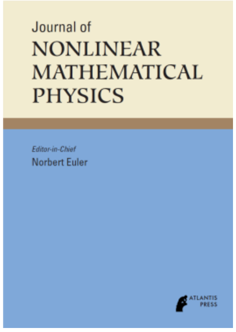

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Xiaorui Hu, Yong Chen

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A DIRECT PROCEDURE ON THE INTEGRABILITY OF NONISOSPECTRAL AND VARIABLE-COEFFICIENT MKdV EQUATION

XIAORUI HU
Software College, East China Normal University
No. 3663 North Zhongshan Road
Shanghai, 200062, P. R. China
lansexiaoer@163.com

YONG CHEN
Shanghai Key Laboratory of Trustworthy Computing
East China Normal University, No. 3663 North Zhongshan Road
Shanghai, 200062, P. R. China
Software College, East China Normal University
No. 3663 North Zhongshan Road, Shanghai, P. R. China
ychen@sei.ecnu.edu.cn

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An elementary and systematic method based on binary Bell polynomials is applied to nonisospectral and variable-coefficient MKdV (vcMKdV) equation. The bilinear representation, bilinear Bäcklund transformation, Lax pair and infinite local conservation laws are obtained step by step, without too much clever guesswork.

Keywords: Binary Bell polynomials; nonisospectral MKdV equation; bilinear Bäcklund transformation; infinite conservation law.

Mathematics Subject Classification: 11T06, 37K10, 37K35

1. Introduction

It is well known that integrability has been an old but significant topic in the field of differential equations. The integrability features of a significant set of soliton equations can be characterized by Hirota bilinear form, Lax pair, infinite symmetry, Painlevé test, Hamiltonian structure, Bäcklund transformation, and so on. Among these direct methods, the bilinear method [7, 8] developed by Hirota has proved particular powerful. On the one hand, once an equation is written in the Hirota bilinear form, many kinds of solutions may be constructed, such as multi-soliton solutions, quasi-periodic wave solutions and rational solutions. On the other hand, a noteworthy feature of bilinear method is that it allows the construction of Bäcklund transformation [3, 17] which may lead to the underlying Lax pairs [2, 14], infinite conservation laws [9, 12, 15], etc. In Hirota's method, one should firstly look for the new dependent variables upon which the original differential equation could be transformed into more tractable bilinear equations. Then, based on the given bilinear form, one may derive the Bäcklund transformation by applying well chosen "exchange formulas" to a particular related ansaze. But, as yet, whether the selection of the Hirota variables or the application of "exchange formulas", there is no general rule and it often needs much more tricks. Recently, Lambert *et al.* proposed a systematic and lucid approach based on the Bell polynomials to study the major integrability of soliton equations [1, 6, 10, 11]. By this alternative method, which only considers dimensional analysis and elementary combinatorics, the corresponding Hirota bilinear form, the bilinear Bäcklund transformation and Lax pair can be straight away disclosed and justified step by step. Lambert *et al.* have successfully applied this method to many constant coefficient nonlinear evolution equations.

In recent years, much more attention has been paid to soliton equations with nonisospectral and variable coefficients, whose spectral parameters are time-dependent. In contrast to constant coefficients cases, these nonisospectral and variable coefficients equations may present a better description for the real phenomena in physical and engineering fields. On the other hand, the difficulty for investigating the complete integrability of these equations would be added. More recently, Fan extends binary Bell polynomials to learn about nonisospectral and variable coefficient KdV equation (vcKdV) and systematically studies its complete integrability [4, 5].

As well known, every solution of the MKdV equation leads, via Miura's transformation, to a solution of the KdV equation, but the converse is not true, so do nonisospectral vcKdV equation and vcMKdV equation. In this paper, we want to apply this method to study the integrability of nonisospectral vcMKdV equation and only start from the equation itself, without any transformation related to other known equations. Here, in contrast to nonisospectral vcKdV equation [5], there are two points we need to emphasize: one is the number of mixing variables exceeds the number of Hirota functions, the other is the derived four-filed constraint system appears not in the conserved form. After dealing with above two problems, we propose a two-field Hirota system, a four-filed bilinear Bäcklund transformation, two-component Lax systems and infinite local conservation laws of nonisospectral vcMKdV equation.

In this paper, we focus on constructing Hirota bilinear form, bilinear Bäcklund transformation, Lax pair and conservation laws of nonisospectral vcMKdV equation. Firstly, with the aid of a dimensionless transformation, we convert the original equation into $\mathcal{Y}$-polynomials, from which the equivalent bilinear form can be written out directly. Then based on the obtained $\mathcal{Y}$-polynomials and starting from a natural four-field ansatz of dependent variables, the bilinear Bäcklund transformation is presented via the identities of mixing variables, which leads to a Lax pair naturally by virtue of the given transformation. Lastly, by combining four $\mathcal{Y}$-polynomials which occurs in the Bäcklund transformation to two equations, the infinite local conservation laws are given.

The paper is arranged as follows. In Sec. 2, we briefly recall the main properties of the multi-dimensional binary Bell polynomials that will be used in this paper. In Sec. 3, based on binary Bell polynomials, the complete integrability of nonisospectral vcMKdV equation is studied. Section 4 is the conclusion.

2. Multi-Dimensional Binary Bell Polynomials

Recently, Lambert *et al.* proposed a generalization of the Bell polynomials, which is called multi-dimensional Bell polynomials or Y-polynomials, defined as follows [1, 6]

$$Y_{n_1x_1,\ldots,n_lx_l}(f) \equiv Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) = e^{-f}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}e^{f}, \tag{2.1}$$

where $f \equiv f(x_1,\ldots,x_n)$ is a C^∞ function with n independent variables and we denote

$$f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l}(f).$$

The most important multi-dimensional binary Bell polynomials, i.e. $\mathcal{Y}$-polynomials, which are related to the standard Hirota's bilinear operator is only a two-field generalization of above Y-polynomials [6]:

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) = Y_{n_1x_1,\ldots,n_lx_l}(f)\Big|_{f_{r_1x_1,\ldots,r_lx_l}=\begin{cases} v_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l=\text{odd},\\ w_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l=\text{even}.\end{cases}} \tag{2.2}$$

To illustrate above generalized formulation, we take $f = f(x,t)$ for example. According to formulation (2.1), one can easily get

$$\begin{gathered} Y_x(f) = f_x, \quad Y_{2x}(f) = f_{2x} + f_x^2,\\ Y_{x,t}(f) = f_{xt} + f_xf_t, \quad Y_{3x}(f) = f_{3x} + 3f_xf_{2x} + f_x^3, \ldots. \end{gathered} \tag{2.3}$$

The corresponding binary Bell polynomials are

$$\begin{gathered} \mathcal{Y}_x(v) = v_x, \quad \mathcal{Y}_{2x}(v,w) = w_{2x} + v_x^2,\\ \mathcal{Y}_{x,t}(v,w) = w_{xt} + v_xv_t, \quad \mathcal{Y}_{3x}(v,w) = v_{3x} + 3v_xw_{2x} + v_x^3, \ldots. \end{gathered} \tag{2.4}$$

On the one hand, the link between $\mathcal{Y}$-polynomials (2.2) and the standard Hirota expressions is given by the follow identity

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}\left(v = \ln\frac{F}{G}, w = \ln FG\right) = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G, \tag{2.5}$$

in which $n_1 + n_2 + \cdots + n_l \geq 1$ and

$$\begin{aligned} &D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G\\ &\quad\equiv (\partial_{x_1}-\partial_{x_1'})^{n_1}\cdots(\partial_{x_l}-\partial_{x_l'})^{n_l}F(x_1,\ldots,x_l)G(x_1',\ldots,x_l')|_{x_1'=x_1,\ldots,x_l'=x_l}. \end{aligned} \tag{2.6}$$

On the other hand, the $\mathcal{Y}$-polynomials (2.2) can be linearizabled by the following formula

$$\begin{gathered} \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v=v, w=v+q)|_{v=\ln\psi} = \psi^{-1}\sum_{s_1=0}^{n_1}\cdots\sum_{s_l=0}^{n_l}\binom{n_1}{s_1}\cdots\binom{n_l}{s_l}\\ \mathcal{Y}_{s_1x_1,\ldots,s_lx_l}(0,q)\cdot\psi_{(n_1-s_1)x_1,\ldots,(n_l-s_l)x_l}, \end{gathered} \tag{2.7}$$

which is the basis for constructing Lax pair. Specially, it is denoted that

$$P_{s_1x_1,\ldots,s_lx_l}(q) = \mathcal{Y}_{s_1x_1,\ldots,s_lx_l}(0,q), \tag{2.8}$$

where $s_1 + \cdots + s_l =$ even and the odd parts included are zero for $v = 0$. Formula (2.8) restricts the Bell recipe to even part partitions:

$$P_0(q) = 1, \quad P_{2x}(q) = q_{2x}, \quad P_{x,t}(q) = q_{xt}, \quad P_{4x}(q) = q_{4x} + 3q_{2x}^2, \ldots. \tag{2.9}$$

According to formula (2.5) and definition (2.8), it is clear that

$$P_{s_1x_1,\ldots,s_lx_l}(q = 2\ln G) = G^{-2}D_{x_1}^{s_1}\cdots D_{x_l}^{s_l}G\cdot G \quad (s_1 + \cdots + s_l = \text{even}). \tag{2.10}$$

The binary Bell polynomials have been found to play an important role in the characterization of bilinear Bäcklund transformations, Lax pairs and infinite conversion laws. About more detailed applications of binary Bell polynomials, one can consult [1, 4–6, 10, 11].

3. Integrability of Nonisospectral vcMKdV Equation

In [5], Fan applied the binary Bell polynomials to consider nonisospectral vcKdV equation

$$q_t + h_1(q_{3x} - 6qq_x) + 4h_2q_x - h_3(2q + xq_x) = 0, \tag{3.1}$$

where $h_1 = h_1(t), h_2 = h_2(t)$ and $h_3 = h_3(t)$ are arbitrary functions of t.

In this section, we will try to extend "mixing variable procedure" to deal with nonisospectral vcMKdV equation, reading

$$u_t + h_1(u_{3x} - 6u^2u_x) + 4h_2u_x - h_3(u + xu_x) = 0. \tag{3.2}$$

Both Eqs. (3.1) and (3.2) play an important role in mathematical physics field. In [13], Lou and Ruan derived the infinite conservation laws of Eqs. (3.1) and (3.2) by means of Miura's method. Symmetry reductions and soliton-like solutions for Eq. (3.2) were proposed and a transformation between Eq. (3.2) and the standard MKdV equation was also found by Yan [16]. Although the known conclusions of the standard MKdV equation can be used to Eq. (3.2) by means of the given transformation, the obtained results may be complicated and indirect. Here, without relying on any additional transformation and only starting from Eq. (3.2) itself, by the direct and unifying scheme based on binary Bell polynomials, we propose its Hirota bilinear form, bilinear Bäcklund transformation, Lax pair and infinite conservation laws.

3.1. *Hirota bilinear representation*

Introducing a dimensionless field v by setting $u = cv_x(c \equiv c(t))$ into Eq. (3.2), after one integration with respect to x, it leads to

$$v_t + h_1(v_{3x} - 2c^2v_x^3) + 4h_2v_x - xh_3v_x + (\ln c)_tv = 0. \tag{3.3}$$

One can see that the derivatives about v in Eq. (3.3) are all odd. Hence, in terms of the introduction of another auxiliary dependent variable w whose derivatives show even, Eq. (3.3) can be rewritten as

$$v_t + h_1(v_{3x} + 3w_{2x}v_x + v_x^3) - 3h_1 v_x\left(w_{2x} + \frac{2c^2+1}{3}v_x^2\right) + (4h_2 - xh_3)v_x + (\ln c)_t v = 0. \tag{3.4}$$

By choosing $c^2 = 1$, we get

$$\mathcal{Y}_t(v) + h_1\mathcal{Y}_{3x}(v,w) + (4h_2 - xh_3)\mathcal{Y}_x(v) - 3h_1\mathcal{Y}_x(v)\mathcal{Y}_{2x}(v,w) = 0. \tag{3.5}$$

Then Eq. (3.5) can be decoupled into the system

$$\begin{cases} \mathcal{Y}_{2x}(v,w) = 0, \\ \mathcal{Y}_t(v) + h_1\mathcal{Y}_{3x}(v,w) + (4h_2 - xh_3)\mathcal{Y}_x(v) = 0. \end{cases} \tag{3.6}$$

On account of Eq. (2.5), one can easily get the bilinear equivalent of system (3.6) via $v = \ln\frac{F}{G}$ and $w = \ln FG$:

$$\begin{cases} D_x^2 F \cdot G = 0, \\ [D_t + h_1 D_x^3 + (4h_2 - xh_3)D_x]F \cdot G = 0. \end{cases} \tag{3.7}$$

Starting from the bilinear form (3.7), one can easily obtain its multi-soliton solutions by regular perturbation method. Here we do not consider them.

In contrast with one-field Hirota bilinear form of Eq. (3.1) in the form of P-polynomials [5], Eq. (3.2) admits a two-field Hirota bilinear representation expressed in $\mathcal{Y}$-polynomials.

3.2. *Bilinear Bäcklund transformation and Lax pair*

Next, with the help of binary Bell polynomials, we will search for the Bäcklund transformation in the bilinear form and the corresponding spectral formulation of Eq. (3.2) based on Eqs. (3.6).

Let (v', w') and (v, w) be two different pairs of solutions of Eqs. (3.6), respectively. Then the corresponding Hirota variables (F', G') are introduced by $v' = \ln\frac{F'}{G'}$ and $w' = \ln F'G'$ similar to the $v = \ln\frac{F}{G}$ and $w = \ln FG$. To proceed, we have the following mixing variables

$$\begin{cases} v_1 = \ln\dfrac{G'}{G}, \quad v_2 = \ln\dfrac{F'}{F}, \quad v_3 = \ln\dfrac{F'}{G}, \quad v_4 = \ln\dfrac{G'}{F}, \\ w_1 = \ln G'G, \quad w_2 = \ln F'F, \quad w_3 = \ln F'G, \quad w_4 = \ln G'F, \end{cases} \tag{3.8}$$

and they lead to the following identities

$$\begin{cases} v' - v = v_2 - v_1 = w_3 - w_4, \quad v' + v = v_3 - v_4 = w_2 - w_1, \\ w' - w = v_1 + v_2 = v_3 + v_4, \quad w' + w = w_1 + w_2 = w_3 + w_4. \end{cases} \tag{3.9}$$

A natural four-field ansatz for Bäcklund transformation is starting from the system (3.6), reading

$$\begin{cases} \mathcal{Y}_{2x}(v',w') - \mathcal{Y}_{2x}(v,w) = 0, \\ \mathcal{Y}_t(v') - \mathcal{Y}_t(v) + h_1[\mathcal{Y}_{3x}(v',w') - \mathcal{Y}_{3x}(v,w)] + (4h_2 - xh_3)[\mathcal{Y}_x(v') - \mathcal{Y}_x(v)] = 0, \end{cases} \tag{3.10}$$

i.e.

$$(w' - w)_{2x} + (v' - v)_x (v' + v)_x = 0 \tag{3.11}$$

and

$$(v' - v)_t + h_1 \left\{ (v' - v)_{3x} + \frac{3}{2}(v' + v)_x (w' - w)_{2x} + \frac{1}{4}(v' - v)_x [6(w' + w)_{2x} \right.$$
$$\left. + 3(v'_x + v_x)^2 + (v'_x - v_x)^2] \right\} + (4h_2 - xh_3)(v' - v)_x = 0. \tag{3.12}$$

In terms of the mixing variables $(v_i, w_i)(i = 1, \ldots, 4)$ in (3.9), Eqs. (3.11) and (3.12) can be rewritten as follows

$$(v_3 + v_4)_{2x} + (v_2 - v_1)_x (v_3 - v_4)_x = 0, \tag{3.13}$$

$$(v_2 - v_1)_t + h_1 \left\{ (v_2 - v_1)_{3x} + \frac{3}{2}(v_3 - v_4)_x (v_3 + v_4)_{2x} + \frac{1}{4}(v_2 - v_1)_x \right.$$
$$\left. \times [6(w_1 + w_2)_{2x} + 3(v_{3x} - v_{4x})^2 + (v_{2x} - v_{1x})^2] \right\} + (4h_2 - xh_3)(v_2 - v_1)_x = 0. \tag{3.14}$$

Keeping Eq. (3.13) under observation, one can decouple it into the following bilinear constraints

$$\mathcal{Y}_x(v_3) = \lambda e^{v_1 - v_2}, \quad \mathcal{Y}_x(v_4) = \mu e^{v_2 - v_1}. \quad (\lambda \equiv \lambda(t), \mu \equiv \mu(t).) \tag{3.15}$$

We impose Eq. (3.15) to the second condition (3.14) and it yields to

$$\mathcal{Y}_t(v_2) - \mathcal{Y}_t(v_1) + h_1[\mathcal{Y}_{3x}(v_2, w_2) - \mathcal{Y}_{3x}(v_1, w_1)]$$
$$+ (4h_2 - xh_3)[\mathcal{Y}_x(v_2) - \mathcal{Y}_x(v_1)] + h_1 R = 0, \tag{3.16}$$

with

$$R = \frac{1}{4}(v_2 - v_1)_x (v_{2x} - v_{1x})^2 + (v_{1,x}^3 - v_{2,x}^3) + \frac{3}{2}(v_2 - v_1)_x (w_1 + w_2)_{2x}$$
$$+ 3w_{1,2x} v_{1,x} - 3w_{2,2x} v_{2,x} + \frac{3}{2}(v_3 - v_4)_x (v_3 + v_4)_{2x} = 3\lambda\mu(v_{2,x} - v_{1,x}).$$

It is clear that Eq. (3.16) can produce the following two $\mathcal{Y}$-basis decompositions

$$\mathcal{Y}_t(v_1) + h_1 \mathcal{Y}_{3x}(v_1, w_1) + (3\lambda\mu h_1 + 4h_2 - xh_3)\mathcal{Y}_x(v_1) = 0, \tag{3.17}$$

$$\mathcal{Y}_t(v_2) + h_1 \mathcal{Y}_{3x}(v_2, w_2) + (3\lambda\mu h_1 + 4h_2 - xh_3)\mathcal{Y}_x(v_2) = 0. \tag{3.18}$$

As a result, by means of relations (2.5) and (3.8), Eqs. (3.15), (3.17) and (3.18) are cast into the following bilinear equivalents

$$D_x F' \cdot G = \lambda F G', \quad D_x G' \cdot F = \mu F' G,$$
$$[D_t + h_1 D_x^3 + (3\lambda\mu h_1 + 4h_2 - xh_3) D_x] G' \cdot G = 0, \tag{3.19}$$
$$[D_t + h_1 D_x^3 + (3\lambda\mu h_1 + 4h_2 - xh_3) D_x] F' \cdot F = 0,$$

which constitutes a bilinear Bäcklund transformation for Eq. (3.2).

Next, starting from Eqs. (3.15), (3.17) and (3.18) again, we look for the Lax pair of Eq. (3.2). Noticing that there exist four pairs of mixing variables $(v_i, w_i)(i = 1, \dots, 4)$ in obtained $\mathcal{Y}$-expressions, we should firstly eliminate two pairs of them through the following relations

$$v = \ln \frac{F}{G} = v_3 - v_2 = w_2 - w_3 = v_1 - v_4 = w_4 - w_1. \tag{3.20}$$

Here we make use of $v_3 = v + v_2$ and $v_4 = v_1 - v$ to eliminate the variables with indices 3 and 4 in Eqs. (3.15) and get

$$\mathcal{Y}_x(v_2) = \lambda e^{v_1 - v_2} - v_x, \quad \mathcal{Y}_x(v_1) = \mu e^{v_2 - v_1} + v_x. \tag{3.21}$$

By virtue of $w_1 = v_1 + w - v, w_2 = v_2 + w + v$ and $v_i = \ln \psi_i (i = 1, 2)$, on account of formula (2.7), the system consisting of Eqs. (3.17), (3.18) and (3.21) is linearized into a two component system:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = M \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = N \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{cases} \lambda_t = \lambda h_3, \\ \mu_t = \mu h_3. \end{cases} \tag{3.22}$$

where

$$M = \begin{pmatrix} v_x & \mu \\ \lambda & -v_x \end{pmatrix}, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

$$\begin{aligned} A &= -\mathcal{Y}_{3x}(v, w)h_1 - (4\lambda\mu h_1 + 4h_2 - xh_3)\mathcal{Y}_x(v), \\ B &= -\mu[(\mathcal{Y}_{2x}(v, w) + 2P_{2x}(w - v) + 4\lambda\mu)h_1 + 4h_2 - xh_3], \\ C &= -\lambda[(\mathcal{Y}_{2x}(v, w) + 2P_{2x}(w + v) + 4\lambda\mu)h_1 + 4h_2 - xh_3]. \end{aligned} \tag{3.23}$$

It is easy to check that the integrability condition $M_t - N_x + [M, N] = 0$, with the non-isospectral conditions about λ and μ in (3.22), is subject to two derivative equations of system (3.6) with respect to x.

By setting $\mu = \lambda$ and $\mathcal{Y}_{2x}(v, w) = 0$ to eliminate w_{2x}, then replacing v_x by $\pm u$ in (3.22), one can find a more natural Lax pair of Eq. (3.2) under the transformation $\phi_1 = \frac{1}{2}(\psi_1 + \psi_2)$ and $\phi_2 = \frac{1}{2}(\psi_1 - \psi_2)$:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & \pm u \\ \pm u & -\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \lambda_t = h_3\lambda,$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

with

$$\begin{aligned} A_1 &= -4h_1\lambda^3 + (2u^2 h_1 - 4h_2 + xh_3)\lambda, \\ B_1 &= \mp[4uh_1\lambda^2 + 2u_x h_1 \lambda + (u_{2x} - 2u^3)h_1 + (4h_2 - xh_3)u], \\ C_1 &= \mp[4uh_1\lambda^2 - 2u_x h_1 \lambda + (u_{2x} - 2u^3)h_1 + (4h_2 - xh_3)u], \end{aligned}$$

where the integrability condition is the exactly vcMKDV equation.

3.3. *Infinite conservation laws*

As another application of binary Bell polynomials, Fan derived the infinite conservation laws of soliton equations through decoupling binary Bell polynomials into a Riccati type equation and a divergence type equation [4, 5].

Here, by setting $\mu = \lambda$, Eqs. (3.17), (3.18) and (3.21) can be rewritten as

$$\begin{aligned} &\mathcal{Y}_x(v_1) = \lambda e^{v_2 - v_1} + v_x, \quad \mathcal{Y}_x(v_2) = \lambda e^{v_1 - v_2} - v_x, \\ &\mathcal{Y}_t(v_1) + h_1 \mathcal{Y}_{3x}(v_1, w_1) + (3\lambda^2 h_1 + 4h_2 - x h_3)\mathcal{Y}_x(v_1) = 0, \\ &\mathcal{Y}_t(v_2) + h_1 \mathcal{Y}_{3x}(v_2, w_2) + (3\lambda^2 h_1 + 4h_2 - x h_3)\mathcal{Y}_x(v_2) = 0. \end{aligned} \tag{3.24}$$

To proceed, one should reduce above system to two equations which are only related to one pair of (v_i, w_i) $(i = 1, 2)$. From the first equation in (3.24), one can obtain

$$v_2 = v_1 + \ln(v_{1,x} - v_x) - \ln \lambda. \tag{3.25}$$

Substituting Eq. (3.25) into the second equation of (3.24), it leads to

$$v_{1,x} + v_{1,2x} - v_{2x} - v_x^2 - \lambda^2 = 0. \tag{3.26}$$

The last two equations of (3.24) which are obtained by decoupling Eq. (3.16) can be combined to an equation, after taking advantage of (3.25) and the relations

$$w_2 = w + v + v_2, \quad w_1 = w - v + v_2, \quad w_{2x} = -v_x^2, \quad \lambda_t = h_3 \lambda, \tag{3.27}$$

reading

$$\begin{aligned} &\partial_t v_{1,x} + \{[v_{1,4x} + 3v_{1,3x} v_{1,x} + 3v_{1,2x}^2 + 3(v_{1,x}^2 - v_x^2 + \lambda^2) v_{1,2x} + 3v_{2x} v_{1,x}^2 \\ &\quad - 3(v_{3x} + 2v_{2x} v_x) v_{1,x} - v_{4x} - 3v_{2x}^2 + 3(v_x^2 - \lambda^2) v_{2x}] h_1 \\ &\quad + 4(v_{1,2x} - v_{2x}) h_2 - (x v_{1,2x} + v_{1,x} - x v_{2x} - v_x) h_3 - v_{xt}\} = 0. \end{aligned} \tag{3.28}$$

Here, for Eq. (3.2), its spectrum parameter λ is dependent of time t. By solving the nonisospectral condition $\lambda_t = h_3 \lambda$, one can obtain

$$\lambda = C e^{\int h_3 dt} \equiv \epsilon g, \tag{3.29}$$

where $C = \epsilon$ is an arbitrary constant and there is $g \equiv g(t) = e^{\int h_3 dt}$ with $g_t = h_3 g$.

In the following, we introduce η by setting

$$v_{1,x} = \eta + \epsilon g. \tag{3.30}$$

Substituting (3.29) and (3.30) into Eq. (3.26) produces

$$\eta^2 + \eta_x + 2\epsilon g \eta - v_{2x} - v_x^2 = 0. \tag{3.31}$$

One can see that Eq. (3.28) is not in our prefer conserved form. We may try to express the brace part of Eq. (3.28) into the form of x-derivative. Luckily, Eq. (3.26) with (3.29), (3.30) and $g_t = h_3 g$ can be used to express Eq. (3.28) as follows

$$\begin{aligned} &-(\eta - v_x)_t + \partial_x[2h_1\eta^3 + (6h_1\epsilon g - 2h_1 v_x)\eta^2 + 2h_1\eta_x\eta + (x h_3 - 4h_1 v_x \epsilon g - 4h_2)\eta \\ &\quad - 2h_1 v_x \eta_x + 2h_1 \epsilon g \eta_x + 4h_2 v_x - x h_3 v_x] = 0. \end{aligned} \tag{3.32}$$

Now it is clear that the conservation laws have been hinted in Eq. (3.32). Next, we expand η in series form of ϵ

$$\eta = \sum_{n=1}^{\infty} I_n(q, q_x, q_y, \ldots)\epsilon^{-n}. \tag{3.33}$$

Firstly, the substitution of (3.33) into Eq. (3.31) would lead to

$$\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n-1} I_k I_{n-k}\right)\epsilon^{-n} + \sum_{n=1}^{\infty} I_{n,x}\epsilon^{-n} + 2gI_1 + 2g\sum_{n=1}^{\infty} I_{n+1}\epsilon^{-n} - (v_{2x} + v_x^2) = 0. \tag{3.34}$$

Collecting the coefficients for power of ϵ and equating them with zero, we then get the recursion for I_n

$$\begin{aligned} I_1 &= \frac{1}{2g}(v_{2x} + v_x^2) = \frac{1}{2g}(\pm u_x + u^2), \\ I_2 &= -\frac{1}{4g^2}(v_{3x} + 2v_x v_{2x}) = -\frac{1}{4g^2}(\pm u_{2x} + 2uu_x), \\ I_{n+1} &= -\frac{1}{2g}\left(I_{n,x} + \sum_{k=1}^{n-1} I_k I_{n-k}\right), \quad (n \geq 2). \end{aligned} \tag{3.35}$$

It is found that Eq. (3.32) embodies the explicit expression of conservation laws

$$I_{n,t} + F_{n,x} = 0, \quad (n = 1, 2, \ldots). \tag{3.36}$$

Applying (3.33) to Eq. (3.32) and collecting the coefficients of each order of ϵ, we have

$$\epsilon^0 : \partial_t v_x + \partial_x[2h_1 g I_{1,x} - 4h_1 v_x g I_1 + 4h_2 v_x - x h_3 v_x] = 0; \tag{3.37}$$

$$\begin{aligned} F_1 = \frac{1}{2g}[&(\pm u_{3x} + 2uu_{2x} - u_x^2 \mp 6u^2 u_x - 3u^4)h_1 \\ &+ 4(\pm u_x + u^2)h_2 - x(\pm u_x + u^2)h_3]; \end{aligned} \tag{3.38}$$

$$\begin{aligned} F_2 = \frac{1}{4g^2}[&(\mp u_{4x} - 2uu_{3x} \pm 6u^2 u_{2x} + 12u^3 u_x \pm 12uu_x^2)h_1 \\ &- 4(\pm u_{2x} + 2uu_x)h_2 + x(\pm u_{2x} + 2uu_x)h_3]; \end{aligned} \tag{3.39}$$

$$\begin{aligned} -F_n = 2h_1 &\sum_{i+j+k=n} I_i I_j I_k + 6h_1 g\sum_{k=1}^{n} I_k I_{n+1-k} - 2h_1 v_x \sum_{k=1}^{n-1} I_k I_{n-k} \\ &+ 2h_1\sum_{k=1}^{n-1} I_{k,x} I_{n-k} + (xh_3 - 4h_2)I_n - 4h_1 v_x g I_{n+1} \\ &- 2h_1 v_x I_{n,x} + 2h_1 g I_{n+1,x}, \quad (n \geq 3). \end{aligned} \tag{3.40}$$

Substituting I_1 into Eq. (3.37) and replacing v_x by $\pm u$, Eq. (3.37) can be rewritten as

$$\partial_t u + \partial_x[(u_{2x} - 2u^3)h_1 + 4uh_2 - xuh_3] = 0.$$

We denote

$$I_0 = u, \quad F_0 = (u_{2x} - 2u^3)h_1 + 4uh_2 - xuh_3.$$

Then $I_{n,t} + F_{n,x} = 0\,(n = 0, 1, 2, \ldots)$ constitute the infinite conservation law equations of nonisospectral vcMKdV equation (3.2), the first of which is exactly Eq. (3.2). One can see that the conserved densities I_n and the fluxes F_n are all local.

Remark. One can see that similar to the case of standard constant coefficient MKdV equation, the conservation density $I_{2k}(k = 1, 2, \ldots)$ expressed in (3.35) of vcMKdV equation must be written as the derivative terms with respect to x while the odd ones I_{2k+1} $(k = 0, 1, 2, \ldots)$ cannot. Hence, here, the even ones of I_n except I_0 are all trivial but the odd ones with I_0 are nontrivial. Whence the nontrivial odd terms $I_{2k+1}(k = 0, 1, 2, \ldots)$ with I_0 given in (3.35) are written out by the recursion formula, it is shown that they are independent.

4. Conclusion

In conclusion, the nonisospectral vcMKdV equation (3.2) is completely integrable. Here, without relying on any transformation related to other known equations and only starting from the original equation itself, we directly derive its bilinear representation, bilinear Bäcklund transformation, Lax pair and infinite local conservation laws with the help of binary Bell polynomials. The obtained integrability properties enable one to seek for kinds of explicit solutions ulteriorly.

For the nonisospectral vcMKdV equation, its Hirota bilinear form cannot be expressed as a single combination of P-polynomials like nonisospectral vcKdV equation. One need to introduce another auxiliary dependent variable to decouple it into two equations about $\mathcal{Y}$-polynomials and obtain a two-field Hirota bilinear system. Due to that, in the following derivation of bilinear Bäcklund transformation, there occur four pairs of mixing variables $(v_i, w_i)(i = 1, \ldots, 4)$ which are more complex than that in nonisospectral vcKdV case only including one pair of mixing variables. With the help of these mixing variables, we obtain a four-filed bilinear Bäcklund transformation and two-component Lax systems. Furthermore, although the derived four-filed constraint system seems not easy to be rewritten in the conserved form, after some flexible combinations and one appropriate integration with respect to x, the infinite conservation laws of nonisospectral vcMKdV equation are found and show local.

Note that applying the binary Bell polynomials method to investigate kinds of integrability of nonlinear system is novel and explanatory. How to relate this algebra approach to more integrability properties, such as symmetries, Hamiltonian structure, etc, is worth to be considered. We also believe that binary Bell polynomials can be extended to discrete equations.

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