c}(y) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{i\zeta y}}{\mu}\left(\frac{S_{21}(\mu, \zeta)}{S_{11}(\mu, \zeta)} - \frac{S_{21}(-\mu, \zeta)}{S_{11}(-\mu, \zeta)}\right)d\zeta, \\ &h_{2d}(y) = \frac{1}{4\pi}\oint \frac{\lambda e^{i\zeta}}{\zeta}\frac{S_{21}(\lambda, \zeta)}{S_{11}(\lambda, \zeta)}d\lambda, \quad h_{2c}(y) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} e^{i\zeta y}\left(\frac{S_{21}(\mu, \zeta)}{S_{11}(\mu, \zeta)} + \frac{S_{21}(-\mu, \zeta)}{S_{11}(-\mu, \zeta)}\right)d\zeta, \\ &\mu = \sqrt{\zeta^2 - \lambda_0^2}. \end{aligned} \tag{2.31}$$

Once we have introduced the basic concepts and tools of the ISM with NVBC by Kawata and Inoue [10], we apply them to obtain *explicit* breather solutions of the mKdV (1.1) with NVBC.

2.2. *Proof of Theorem 1.1*

We will follow the next steps, presented in the subsections below.

2.2.1. *Gelfand–Levitan equation for the mKdV*

We first calculate, from (2.13), the temporal evolution of the elements of the scattering matrix (S_{ij}) in the case of mKdV. For that purpose (see [10, p. 1723]), we select $N = 3$ and $a_0 = a_2 = 0$, $a_1 = 2ib^2$, $a_3 = -4i$ in (2.13), so that:

$$S_{11}(\lambda,\zeta;t) = S_{11}(\lambda,\zeta;0), \quad S_{12}(\lambda,\zeta;t) = S_{12}(\lambda,\zeta;0)e^{-4i\zeta(2\lambda^2-b^2)t},$$

$$S_{21}(\lambda,\zeta;t) = S_{21}(\lambda,\zeta;0)e^{4i\zeta(2\lambda^2-b^2)t}, \quad S_{22}(\lambda,\zeta;t) = S_{22}(\lambda,\zeta;0).$$

Now, assuming that zeros of $S_{11}(\lambda,\zeta;0)$ in the region $\operatorname{Im}\zeta > 0$ are given by

$$\lambda_j, \zeta_j = \sqrt{\lambda_j^2 - \lambda_0^2}, \quad j = 1, 2, \tag{2.32}$$

and the expressions (2.30) and (2.31), the Gelfand–Levitan equation (2.21) for the function $K^+(s,y)$ is rewritten as

$$\begin{aligned} &K^+(s,y)\cdot\begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{2}\begin{pmatrix}c_j\\ \tilde{c}_j\end{pmatrix} e^{i\zeta_j(s+y)} - H_c(s+y)\begin{pmatrix}0\\1\end{pmatrix} \\ &\quad - \int_s^{+\infty} K^+(s,y')\sum_{j=1}^{2}\begin{pmatrix}c_j\\ \tilde{c}_j\end{pmatrix} e^{i\zeta_j(y+y')}dy' + \int_s^{+\infty} K^+(s,y')H_c(s+y')\begin{pmatrix}0\\1\end{pmatrix}dy' = 0, \end{aligned} \tag{2.33}$$

where

$$c_j = \frac{i}{2}(\lambda_j - \zeta_j)m(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}, \tag{2.34}$$

$$\tilde{c}_j = \frac{b}{2}m(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}, \tag{2.35}$$

$$m(\lambda) = \frac{S_{21}(\lambda,\zeta;0)}{\zeta\frac{d}{d\lambda}S_{11}(\lambda,\zeta;0)}. \tag{2.36}$$

We rewrite (2.33) taking into account that $m(-\lambda^*) = m(\lambda)^*$, that $(q^-/q^+)S_{11}(\lambda,\zeta) = S_{11}(-\lambda,\zeta)$ (see [10, p. 1728]), and assuming

(a) Continuous component

$$H_c(y;0) = 0. \tag{2.37}$$

(b) Zeros of $S_{11}(\lambda, \zeta)$ in the region $\mathrm{Im}(\zeta) > 0$ are (λ_j, ζ_j) and $(-\lambda_j, \zeta_j)$, $j = 1, 2$, where

$$\zeta_j = \sqrt{\lambda_j^2 + b^2}, \quad \lambda_1 = \alpha + i\beta, \quad \lambda_2 = -\lambda_1^* = -\alpha + i\beta. \tag{2.38}$$

Then (2.33) becomes

$$K^+(s,y)\begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{2}\begin{pmatrix}-i\zeta_j m(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}\\ bm(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}\end{pmatrix}e^{i\zeta_j(s+y)}$$
$$-\int_s^{+\infty} K^+(s,y')\sum_{j=1}^{2}\begin{pmatrix}-i\zeta_j m(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}\\ bm(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}\end{pmatrix}e^{i\zeta_j(y+y')}dy' = 0. \tag{2.39}$$

We choose a representation of $\begin{pmatrix}K_{12}^+\\K_{22}^+\end{pmatrix}$ as follows

$$\begin{pmatrix}K_{12}^+(s,y)\\K_{22}^+(s,y)\end{pmatrix} = \sum_{j=1}^{2}\begin{pmatrix}K_j(s)\\ \tilde{K}_j(s)\end{pmatrix}e^{i\zeta_j y}, \tag{2.40}$$

and the matrix $K^{\pm}$ writes as follows

$$K^+(s,y) = \begin{pmatrix}K_{11}^+(s,y) & K_{12}^+(s,y)\\ K_{21}^+(s,y) & K_{22}^+(s,y)\end{pmatrix} = \begin{pmatrix}K_{22}^+(s,y) & K_{12}^+(s,y)\\ -K_{12}^+(s,y) & K_{22}^+(s,y)\end{pmatrix}. \tag{2.41}$$

Now, substituting (2.40) and (2.41) into Eqs. (2.19), the potential q can be obtained as follows

$$q^2(s,t) = b^2 - 2\frac{dK_{22}^+(s,s)}{ds}. \tag{2.42}$$

Now, taking into account the Eq. (2.40) and (2.41) in (2.39), we obtain the system (for $j = 1, 2$)

$$\begin{aligned} K_j(s) + \sum_{n=1}^{2} K_n(s)\tilde{a}_j\frac{e^{i(\zeta_j+\zeta_n)s}}{i(\zeta_j+\zeta_n)} + \sum_{n=1}^{2}\tilde{K}_n(s)a_j\frac{e^{i(\zeta_j+\zeta_n)s}}{i(\zeta_j+\zeta_n)} + a_j e^{i\zeta_j s} = 0,\\ \tilde{K}_j(s) + \sum_{n=1}^{2}\tilde{K}_n(s)\tilde{a}_j\frac{e^{i(\zeta_j+\zeta_n)s}}{i(\zeta_j+\zeta_n)} - \sum_{n=1}^{2}K_n(s)a_j\frac{e^{i(\zeta_j+\zeta_n)s}}{i(\zeta_j+\zeta_n)} + \tilde{a}_j e^{i\zeta_j s} = 0, \end{aligned} \tag{2.43}$$

where

$$a_j = -i\zeta_j m(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}, \quad \tilde{a}_j = bm(\lambda_j)e^{4i\zeta_j(2\lambda_j^2-b^2)t}. \tag{2.44}$$

This system can be rewritten in a matricial form as

$$\begin{pmatrix} E & B \\ -B & E \end{pmatrix} \begin{pmatrix} \vec{K} \\ \vec{\tilde{K}} \end{pmatrix} = \begin{pmatrix} \vec{A} \\ \vec{\tilde{A}} \end{pmatrix}, \tag{2.45}$$

where

$$\vec{K} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \quad \vec{\tilde{K}} = \begin{pmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{pmatrix}, \quad \vec{A} = \begin{pmatrix} -a_1 e^{i\zeta_1 s} \\ -a_2 e^{i\zeta_2 s} \end{pmatrix}, \quad \vec{\tilde{A}} = \begin{pmatrix} -\tilde{a}_1 e^{i\zeta_1 s} \\ \tilde{a}_2 e^{i\zeta_2 s} \end{pmatrix},$$

$$E = (E_{jn}) = \delta_{jn} + \tilde{a}_j \frac{e^{i(\zeta_j+\zeta_n)s}}{i(\zeta_j+\zeta_n)}, \quad j,n = 1,2, \tag{2.46}$$

$$B = (B_{jn}) = a_j \frac{e^{i(\zeta_j+\zeta_n)s}}{i(\zeta_j+\zeta_n)}, \quad j,n = 1,2. \tag{2.47}$$

Defining the determinant of the coefficient matrix by

$$\Delta = \begin{vmatrix} E & B \\ -B & E \end{vmatrix}, \tag{2.48}$$

and using (2.40), we obtain that the solution of the system (2.43) is

$$K_{22}^{+}(s,s) = -\frac{1}{2}\frac{d}{ds}(\log \Delta). \tag{2.49}$$

From (2.42), we remember that the mKdV solution is written as $k^2(s,t) = b^2 - 2\frac{d}{ds}K_{22}(s,s)$. Inserting (2.49) into this relation, we finally obtain the expression of the mKdV solution as

$$k^2(s,t) = b^2 + \frac{d^2(\log \Delta)}{ds^2}. \tag{2.50}$$

2.2.2. *Matricial expression for the breather solution of mKdV with NVBC*

Now, our aim is to obtain from (2.50) an *explicit* expression of the focusing mKdV breather solution k. With this goal in mind, we rewrite the determinant (2.48) in the four-dimensional case (as it corresponds to the breather case) as a product of two simpler 2×2 determinants. Before that, we remark the following facts about the roots (ζ_1, ζ_2) and the temporal dependence of the coefficient S_{11} of the scattering matrix. First, recall that in the breather case with nonvanishing boundary condition, the roots of $S_{11}(\lambda, \zeta)$ are given by

$$\zeta_j = \sqrt{\lambda_j^2 + b^2}, \quad j = 1,2, \quad \lambda_1 = \alpha + i\beta, \quad \lambda_2 = -\lambda_1^* = -\alpha + i\beta. \tag{2.51}$$

We express $\zeta_i, i = 1,2$ as complex numbers as

$$\begin{aligned} \zeta_1 &= \tilde{\alpha}(\alpha,\beta,b) + i\tilde{\beta}(\alpha,\beta,b), \\ \zeta_2 &= -\tilde{\alpha}(\alpha,\beta,b) + i\tilde{\beta}(\alpha,\beta,b), \end{aligned} \tag{2.52}$$

where we only consider the region $\mathrm{Im}(\zeta) > 0$ so that

$$\tilde{\alpha}(\alpha,\beta,b) = \sqrt[4]{(\alpha^2-\beta^2+b^2)^2+4\alpha^2\beta^2}\cos\left(\frac{1}{2}\arctan\left(\frac{2\alpha\beta}{\alpha^2-\beta^2+b^2}\right)\right), \tag{2.53}$$

$$\tilde{\beta}(\alpha,\beta,b) = \sqrt[4]{(\alpha^2-\beta^2+b^2)^2+4\alpha^2\beta^2}\sin\left(\frac{1}{2}\arctan\left(\frac{2\alpha\beta}{\alpha^2-\beta^2+b^2}\right)\right), \tag{2.54}$$

which satisfy

$$\tilde{\alpha}(\alpha,\beta,0) = \sqrt{\alpha^2+\beta^2}\cos\left(\arctan\left(\frac{\beta}{\alpha}\right)\right) = \mathrm{Re}(\alpha+i\beta) = \alpha,$$
$$\tilde{\beta}(\alpha,\beta,0) = \sqrt{\alpha^2+\beta^2}\sin\left(\arctan\left(\frac{\beta}{\alpha}\right)\right) = \mathrm{Im}(\alpha+i\beta) = \beta.$$

Now, by using (2.52), the exponents of the temporal dependence given by (2.44) are rewritten as

$$\begin{aligned} 4i\zeta_1(2\lambda_1^2-b^2) &= -\tilde{\gamma}(\alpha,\beta,b)+i\tilde{\delta}(\alpha,\beta,b),\\ 4i\zeta_2(2\lambda_2^2-b^2) &= -\tilde{\gamma}(\alpha,\beta,b)-i\tilde{\delta}(\alpha,\beta,b), \end{aligned} \tag{2.55}$$

and, again, we only consider the region $\mathrm{Im}(\zeta) > 0$ so that

$$\tilde{\gamma}(\alpha,\beta,b) = 8\tilde{\beta}(3\tilde{\alpha}^2-\tilde{\beta}^2)-12b^2\tilde{\beta}, \tag{2.56}$$

$$\tilde{\delta}(\alpha,\beta,b) = 8\tilde{\alpha}(\tilde{\alpha}^2-3\tilde{\beta}^2)-12b^2\tilde{\alpha}, \tag{2.57}$$

which satisfy

$$\tilde{\gamma}(\alpha,\beta,0) = 8\beta(3\alpha^2-\beta^2) = \gamma, \tag{2.58}$$

$$\tilde{\delta}(\alpha,\beta,0) = 8\alpha(\alpha^2-3\beta^2) = \delta. \tag{2.59}$$

For the sake of simplicity and without loss of generality, we rename $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}$ as $\alpha,\beta,\gamma,\delta$, respectively. Now, we are able to rewrite the determinant Δ as a product of two simpler 2×2 determinants.

$$\begin{aligned} \Delta &= \det((I-iM-ibN)\cdot(I+iM-ibN))\\ &= \det(I-iM-ibN)\det(I+iM-ibN)\\ &= \mathrm{Re}\{\det(I-iM(s,t)-ibN(s,t))\}^2+\mathrm{Im}\{\det(I-iM(s,t)-ibN(s,t))\}^2, \end{aligned} \tag{2.60}$$

where

$$I \equiv \mathbb{1}_{2x2} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},$$

$$M(s,t) = \begin{pmatrix} \dfrac{-me^{2i(\alpha+i\beta)s+(-\gamma+i\delta)t}}{2} & \dfrac{i(\alpha+i\beta)me^{-2\beta s+(-\gamma+i\delta)t}}{2\beta}\\ \dfrac{i(-\alpha+i\beta)m^*e^{-2\beta s-(\gamma+i\delta)t}}{2\beta} & \dfrac{-m^*e^{2i(-\alpha+i\beta)s-(\gamma+i\delta)t}}{2} \end{pmatrix},$$

$$N(s,t) = \begin{pmatrix} \dfrac{me^{2i(\alpha+i\beta)s+(-\gamma+i\delta)t}}{2(\alpha+i\beta)} & \dfrac{-ime^{-2\beta s+(-\gamma+i\delta)t}}{2\beta} \\ \dfrac{-im^*e^{-2\beta s-(\gamma+i\delta)t}}{2\beta} & \dfrac{m^*e^{2i(-\alpha+i\beta)s-(\gamma+i\delta)t}}{2(-\alpha+i\beta)} \end{pmatrix}. \tag{2.61}$$

Then, substituting (2.60) in (2.50), and resorting to the identity

$$2\frac{\partial}{\partial s}\arctan\left(\frac{z(s)}{w(s)}\right) = i\frac{\partial}{\partial s}\log\left(\frac{w(s)-z(s)}{w(s)+z(s)}\right), \tag{2.62}$$

we get

$$\begin{aligned} k^2(s,t) &= b^2 + \frac{d^2}{ds^2}\log\Big[\mathrm{Re}\{\det(I - iM(s,t) - ibN(s,t))\}^2 \\ &\qquad + \mathrm{Im}\{\det(I - iM(s,t) - ibN(s,t))\}^2\Big] \\ &= \left(b + i\frac{\partial}{\partial s}\log\left[\frac{\mathrm{Re}\{\det(I - iM(s,t) - ibN(s,t))\} - i\,\mathrm{Im}\{\det(I - iM(s,t) - ibN(s,t))\}}{\mathrm{Re}\{\det(I - iM(s,t) - ibN(s,t))\} + i\,\mathrm{Im}\{\det(I - iM(s,t) - ibN(s,t))\}}\right]\right)^2 \\ &= \left(b + 2\frac{\partial}{\partial x}\arctan\left[\frac{\mathrm{Im}\{\det(I - iM(s,t) - ibN(s,t))\}}{\mathrm{Re}\{\det(I - iM(s,t) - ibN(s,t))\}}\right]\right)^2, \end{aligned} \tag{2.63}$$

which gives directly the matricial expression for the breather solution of the focusing mKdV with nonvanishing boundary value:

$$k(s,t) = b + 2\frac{\partial}{\partial s}\arctan\left[\frac{\mathrm{Im}\{\det(I - iM(s,t) - ibN(s,t))\}}{\mathrm{Re}\{\det(I - iM(s,t) - ibN(s,t))\}}\right]. \tag{2.64}$$

2.2.3. *Final expression for the breather solution with NVBC*

We first calculate the determinant in (2.64) (we assume $m = |m|e^{i\phi}$):

$$\begin{aligned} f(s,t) + ig(s,t) &\equiv \det(I - iM(s,t) - ibN(s,t)) \\ &= 1 + \left(1 - \frac{b^2}{\alpha^2+\beta^2}\right)\frac{\alpha^2|m|^2}{4\beta^2}e^{-4\beta s-2\gamma t} + i\frac{m}{2}e^{2i(\alpha+i\beta)s+(-\gamma+i\delta)t} \\ &\quad + i\frac{m^*}{2}e^{2i(-\alpha+i\beta)s-(\gamma+i\delta)t} + i\frac{b\alpha^2|m|^2}{2\beta(\alpha^2+\beta^2)}e^{-4\beta s-2\gamma t} \\ &\quad + \frac{b|m|}{\alpha^2+\beta^2}e^{-2\beta s-\gamma t}(\alpha\sin(2\alpha s+\delta t+\phi) - \beta\cos(2\alpha s+\delta t+\phi)), \end{aligned} \tag{2.65}$$

so that

$$g(s,t) = \frac{m}{2}e^{2i(\alpha+i\beta)s+(-\gamma+i\delta)t} + \frac{m^*}{2}e^{2i(-\alpha+i\beta)s-(\gamma+i\delta)t} + \frac{b\alpha^2|m|^2}{2\beta(\alpha^2+\beta^2))}e^{-4\beta s-2\gamma t}, \tag{2.66a}$$

$$f(s,t) = 1 + \left(1 - \frac{b^2}{\alpha^2+\beta^2}\right)\frac{\alpha^2|m|^2}{4\beta^2}e^{-4\beta s-2\gamma t} + \frac{b|m|}{\alpha^2+\beta^2}e^{-2\beta s-\gamma t}(\alpha\sin(2\alpha x+\delta t+\phi) - \beta\cos(2\alpha s+\delta t+\phi)). \tag{2.66b}$$

By defining $e^{-2\psi} = (1 - \frac{b^2}{\alpha^2+\beta^2})\frac{\alpha^2|m|^2}{4\beta^2}$, the expression of g and$f$ simplifies to

$$g(s,t) = 2e^{-2\beta s-\gamma t-\psi}\tilde{g}(s,t), \tag{2.67a}$$

$$f(s,t) = 2e^{-2\beta s-\gamma t-\psi}\tilde{f}(s,t), \tag{2.67b}$$

where

$$\tilde{g}(s,t) = \frac{\beta}{\alpha}\sqrt{\frac{\alpha^2+\beta^2}{\alpha^2+\beta^2-b^2}}\cos(2\alpha s+\delta t+\phi) + \frac{b\beta}{\alpha^2+\beta^2-b^2}(\cosh(2\beta s+\gamma t+\psi) - \sinh(2\beta s+\gamma t+\psi)), \tag{2.68a}$$

$$\tilde{f}(s,t) = \cosh(2\beta s+\gamma t+\psi) + \frac{b\beta}{\alpha\sqrt{\alpha^2+\beta^2-b^2}}\sin\left(2\alpha s+\delta t+\phi - \arctan\left(\frac{\beta}{\alpha}\right)\right). \tag{2.68b}$$

Hence, the explicit expression for the breather solution of the focusing mKdV with nonvanishing boundary value b (or *b-breather*) is:

$$k(s,t) = b + 2\frac{\partial}{\partial s}\arctan\left[\frac{\mathrm{Im}[f(s,t)+ig(s,t)]}{\mathrm{Re}[f(s,t)+ig(s,t)]}\right] = b + 2\frac{\partial}{\partial s}\arctan\left[\frac{\tilde{g}(s,t)}{\tilde{f}(s,t)}\right], \tag{2.69}$$

with $\tilde{f}, \tilde{g}$ given by (2.68). This completes the proof of Theorem 1.1 (up to translations in time and space).

Remark. Note that if we take the formal limit $b \to 0$, (2.69) reduces to the well known breather solution (1.2) of the focusing mKdV equation (see Wadati [20], up to translations in time and space).

Once proved the main result, it is possible to obtain the generalization with a nonvanishing boundary value at infinity of the *double pole* solution presented by M. Wadati *et al.* [17].

Proof of Corollary 1.2. First of all, perform the translation $\tilde{y} = y - \arctan(\frac{\beta}{\alpha})$, where $y = 2\alpha s + \delta t + \phi$ is the argument of the oscillatory functions in (2.68), and calculate the formal limit $\alpha \to 0$. Then, the generalization of the *double pole* solution presented by Wadati *et al.* [17] is given by the explicit formula

$$k(s,t) = b + 2\frac{\partial}{\partial s}\arctan\left(\frac{\tilde{G}(s,t)}{\tilde{F}(s,t)}\right), \tag{2.70}$$

where

$$\tilde{G}(s,t) = \frac{\beta(1-2\beta(s-6(2\beta^2+b^2)t))}{\sqrt{\beta^2-b^2}} + \frac{b\beta\,(\cosh(y)-\sinh(y))}{\beta^2-b^2},$$
$$\tilde{F}(s,t) = \cosh(y) + \frac{2b\beta(s-6(2\beta^2+b^2)t))}{\sqrt{\beta^2-b^2}}, \quad y = \beta(2s-4(2\beta^2+3b^2)t).$$

Taking into account the point-wise convergence of (2.70) to b when time goes to[c] $t \to +\infty$, it is possible to guess its asymptotic form at the mentioned limit, which is

$$k(s,t) \approx b + \frac{2\beta^2}{\sqrt{\beta^2+b^2}}\operatorname{sech}(y-\delta_+) - \frac{2\beta^2}{\sqrt{\beta^2+b^2}}\operatorname{sech}(y-\delta_-), \tag{2.71}$$

with

$$\delta_\pm = \beta\log\left(12\beta\left(\frac{\beta^2-b^2}{\beta^2+b^2}\right)^{\pm\frac{1}{2}}(4\beta^2+2b^2)t\right). \tag{2.72}$$

The phases $\delta_\pm$ determine how evolves the distance between the soliton and the antisoliton of (2.70). □

3. Summary and Remarks

In this paper we have obtained the breather solution of the focusing mKdV equation with nonvanishing boundary condition at infinity (2.69) by using the inverse scattering method for potentials that are not trivial at infinity as it was devised by Kawata and Inoue in [10]. As far as we know, it has not been reported before a systematic work on the obtention of this kind of breather solutions of mKdV under the ISM. These solutions play an important role in the construction of closed curves with localized perturbations, which evolve under the mKdV flow of curves (see [2]). We have also generalized the *double pole* solution found by Wadati *et al.* [17] to the case when it takes nontrivial values b at infinity. We have shown that even in this generalization, the distance between humps grows proportionally to $\log(t)$, as the formula (2.72) shows. Moreover, the associated closed curve to this (*double pole* curvature) solution is a closed curve with two loops. These two loops enclose asymptotically the same area, they point in- and outward respectively the closed curve, travel in the same direction and the distance between them grows slowly (proportionally to $\log(t)$ as t goes to ∞) (see [2]). We think that the asymptotic property (2.72) could be useful to check the accuracy of numerical methods (e.g. difference and pseudo spectral methods) for t big enough, as it was shown in Alejo, Gorria and Vega [3] when $b = 0$.

[c]The case $t \to -\infty$ is equivalent.

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