High-Frequency Asymptotics for the Helmholtz Equation in a Half-Plane

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HIGH-FREQUENCY ASYMPTOTICS FOR THE HELMHOLTZ EQUATION IN A HALF-PLANE

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Base on the integral representations of the solution being derived via Fokas' transform method, the high-frequency asymptotics for the solution of the Helmholtz equation, in a half-plane and subject to the Neumann condition is discussed. For the case of piecewise constant boundary data, full asymptotic expansions of the solution are obtained by using Watson's lemma and the method of steepest descents for definite integrals.

Keywords: High-frequency asymptotics; Fokas' transform method; method of steepest descents; Helmholtz equation; Neumann condition.

Mathematics Subject Classification: 35B40, 35C15, 35J05, 41A60

1. Introduction

There is huge mathematical and engineering interest in acoustic and electromagnetic wave scattering problems, driven by many applications such as modeling radar, sonar, acoustic noise barriers, atmospheric particle scattering, ultrasound and VLSI [4]. Many problems of scattering of time-harmonic acoustic or electromagnetic waves can be formulated as the Helmholtz equation, supplemented with appropriate boundary conditions. Much effort has been put into the development of efficient numerical schemes and approximate methods to deal with the Helmholtz equation involving high wave-numbers (i.e., high-frequencies) [1–3, 5, 9, 10, 13, 14, 16–18]. It is noted in [5, 15], that a question yet to be fully resolved, is to obtain accurate approximations of the solutions with a reasonable computational cost in the high-frequency case. Therefore, it seems desirable, and of difficulty, to consider the high-frequency asymptotics of the solutions to the Helmholtz equation. This is one of the main motives of the present investigation. Applying the theory of asymptotic analysis [6, 19, 22],

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one may achieve a high degree of accuracy with only a few leading terms in the asymptotic expansions of the solution being involved.

A novel flexible approach was introduced by Fokas in the late 1990s, to solve the initial/boundary value problems for various two-dimensional linear and integrable non-linear partial differential equations (PDEs). In particular, the method has been applied to the Laplace equation in a convex polygon and to other linear two-dimensional PDEs and nonlinear PDEs, including the Helmholtz and modified Helmholtz equations. Readers are referred to [8] for a systematic exposition of the method, its various applications and more references therein.

For linear PDEs, the new method of Fokas can be viewed as an extension of the classical transform method that produces the analogue of Green's integral representation (IR) in the transform (or Fourier) space. Indeed, whereas Green's IR given the solution as an integral involving the boundary values, the new IR given the solution as an integral involving transforms of the boundary values [21]. For some specific domains it is possible to remove either the Dirichlet or the Neumann data, thus providing an integral representation of the Neumann or the Dirichlet problem of the corresponding linear PDE. Various asymptotic methods for definite integrals might then be applied to derive asymptotic formulas of the solutions. As mentioned in [21], the solutions obtained by the Fokas method has two significant features: (i) The integrals can be deformed to involve exponentially decaying integrands. (ii) The expression is uniformly convergent at the boundary of the domain. These features give rise to both analytical and numerical advantages in comparison with classical methods [7]. It will be shown here and see [11, 12, 20] that the novel integral representations are suitable for the numerical evaluation and more convenient to extract the asymptotic behavior of the solution. This is a consequence of the fact that it is possible, using simple contour deformations in the complex k-plane, to obtain integrals involving integrands with a strong decay for large k.

The objective of the present paper is to consider the following Neumann boundary value problem of the Helmholtz equation in the upper half-plane Ω:

$$\triangle q(z, \bar{z}) + 4\beta^2 q(z, \bar{z}) = 0, \quad z \in \Omega, \tag{1.1}$$

$$\frac{\partial q}{\partial \mathbf{n}} = h(z), \quad z \in \Gamma, \tag{1.2}$$

where $\mathbf{n}$ is the outer normal vector, $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ is the usual Laplace operator, $z = x + iy$ and $\bar{z} = x - iy$, $\Gamma = \partial\Omega$ is the closed real line, h decay sufficiently at infinity (e.g. $h \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$).

To ensure the unique solvability, we assume that the following radiation condition at infinity is satisfied (cf. [21]):

$$\frac{\partial q}{\partial r} - 2i\beta q = O(r^{-3/2}), \quad q = O(r^{-1/2}), \quad \text{as}$$
$$z = re^{i\varphi} \in \Omega, \quad r \to +\infty. \tag{1.3}$$

As a first step, we given the integral representation for the solution of the Neumann boundary value problem (1.1)–(1.3) for general h derived by Fokas' transform method. This is actually a special case in [21, Chap. 4].

And then, we focus on the high-frequency asymptotics with respect to a specific Neumann data, namely,

$$\frac{\partial q}{\partial \mathbf{n}} = h(z) := \begin{cases} E, & z \in [a, b], \\ 0, & z \in \Gamma \backslash [a, b], \end{cases} \tag{1.4}$$

where $[a, b]$ refers to a finite interval and E is an arbitrary constant.

We need to give a note that the problem considered here is not a realistic scattering problem, but serves as an illustrative example of the asymptotic investigation via the Fokas method. We choose the illustrative example (1.1) and (1.3)–(1.4) to keep the derivation as non-technical as possible. The results can actually be generalized and may involve an oscillatory double integral. To deal with such kind of problems, one may need a higher-dimensional version of the steepest descents method, and it is rather difficult. We will focus on this research elsewhere.

The rest of the paper is arranged as follows: In Sec. 2, as a special case in [21, Chap. 4], the integral representation to an expression of the solution to the boundary value problem (1.1)–(1.3) is given. The integral representation of the solution to the boundary value problem (1.1), (1.3) and (1.4) is obtained accordingly. In Sec. 3, using the method of steepest descents and Watson's lemma (see, e.g., [6, 19, 22]), for large frequency/wave-number β, full asymptotic expansions of the solution for the Helmholtz equation are obtained. The main point-wise asymptotic results are stated in Theorem 1 in this section. Several remarks are given in the last section, Sec. 4.

2. The Integral Representation for the Helmholtz Equation

The method of Fokas has two basic ingredients: one is the integral representation and other is the global relation. The integral representations for linear PDEs obtained by the Fokas method can be derived in several ways including (a) spectral analysis of the appropriate Lax pair; (b) substituting particular IRs of the fundamental solution (obtained by classical transforms) into Green's IR. The interested reader is referred to [8, Chap. 11; 21] for derived in detail.

For the Neumann boundary value problem (1.1)–(1.3), this is actually a special case in [21, Chap. 4] with $\alpha = \pi/2, \gamma = 0$. From that, we have the following lemma.

Lemma 1. *Assume that the function $q(x, y)$ solves the Helmholtz equation (1.1) in the upper half z-plane Ω, and satisfies the Neumann boundary conditions (1.2) and the radiation condition (1.3), then the integral representation is valid:*

$$q(z, \bar{z}) = \frac{1}{2\pi} \int_L e^{i\beta(kz + \frac{\bar{z}}{k})} [N(-ik)] \frac{dk}{k}, \tag{2.1}$$

where

$$N(k) = \int_{-\infty}^{\infty} e^{\beta(k - \frac{1}{k})s} h(s) ds, \tag{2.2}$$

and the curves L, oriented as in Fig. 1, is defined as

$$L = \{|k| = 1, \pi \le \arg k \le 2\pi\} \cup \{|k| \le 1, \arg k = \pi\} \cup \{|k| \ge 1, \arg k = 0\} \tag{2.3}$$
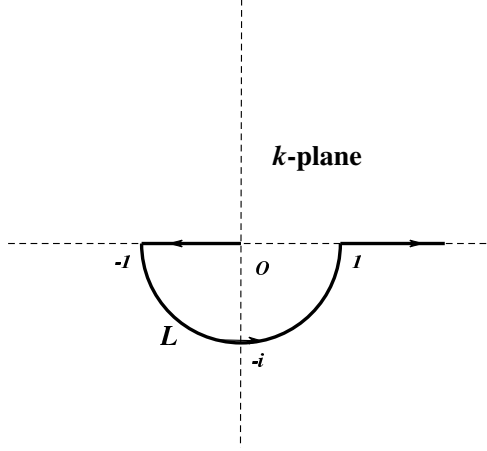
Fig. 1. The curves L.

Accordingly, when specify (1.2) to the piecewise constant Neumann data (1.4), we have the following lemma.

Lemma 2. *Assume that the function $q(x, y)$ solves the Helmholtz equation (1.1) in the upper half z-plane Ω, and satisfies the Neumann boundary conditions (1.4) and the radiation condition (1.3) at ∞, then the integral representation is valid:*

$$q(x, y) = \frac{E}{2\pi\beta i} \int_L e^{i\beta(kz + \frac{\bar{z}}{k})} [e^{-i\beta a(k + \frac{1}{k})} - e^{-i\beta b(k + \frac{1}{k})}] \frac{dk}{k^2 + 1}. \qquad (2.4)$$

3. Asymptotic Approximations

Our goal in this section is to study the asymptotic behavior of the solution to the Neumann problem of the Helmholtz equation, as the frequency approaches to infinity. This large-β asymptotic analysis will be based on the integral representation (2.4), and will be carried out by using the method of steepest descents.
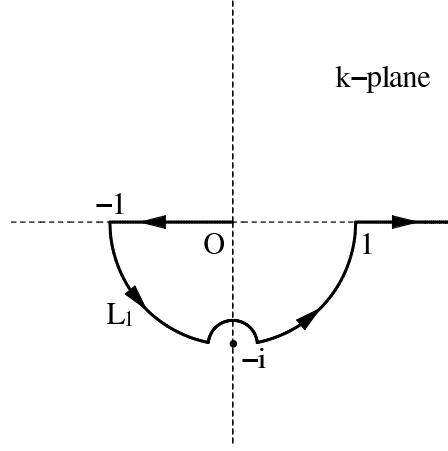
First we note that to separate the integrand in (2.4) into pieces, the curve L is dented to keep away from $k = -i$; cf. Fig. 2. We denote this dented curve by L_1. Therefore, $q(z)$ in (2.4) can be expressed as the sum of two integrals of the form

$$I(\beta) = \frac{E}{2\pi\beta i} \int_{L_1} e^{i\beta[k(z + A) + \frac{1}{k}\overline{(z + A)}]} \frac{dk}{k^2 + 1}, \qquad (3.1)$$

where $A = -a, -b$. We denote

$$\phi(k) = i\left[k(z + A) + \frac{1}{k}\overline{(z + A)}\right], \quad k = u + iv, \quad z = x + iy, \quad z + A = re^{i\theta}. \qquad (3.2)$$

The phase function $\phi(k)$ has a pair of saddle points $k = \pm e^{-i\theta}$ with $\theta \in (0, \pi)$, determined by $\phi'(k) = 0$, lying symmetrically on the unit circle.

Fig. 2. The dented curve L_1.

The steepest descent path passing through the saddle point k_+ is determined by requiring that

$$\operatorname{Im}\phi(k) = \operatorname{Im}\phi(k_+) : \left(1 + \frac{1}{u^2 + v^2}\right)[(x+A)u - yv] = 2\sqrt{(x+A)^2 + y^2},$$

and that $\operatorname{Re}\phi(k)$ decreases as k goes away from the saddle; cf. [6, 19, 22] for the definition and for basic background of the method of steepest descents. For $\theta \in (0, \pi/2)$, the path is the curve Γ connecting the origin and the infinite, as illustrated in Fig. 3. Furthermore, we need to decide how Γ turns to the origin and infinite. Writing in polar coordinates $k = |k|e^{i\varphi_1}$, one has $\Gamma : (|k| + 1/|k|)\cos(\varphi_1 + \theta) = 2$, while $\operatorname{Re}\phi = -r(|k| - 1/|k|)\sin(\varphi_1 + \theta) \to -\infty$ as k turns to 0 or ∞ along Γ. Hence $\varphi_1 = \arg k \to -\pi/2 - \theta \in (-\pi, -\pi/2)$ as k turns to 0 on Γ, and $\varphi_1 \to \pi/2 - \theta \in (0, \pi/2)$ as k turns to infinity on Γ. Hence we can deform the integration path L_1 in (3.1) into the oriented steepest descent path Γ by applying Cauchy's integral theorem. Hence we have

$$I(\beta) = \frac{E}{2\pi\beta i} \int_\Gamma e^{i\beta[k(z+A) + \frac{1}{k}\overline{(z+A)}]} \frac{dk}{k^2 + 1}, \quad \theta = \arg(z + A) \in (0, \pi/2). \tag{3.3}$$
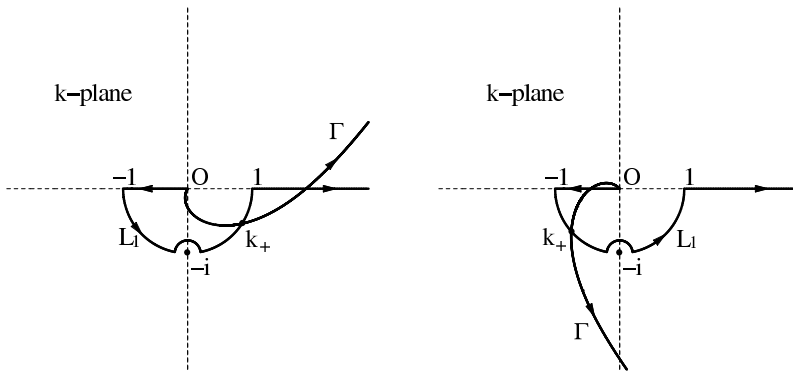

Fig. 3. The steepest descent path Γ into which the integration path L_1 deformed: $0 < \theta < \pi/2$ (left), $\pi/2 < \theta < \pi$ (right).

The situation when $\theta \in (\pi/2, \pi)$ can be treated similarly. This time we still denote by Γ the steepest descent path passing through k_+, illustrated also in Fig. 3. As $|k| \to 0$ on Γ, $\varphi_1 = \arg k \to -\pi/2 - \theta$, lying in $(-3\pi/2, -\pi)$, the second quadrant, while $|k| \to +\infty$, $k \in \Gamma$, $\varphi_1 \to \pi/2 - \theta \in (-\pi/2, 0)$, the fourth quadrant. Again the integration path L_1 can be deformed to the steepest descent path Γ since the integrand in (3.1) is exponentially small as $|k| \to 0$ for $\varphi_1 \in [-\pi/2 - \theta, -\pi]$, and as $|k| \to +\infty$ for $\varphi_1 \in [\pi/2 - \theta, 0]$. But this time a contribution from the simple pole at $k = -i$ should be picked up, namely, we have

$$I(\beta) = \frac{E}{2\beta i} e^{2i\beta r \sin\theta} + \frac{E}{2\pi\beta i} \int_\Gamma e^{i\beta[k(z+A) + \frac{1}{k}\overline{(z+A)}]} \frac{dk}{k^2 + 1},$$

$$\theta = \arg(z + A) \in (\pi/2, \pi).$$

$$(3.4)$$

To obtain the large-β asymptotic approximation of $I(\beta)$ in (3.1), recalling that $\theta = \arg(z + A) \in (0, \pi)$, we divide our discussion into three cases, namely (i) $0 < \theta < \pi/2$, (ii) $\theta = \pi/2$, and (iii) $\frac{\pi}{2} < \theta < \pi$. We use the method of steepest descents to derive asymptotic formulas; cf. [6, 19, 22].

Case (i) $0 < \theta < \pi/2$. In this case, we start from (3.3) and employ a standard argument of Watson's lemma. Firstly, we introduce a new integration variable

$$\tau := \phi(k_+) - \phi(k) = 2ir - ir\left(ke^{i\theta} + \frac{1}{k}e^{-i\theta}\right), \quad k \in \Gamma. \tag{3.5}$$

It is readily seen that $\tau \in [0, \infty)$ on the steepest descent path, and for each $\tau \in (0, \infty)$ there is a pair of points on Γ, say, k_1 and k_2, satisfying (3.5). To be precise, we may specify

$$k_1 = e^{-i\theta}\left[1 - \frac{\tau}{2ir} + \sqrt{\left(1 - \frac{\tau}{2ir}\right)^2 - 1}\right],$$

$$k_2 = e^{-i\theta}\left[1 - \frac{\tau}{2ir} - \sqrt{\left(1 - \frac{\tau}{2ir}\right)^2 - 1}\right]. \tag{3.6}$$

The branches are chosen such that $\sqrt{(1 - \frac{\tau}{2ir})^2 - 1} = e^{\frac{\pi}{4}i}\sqrt{\frac{\tau}{2r}} + O(\tau^{\frac{3}{2}})$ as $\tau \to 0^+$. Thus for $\tau \in (0, \infty)$, k_1 lies on the half steepest descent path connecting k_+ and ∞, while k_2 on the half joining k_+ and the origin. Now we have

$$I(\beta) = \frac{E}{2\pi\beta i} \int_\Gamma e^{\beta\phi(k)} \frac{dk}{k^2 + 1} = \frac{E}{2\pi\beta i} e^{2r\beta i} \int_0^\infty \Phi(\tau) e^{-\beta\tau} d\tau, \tag{3.7}$$

where

$$\Phi(\tau) = \frac{1}{k_1^2 + 1} \frac{dk_1}{d\tau} - \frac{1}{k_2^2 + 1} \frac{dk_2}{d\tau}, \tag{3.8}$$

which can be expanded into convergent series for small τ in the form

$$\Phi(\tau) = \tau^{-1/2} \sum_{l=0}^\infty c_l \tau^l, \quad \tau \in (0, \delta r), \tag{3.9}$$

where δ is a positive constant, and the coefficients $c_l = c_l(r, \theta)$ can be evaluated in view of (3.6), of which the first few are

$$c_0 = \frac{1+i}{2\sqrt{2}r^{1/2}\cos\theta}, \quad \text{and} \quad c_1 = \frac{-1+i}{\sqrt{2}r^{3/2}}\left(\frac{3}{16\cos\theta} - \frac{1}{2\cos^3\theta}\right). \tag{3.10}$$

Waston's lemma implies that all contribution to the asymptotic behavior of the integrals in (3.7) comes from the saddle $k = k_+$. Substituting (3.9) into (3.7) yields the full asymptotic expansion

$$I(\beta) = \frac{E}{2\pi\beta i}\int_\Gamma e^{\beta\phi(k)}\frac{dk}{k^2+1} \sim \frac{E}{2\pi\beta i}e^{2r\beta i}\sum_{l=0}^\infty \frac{c_l\Gamma(l+1/2)}{\beta^{l+1/2}}, \quad \beta \to +\infty, \tag{3.11}$$

where $\Gamma(t)$ is the usual Gamma function, $\Gamma(l+\frac{1}{2}) = (l-\frac{1}{2})(l-\frac{3}{2})\cdots\frac{1}{2}\sqrt{\pi}$.

Case (ii) $\theta = \pi/2$. From (3.1) and (3.2), in this case, we have

$$I(\beta) = \frac{E}{2\pi\beta i}\int_{L_1} e^{-\beta r(k-\frac{1}{k})}\frac{dk}{k^2+1}; \tag{3.12}$$

cf. Fig. 3 for the integration path L_1, dented at $k = -i$. The situation could be complicated since we have a pole of the integrand and a saddle of the phase function, both locate at $k = -i$. Denoting by Γ_ε the half circle centered at $k = -i$, where ε is the radius. Then we observe that

$$J := \int_{L_1\backslash\Gamma_\varepsilon} e^{-\beta r(k-\frac{1}{k})}\frac{dk}{k^2+1} = 0.$$

This fact can be seen by making a change of variable $k \to -1/k$, which maps $L_1\backslash\Gamma_\varepsilon$ onto itself, with orientation being reversed, and keeps the integrand unchanged, thus implies $J = -J$. Hence we need only pick up the contribution from the circular part around $k = -i$, and we have

$$I(\beta) = \frac{E}{4\beta i}e^{2i\beta r} = \frac{E}{4\beta i}e^{2y\beta i}. \tag{3.13}$$

Case (iii) $\frac{\pi}{2} < \theta < \pi$. The derivation is now based on (3.4). Similar to the discussion in Case (i), one obtains

$$I(\beta) \sim \frac{E}{2\beta i}e^{2y\beta i} + \frac{E}{2\pi\beta i}e^{2r\beta i}\sum_{l=0}^\infty \frac{c_l\Gamma(l+1/2)}{\beta^{l+1/2}}, \quad \beta \to +\infty, \tag{3.14}$$

where the coefficients $c_l = c_l(r, \theta)$ are still given in (3.8)–(3.10).

We are now in a position to derive the asymptotic approximation for $q(x, y)$ in (2.4) for fixed $z = x + iy \in \Omega$. We denote

$$z - a = r_1 e^{i\theta_1}, \quad z - b = r_2 e^{i\theta_2}. \tag{3.15}$$

Obviously we have $0 < \theta_1 < \theta_2 < \pi$. We divide the half z-plane Ω into three regions, namely, *I*, *II* and *III*, defined by $\text{Re } z \in (-\infty, a)$, $\text{Re } z \in (a, b)$ and $\text{Re } z \in (b, \infty)$, respectively; cf. Fig. 4. Then it is clear that when $z \in III$, $\theta_j \in (0, \pi/2)$ for $j = 1, 2$; for $z \in II$, $\theta_2 \in (\pi/2, \pi)$ and $\theta_1 \in (0, \pi/2)$; and for $z \in I$, $\theta_1, \theta_2 \in (\pi/2, \pi)$. On the boundaries, when $\text{Re } z = b$, $\theta_2 = \pi/2$, while $\theta_1 = \pi/2$ as $\text{Re } z = a$.
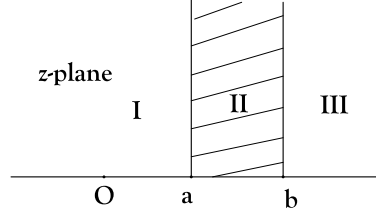
Fig. 4. The regions *I*, *II* and *III* in the complex half *z*-plane.

Combining (3.11), (3.13) and (3.14) with (2.4) yields the following theorem.

Theorem 1. *For large positive β (i.e., high frequency and equivalently, large wavenumbers), the asymptotic expansions of the solution $q(x, y)$ to the boundary value problem (1.1), (1.3) and (1.4) of the Helmholtz equation hold.*

$$q(x,y) \sim \frac{E}{2\pi i}\sum_{j=1}^{2}(-1)^{j+1}e^{2r_j\beta i}\sum_{l=0}^{\infty}\frac{c_l(r_j,\theta_j)\Gamma(l+\frac{1}{2})}{\beta^{l+\frac{3}{2}}} + \begin{cases} 0, & z \in I; \\ -\dfrac{E}{2\beta i}e^{2y\beta i}, & z \in II; \\ 0, & z \in III, \end{cases} \quad (3.16)$$

as $\beta \to +\infty$, where r_j and θ_j are defined in (3.15). Two exceptional cases are

$$q(x,y) \sim -\frac{E}{2\pi i}e^{2r_2\beta i}\sum_{l=0}^{\infty}\frac{c_l(r_2,\theta_2)\Gamma(l+\frac{1}{2})}{\beta^{l+\frac{3}{2}}} - \frac{E}{4\beta i}e^{2y\beta i}, \quad (3.17)$$

as $\beta \to +\infty$ and $\operatorname{Re} z = a$, and

$$q(x,y) \sim \frac{E}{2\pi i}e^{2r_1\beta i}\sum_{l=0}^{\infty}\frac{c_l(r_1,\theta_1)\Gamma(l+\frac{1}{2})}{\beta^{l+\frac{3}{2}}} - \frac{E}{4\beta i}e^{2y\beta i}, \quad (3.18)$$

as $\beta \to +\infty$ and $\operatorname{Re} z = b$.

Picking up the main contribution to the asymptotic behavior, for fixed $z = x + iy \in \Omega$, we have the following piecewise approximation:

$$q(x,y) = O(\beta^{-3/2}) + \begin{cases} 0, & \operatorname{Re} z \in (-\infty, a), \text{ i.e., } z \in I; \\ -\dfrac{E}{4\beta i}e^{2y\beta i}, & \operatorname{Re} z = a; \\ -\dfrac{E}{2\beta i}e^{2y\beta i}, & \operatorname{Re} z \in (a, b), \text{ i.e., } z \in II; \\ -\dfrac{E}{4\beta i}e^{2y\beta i}, & \operatorname{Re} z = b; \\ 0, & \operatorname{Re} z \in (b, \infty), \text{ i.e., } z \in III. \end{cases} \quad (3.19)$$

One interesting phenomenon is that the exponential terms switch on and off as z crosses the vertical half lines $\operatorname{Re} z = a$ and $\operatorname{Re} z = b$.

4. Conclusions

A new transform method for solving linear and certain nonlinear PDEs was introduced in the late 1990's by Fokas [8]. For linear PDEs it can be viewed as an extension of the classical transform method that produces the analogue of Green's Integral Representation (IR) in the transform (or Fourier) space [21].

In the present paper, based on the integral representations of the solution to the Helmholtz equation, in a half-plane and subject to Neumann condition obtained by Spence [21], full high-frequency asymptotic expansions for a specific Neumann condition (piecewise constant boundary data) are derived by using Watson's lemma and the method of steepest descents for definite integrals. The asymptotic expansion (3.16) in Theorem 1 is uniform in compact subsets of I, II and III, but not in neighborhoods of the boundaries. The piecewise results have jumps along two vertical lines. This can be explained mathematically, the reason is that there is a coalescing of saddle point with a simple pole in the integrand.

Acknowledgments

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