International Journal of Computational Intelligence Systems
Vol. **14(1)**, 2021, *pp.* 701–714
DOI: https://doi.org/10.2991/ijcis.d.201230.001; ISSN: 1875-6891; eISSN: 1875-6883
https://www.atlantis-press.com/journals/ijcis/

Research Article

Granule Description of Object (Attribute)-Oriented Linguistic Concept Lattice Based on Dominance Relation

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ARTICLE INFO

Article History
Received 15 Jul 2020
Accepted 06 Dec 2020

Keywords
Concept lattice
Object (attribute)-oriented linguistic concept lattice
Dominance relation
Granular computing
Linguistic term set

ABSTRACT

Concept lattice, as an effective tool for knowledge acquisition and data analysis, has been successfully used in many fields. Aiming at the problem of groups for uncertain information in the linguistic environment, this paper mainly focuses on the granule description with linguistic concept lattice from the two aspects of object-oriented and attribute-oriented. Specifically, we present an object-oriented linguistic concept lattice to describe the linguistic dominance relation between attributes, the corresponding attribute dominant class is obtained meanwhile. Furthermore, we devote to constructing an object-oriented linguistic granular concept lattice by using the equivalence relations on the set of objects, combining the idea of granular computing. In order to deal with the linguistic information under the different importance of objects, the notion of attribute-oriented linguistic concept lattice based on object dominance relation is put forward. What's more, we classify the attribute set so as to generate the attribute-oriented linguistic granular concepts with respect to a set of attributes. Besides, the effectiveness of the object-oriented linguistic concept lattice, object-oriented linguistic granular concept lattice, attribute-oriented linguistic concept lattice, attribute-oriented linguistic granular concept lattice is illustrated by the comparative analysis with other methods.

1. INTRODUCTION

Formal concept analysis, also called concept lattice, was first proposed by German mathematician Wille [1]. As the basis of formal concept analysis, the formal context can express the binary relation between objects and attributes while concept lattice can be constructed to visualize the hidden information in the formal context by defining the formal concept shorted as a concept. Over the past 20 years, scholars have conducted multi-angle and deep research on concept lattice, the research results of concept lattice are becoming more and more abundant [2–7]. Formal concept analysis with the advantage of providing a strong formal theory, has been widely used in various fields, such as information retrieval [8], semantic web [9,10], cognitive computing [11–13], three-way decisions [14,15] and knowledge engineering [16].

Considering that formal concept analysis and rough set are highly complementary, and rough set can analyze incomplete information such as inaccuracy, inconsistency and incompleteness, many scholars have studied the issue of knowledge discovery by combining the two theories to research and compare from multiple aspects [17,18]. Dntsch and Gediga [19,20] defined the model-style operator based on the binary relation, then the attribute-oriented concept lattice was established by the upper approximation operator. Yao [21,22] introduced rough set theory into the concept lattice and proposed object-oriented concept lattice. Based on this, the research of object (attribute)-oriented concept lattices has been mainly reflected in construction algorithms, attribute reduction and rule extraction. In order to reduce the time complexity, Ma *et al.* [23] presented a novel approach to acquire object-oriented concept lattice by discussing the properties of layered extension sets. Shao *et al.* [24] studied the reconstruction of attribute (object)-oriented concept lattice after coarsening attribute granularity based on granularity tree by using zoom algorithms. Qin *et al.* [25] adopted some equivalence relations and discernibility attributes to improve the attribute reduction and rule acquisition of formal decision context based on object (attribute)-oriented concept lattices. Kumar [26] focused on drawing the object and attribute based m-polar fuzzy concept lattice using the projection operator to discover some useful patterns for solving the particular issue of a given problem. Inspired by the above research, Zhi *et al.* [27] introduced three-way decision ideas to describe several connections among three-way concept analysis models where the dual concepts could cater the specific applications of international import and export transactions.

In recent years, with the rapid development of granular computing (GrC), the research of formal concept analysis has also advanced to a higher level. The core idea of GrC comes from Zadeh's fuzzy information granules [28]. Qian *et al.* [29,30] extended the classic rough set model to a multi-granularity rough set, and obtained a rough set characterized by multiple upper and lower approximations of

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equivalence relations. Subsequently, GrC has attracted more and more attention in different fields [31–33], among them, the combination of GrC and formal concept analysis is particularly emerging. Xu and Li [34] discussed a novel GrC method of machine learning by using formal concept description of information granules. Belohlavek *et al.* [35] established a simple formal structure to provide users with different levels of attribute granularity. In order to reduce the time complexity and save the storage space in formal concept analysis, Mi *et al.* [36] further studied a simple discernibility matrix combining the object concept and the attribute concept with granular concept. Long *et al.* [37] proposed a dynamic updating method of attribute-induced three-way granular concept by deleting multiple objects and attributes in the formal context, which improved the efficiency and flexibility of construction for three-way concepts.

Many aspects of different activities in the real world cannot be described in a quantitative information, but rather in a qualitative one, for example, evaluating a school's educational resources may be "very good", "quite high" in quality of teaching, "a bit remote" in environment. Therefore, it is necessary to study the approaches with linguistic information. Affected by the research of computing with words (CWWs) [38,39], many related linguistic term models have been proposed [40–44]. Xu *et al.* [45] studied linguistic information from the perspective of formal concept analysis, which realized the combination of multiple formal sub-contexts to support the lattice-valued concept lattices. In order to solve the problems of reasoning under uncertainty, Zou *et al.* [46,47] proposed fuzzy linguistic concept lattice and knowledge reduction method via introducing linguistic terms into formal concept analysis. However, fuzzy linguistic concept lattice cannot solve some specific problems. For example, for the Mathematics, Physics and Chemistry scores of several students, it is easy to get the common level of these students based on the fuzzy linguistic concept lattice while we cannot get the highest scores of these students. Meanwhile, it is difficult to obtain information such as which subject is better for these students or which student is better. In this way, how to deal with linguistic information with dominance relation becomes particularly important [48–50]. In view that granular computing can effectively process a large amount of complex information, this article focuses on studying granular concepts with dominance relation in fuzzy linguistic formal context. We construct object (attribute)-oriented linguistic concept lattice and granular concept lattice to discuss the information of group linguistic value with dominance relation. It will further enrich the concept lattice theory of linguistic formal concept analysis.

The remainder of this paper is organized as follows: In order to make the paper self-contained, basic notions of ordered information system (IS), object (attribute)-oriented concept lattice and linguistic term set are briefly reviewed in Section 2. In Section 3, we propose an object-oriented linguistic concept lattice and provide the construction algorithm of object-oriented linguistic concept lattice. The object-oriented linguistic granular concept lattice which is granularization of object-oriented linguistic concept lattice based on equivalence relation is presented in Section 4. In Section 5, we propose the notions of attribute-oriented linguistic concept based on object dominance relation. The related attribute granulation method for linguistic formal context based on object dominance relation is presented in Section 6. In Section 7, we make some comparative analysis to illustrate the effectiveness of the proposed methods in this paper. It is concluded with a summary and the prospects for further research.

2. PRELIMINARIES

In this section, we briefly review the concepts of ordered IS, formal context and linguistic term set, seeing the following definitions for more details.

An IS is a quadruple $I = (U, AT, V, f)$, where U is a finite nonempty set of objects and AT is a finite nonempty set of attributes, $V = \bigcup_{a\in AT} V_a$ and V_a is a domain of attribute a, $f : U \times AT \to V$ is a total function such that $f(x, a) \in V_a$ for every $a \in AT$, $x \in U$, called an information function.

Definition 1. [48] An IS is called an ordered information system (OIS) if all condition attributes are criterions.

It is assumed that the domain of a criterion $a \in AT$ is completely preordered by an outranking relation $\geqslant_a$; $x \geqslant_a y$ means that x is at least as good as (outranks) y with respect to criterion a. In the following, without any loss of generality, we consider a condition criterion having a numerial domain, that is, $x \geqslant_a y \Leftrightarrow f(x, a) \geqslant f(y, a)$ (according to increasing preference) or $x \geqslant_a y \Leftrightarrow f(x, a) \leqslant f(y, a)$ (according to decreasing preference), where $a \in AT$, $x, y \in U$. For a subset of attributes $A \subseteq AT$, we define $x \geqslant_A y \Leftrightarrow \forall a \in A, x \geqslant_a y$. That is, x is at least as good as y with respect to all attributes in A. In general, the domain of the condition criterion may be also discrete, but the preference order between its values has to be provided.

The dominance relation that identifies granules of knowledge is defined as follows:

For a given OIS, we say that x dominates y with respect to $A \subseteq AT$ if $x \geqslant_A y$, and denoted by $xR_A^{\geqslant}y$. Namely,

$$R_A^{\geqslant} = \{(y, x) \in U \times U | y \geqslant_A x\} \tag{1}$$

If $(y, x) \in R_A^{\geqslant}$, then y dominates x with respect to A.

Given $A \subseteq AT$ and $A = A_1 \bigcup A_2$, where attributes set A_1 according to increasing preference, A_2 according to decreasing preference. The granules of knowledge induced by the dominance relation $R_A^{\geqslant}$ are the set of objects dominating x.

$$\begin{aligned}[x]_A^{\geqslant} = \{y \in U | & f(y, a_1) \geqslant f(x, a_1)(\forall a_1 \in A_1), \\ & f(y, a_2) \leqslant f(x, a_2)(\forall a_2 \in A_2)\} \\ = \{y \in U | & (y, x) \in R_A^{\geqslant}\}\end{aligned} \tag{2}$$

and the set of objects dominated by x.

$$\begin{aligned}[x]_A^{\leqslant} = \{y \in U | & f(y, a_1) \leqslant f(x, a_1)(\forall a_1 \in A_1), \\ & f(y, a_2) \geqslant f(x, a_2)(\forall a_2 \in A_2)\} \\ = \{y \in U | & (x, y) \in R_A^{\geqslant}\}\end{aligned} \tag{3}$$

which are called the A-dominating set and the A-dominated set with respect to $x \in U$, respectively.

Definition 2. [22] A triple $K = (G, M, I)$ is called a formal context, if G (the collection of objects) and M (the collection of attributes) are two finite nonempty sets and I (subset of cartesian product $G \times M$) represents the binary relation between G and M, where 1

denotes that object has attribute and 0 denotes that object does not have the attribute.

Definition 3. [19,22] Let $K = (G, M, I)$ be a formal context, $X \subseteq G$, $B \subseteq M$. Operators $\uparrow$ and $\downarrow$ are defined as follows:

$\uparrow,\downarrow, 2^G \to 2^M$:

$$X^{\uparrow} = XI = \{b \in M | \exists x \in G, (xIb \wedge (x \in X))\} \quad (4)$$
$$X^{\downarrow} = \{b \in M | \forall x \in G, (xIb \Rightarrow (x \in X))\}$$

$\uparrow,\downarrow, 2^M \to 2^G$:

$$B^{\uparrow} = IB = \{x \in G | \exists b \in M, (xIb \wedge (b \in B))\} \quad (5)$$
$$B^{\downarrow} = \{x \in G | \forall b \in M, (xIb \Rightarrow (b \in B))\}$$

Definition 4. [19,22] Let $K = (G, M, I)$ be a formal context. A pair $(X, B)(X \subseteq G, B \subseteq M)$ is called an object-oriented concept if $X = B^{\uparrow}$ and $X^{\downarrow} = B$. Similarly, (X, B) is called attribute-oriented concept if $X^{\uparrow} = B$ and $X = B^{\downarrow}$. X and B are called the extent and intent of the object (attribute)-oriented concept respectively.

Let (X_1, B_1) and (X_2, B_2) be the object(attribute)-oriented concepts. The partial order $\leqslant$ among them is defined by

$$(X_1, B_1) \leqslant (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \subseteq B_2) \quad (6)$$

Let $K = (G, M, I)$ be a formal context. Obviously, the complete set of object (attribute)-oriented concepts forms a complete lattice denoted by $L_O(G, M, I)$ and $L_A(G, M, I)$ respectively.

In real life, most people prefer to use "good," "very good," "very poor" and other linguistic values to describe things because of the ambiguous environment. In order to deal with the uncertain problems with linguistic values, Herrera *et al.* [51,52] proposed a linguistic term set as follows:

Let $S = \{s_i | i = 0, 1, \ldots, g\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics:

1. Order relation: $s_i > s_j$, if $i > j$.
2. Negation operator: $Neg(s_i) = s_j$, where $j = g - i$.
3. Maximization operator: $\max\{s_i, s_j\} = s_i$, if $i \geqslant j$.
4. Minimization operator: $\min\{s_i, s_j\} = s_j$, if $i \geqslant j$.

For example, a set of seven terms of S could be given as S = {s_0 = extremely poor, s_1 = very poor, $_2$ = poor, s_3 = medium, s_4 = good, s_5 = very good, s_6 = extremely good}.

3. THE CONSTRUCTION OF OBJECT-ORIENTED LINGUISTIC CONCEPT LATTICE

3.1. Object-Oriented Linguistic Concept Lattice

In formal concept analysis, the traditional formal context is mostly composed of 0 and 1. However, due to the uncertainty and complexity of the environment, the data obtained is sometimes linguistic values. In order to avoid the problem of missing information in the process of linguistic information conversion, this section studies the object-oriented linguistic concept lattice based on formal context with linguistic terms, which is under dominance relation from the perspective of attribute importance.

Definition 5. [46] Let (U, A, S) be a fuzzy linguistic formal context, where $U = \{x_i | i \in 1, 2, \ldots, n\}$ is a nonempty finite set of objects, $A = \{a_j | j \in 1, 2, \ldots, m\}$ is a nonempty finite set of attributes and $S = \{s_l | l \in 0, 1, \ldots, g\}$ is a relation between U and A, which is a subset of Cartesian product $U \times A$, such that $S(x, a) = s_l$.

Let (U, A, S) be a fuzzy linguistic formal context, for any $x \in U$, $a_j, a_k \in A (j, k \in 1, 2, \ldots, m)$, we can establish an attribute partial order relation $\leqslant_x$, in which $a_k \leqslant_x a_j$ indicates that the linguistic values corresponding to attribute a_j dominant a_k with respect to object x, i.e., $S(x, a_k) \leqslant S(x, a_j)$. For simplicity, the partial order relation on $a_k \leqslant_x a_j$ is abbreviated as a_{jk}.

Therefore, when there exists a_{jk} in a formal context, if $S(x, a_k) \leqslant S(x, a_j)$, we use $S(x, a_j)$ to represent the relation between x and a_{jk}. If $S(x, a_k) > S(x, a_j)$, we use " $\Diamond$" to represent that the linguistic values of attribute a_j don't dominant attribute a_k with respect to object x which we won't consider in this article. In this way, we can get a formal context based on dominance relation with linguistic values.

Definition 6. A triple $(U, M, S^{\Diamond})$ is called a linguistic formal context based on attribute dominance relation, where $M = \{M_j | j \in 1, 2, \ldots, m\}$, $M_j = \{a_{jk} | k \in 1, 2, \ldots, m\}$ denotes a set of attributes with a partial order relation, $S^{\Diamond}$ is a relation between U and M, such that $S^{\Diamond}(x, a_{jk}) = \{s_l \in S \text{ or } \Diamond | x \in U, a_{jk} \in M\}$.

The linguistic formal context based on attribute dominance relation divides the attributes into a number of small blocks $M_j (j = 1, 2, \ldots, m)$ with dominance relation. Each small block represents the importance of the attribute a_j compared with other attributes. Therefore, the attributes of the same block are comparable while not all of the blocks have a dominance relation.

Theorem 1. *Let $(U, M, S^{\Diamond})$ be a linguistic formal context based on attribute dominance relation. For any $a_{jk} \in M$, if $j = k$, then the linguistic values under a_{jk} is the same as a_j, i.e., $S^{\Diamond}(x, a_{jk}) = S(x, a_j)$ with respect to object x.*

Definition 7. Let $(U, M, S^{\Diamond})$ be a linguistic formal context based on attribute dominance relation, for any $a \in M$, the attribute dominant class $[a]_U^{\leqslant}$ can be defined as follows

$$[a]_U^{\leqslant} = \{b \in M | S(x, a) \leqslant S(x, b), \forall x \in U\} \quad (7)$$

In practical applications, it is very general to study whether objects possess attributes in common in a formal context. For example,

teachers often need to be familiar with the knowledge of each student and focus on the explanation through the common learning problems of students in schools. However, if we want to hold a group subject competition for a class, we need to organize the performance of each member of the group, whether there are enough outstanding students in each subject to compete with others. In other words, not only the individual abilities of the students are very important, but also the comprehensive abilities of the group. As mentioned above, we give the following Example 1.

Example 1. Consider the grades of four students in a class, the details about these students are shown with a fuzzy linguistic formal context (U, A, S) in Table 1, where $U = \{x_1, x_2, x_3, x_4\}$ is the set of students, $A = \{a_1 =$ Mathematics, $a_2 =$ Physics, $a_3 =$ Chemistry, $a_4 =$ Biology$\}$ is the set of subjects and $S = \{s_0 =$ failing, $s_1 =$ passing, $s_2 =$ medium, $s_3 =$ good, $s_4 =$ excellent$\}$ is the relation between U and A. We can obtain a linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$, combining the dominance relation with respect to the attributes as shown in Table 2, where $M = \{M_1, M_2, M_3, M_4\} = \{\{a_{11}, a_{12}, a_{13}, a_{14}\}, \{a_{21}, a_{22}, a_{23}, a_{24}\}, \{a_{31}, a_{32}, a_{33}, a_{34}\}, \{a_{41}, a_{42}, a_{43}, a_{44}\}\}$.

From Table 2, we can obviously find some attributes with the same linguistic values. In order to simplify the operation, a new updated linguistic formal context based on attribute dominance relation is shown in Table 3.

We can conclude from Tables 1–3, $S(x_1, a_1) = s_3$ means that student x_1 has a good Mathematics (a_1) score in the fuzzy linguistic formal context (U, A, S). Meanwhile, in the linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$, $S^{\diamondsuit}(x_3, a_{12}) = s_2$ represents that student x_3 has a medium Mathematics score and his Mathematics score is better than his Physics (a_2) score while $S^{\diamondsuit}(x_2, a_{12}) = \diamondsuit$ means that the Mathematics score of student x_2 is worse than his Physics score. For the object set $E = \{x_1, x_2, x_3\}$ in Table 2, we have $[a_{44}]_E^{\preccurlyeq} = \{a_{11}, a_{14}, a_{32}, a_{33}, a_{34}, a_{44}\}$, i.e., their Mathematics and Chemistry (a_3) scores are generally greater than or equal to their Biological (a_4) scores for students x_1, x_2 and x_3. It can be found that the linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$ can not only express the information in fuzzy linguistic formal context (U, A, S) but also describe the partial order relation between attributes.

For the linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$, we record U_S as the set which is the subset of U defined in S.

Definition 8. Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, $\lambda \in S$ be a credibility threshold, for any $X \subseteq U_S$, $B \subseteq M$, $s \in S$, the operators $'$, $*$ and $\triangle$ can be defined as follows:

$$x' = \{a \in M | \forall x \in U, S^{\diamondsuit}(x, a) \geqslant \lambda\}, \tag{8}$$

Table 1 Fuzzy linguistic formal context (U, A, S).

U/A	a_1	a_2	a_3	a_4
x_1	s_3	s_2	s_2	s_1
x_2	s_1	s_3	s_4	s_0
x_3	s_2	s_0	s_3	s_2
x_4	s_3	s_2	s_4	s_3

$$a' = \left\{\frac{s}{x} \in U_S | \forall a \in M, S^{\diamondsuit}(x, a) \geqslant \lambda\right\}, \tag{9}$$

$$X^* = \{a \in M | a' \subseteq X\}, \tag{10}$$

$$B^{\triangle} = \vee\left\{\frac{s}{x} \in U_S | x' \cap B \neq \emptyset\right\}. \tag{11}$$

Remark 1. X^* is the set of attributes such that linguistic values corresponding to any object that satisfy one of them is necessarily in X, $B^{\triangle}$ denotes a set of maximum linguistic values between attributes and objects, which possesses at least one attribute in B.

Definition 9. Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation. For any $X \subseteq U_S$, $B \subseteq M$, if there exist $X^* = B$ and $B^{\triangle} = X$, then (X, B) is called object-oriented linguistic concept based on attribute dominance relation, abbreviated as object-oriented linguistic concept. X and B are called the extent and intent of the object-oriented linguistic concept, respectively.

The set of all object-oriented linguistic concepts of $(U, M, S^{\diamondsuit})$ is denoted by $LL_O(U, M, S^{\diamondsuit})$, which called the object-oriented linguistic concept lattice of $(U, M, S^{\diamondsuit})$. For $(X_1, B_1), (X_2, B_2) \in LL_O(U, M, S^{\diamondsuit})$, the partial order relation $\leqslant$ of object-oriented linguistic concepts is defined by

$$(X_1, B_1) \leqslant (X_2, B_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow B_1 \subseteq B_2) \tag{12}$$

Remark 2. For any $a \in B, X_1 \subseteq X_2$ indicates that the linguistic value corresponding to each object under X_2 is greater than or equal to the linguistic value under X_1, and the objects in X_1 are included in the objects in X_2.

Proposition 1. *Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation. For any $X, X_1, X_2 \subseteq U_S, B, B_1, B_2 \subseteq M$. Then*

1. $B_1 \subseteq B_2 \Rightarrow B_1^{\triangle} \subseteq B_2^{\triangle}, X_1 \subseteq X_2 \Rightarrow X_1^* \subseteq X_2^*$;
2. $X^{*\triangle} \subseteq X, B \subseteq B^{\triangle *}$;
3. $X^{*\triangle *} = X^*, B^{\triangle} = B^{\triangle * \triangle}$;
4. $(X_1 \cap X_2)^* = X_1^* \cap X_2^*, (B_1 \cup B_2)^{\triangle} = B_1^{\triangle} \cup B_2^{\triangle}$.

Proof. We only prove $X^{*\triangle *} = X^*$, others can be proved analogously.

Suppose that $a \in X^{*\triangle *}$, then $a' \subseteq X^{*\triangle}$, i.e., for any $x \in U$ when $\frac{s}{x} \in a'$, it satisfies $x' \cap X^* \neq \emptyset$. There exists an attribute b such that $b \in x'$ and $b \in X^*$ if $\frac{s}{x} \in a'$, which means that if $\frac{s}{x} \in a'$, we have that $b \in x'$ and $b' \subseteq X$. Since $b \in x'$ if and only if $\frac{s}{x} \in b'$, we can find that $x \in X$ in the case of $\frac{s}{x} \in a'$ such that $a' \subseteq X$. Thus, $a \in X^*$, we can obtain $X^{*\triangle *} \subseteq X^*$.

On the contrary, suppose that $a \in X^*$, then $a' \subseteq X$. It implies that for any $\frac{s}{x} \in a', a \in X^*$, i.e., $a' \subseteq \left\{\frac{s}{x} | x' \cap X^* \neq \emptyset\right\} = X^{*\triangle}$, thus, $a \in X^{*\triangle *}$ such that $X^* \subseteq X^{*\triangle *}$.

Hence, we conclude that $X^{*\triangle *} = X^*$.

Table 2 Linguistic formal context based on attribute dominance relation $(U, M, S^{\Diamond})$.

U/M	M_1				M_2				M_3				M_4			
	a_{11}	a_{12}	a_{13}	a_{14}	a_{21}	a_{22}	a_{23}	a_{24}	a_{31}	a_{32}	a_{33}	a_{34}	a_{41}	a_{42}	a_{43}	a_{44}
x_1	s_3	s_3	s_3	s_3	$\Diamond$	s_2	s_2	s_2	$\Diamond$	s_2	s_2	s_2	$\Diamond$	$\Diamond$	$\Diamond$	s_1
x_2	s_1	$\Diamond$	$\Diamond$	s_1	s_3	s_3	$\Diamond$	s_3	s_4	s_4	s_4	s_4	$\Diamond$	$\Diamond$	$\Diamond$	s_0
x_3	s_2	s_2	$\Diamond$	s_2	$\Diamond$	s_0	$\Diamond$	$\Diamond$	s_3	s_3	s_3	s_3	s_2	s_2	$\Diamond$	s_2
x_4	s_3	s_3	$\Diamond$	s_3	$\Diamond$	s_2	$\Diamond$	$\Diamond$	s_4	s_4	s_4	s_4	s_3	s_3	$\Diamond$	s_3

Table 3 The updated linguistic formal context based on attribute dominance relation.

U/M	a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{24}	a_{31}	a_{33}	a_{41}	a_{44}
x_1	s_3	s_3	s_3	$\Diamond$	s_2	s_2	s_2	$\Diamond$	s_2	$\Diamond$	s_1
x_2	s_1	$\Diamond$	$\Diamond$	s_3	s_3	$\Diamond$	s_3	s_4	s_4	$\Diamond$	s_0
x_3	s_2	s_2	$\Diamond$	$\Diamond$	s_0	$\Diamond$	$\Diamond$	s_3	s_3	s_2	s_2
x_4	s_3	s_3	$\Diamond$	$\Diamond$	s_2	$\Diamond$	$\Diamond$	s_4	s_4	s_3	s_3

Proposition 2. *The object-oriented linguistic concept lattice $LL_O(U, M, S^{\Diamond})$ is a complete lattice. For any $(X_1, B_1), (X_2, B_2) \in LL_O(U, M, S^{\Diamond})$, the infimum and supermum are given by*

$$(X_1, B_1) \vee (X_2, B_2) = (X_1 \cup X_2, (B_1 \cup B_2)^{\triangle *}) \tag{13}$$
$$(X_1, B_1) \wedge (X_2, B_2) = ((X_1 \cap X_2)^{*\triangle}, B_1 \cap B_2)$$

Theorem 2. *Let $(U, M, S^{\Diamond})$ be a linguistic formal context based on attribute dominance relation. $\forall a, b \in M, E \subseteq U, S(x, a) \leqslant S(x, b)$ if and only if $a^{\triangle E} \subseteq b^{\triangle E}$ for each $x \in E$.*

Proof. According to dominance relation and approximation operator in linguistic formal context based on attribute dominance relation, it is obviously proved and the details are omitted here.

Corollary 1. *Let $(U, M, S^{\Diamond})$ be a linguistic formal context based on attribute dominance relation. For any $a, b \in M, E \subseteq U, b \in [a]_E^{\leqslant}$ if and only if $a^{\triangle E} \subseteq b^{\triangle E}$.*

Proof. It can be proofed by Definition 7 and Theorem 2 easily.

3.2. The Construction Algorithm of Object-Oriented Linguistic Concept Lattice

The main idea of the construction algorithm for object-oriented linguistic concept lattice is as follows. Firstly, transform the fuzzy linguistic formal context into a linguistic formal context based on attribute dominance relation. Secondly, start from the minimum object-oriented linguistic concept, the type of the concepts is judged from the bottom to the top obviously. Thirdly, find the corresponding extent according to the different combinations of attributes, and then generate the valid concepts by updating, deleting, until the maximal concept is found. Finally, the object-oriented linguistic concept lattice is drawn.

Based on the construction algorithm of object-oriented linguistic concept lattice, we make some analysis on its time complexity. Comparing the linguistic values relation corresponding to each attribute according to the dominance relation between attributes, the running steps 1-2 to get a linguistic formal context based on attribute dominance relation take $O(|M|)$. Running steps 3-8 take at most $O(|M||U|)$, and running steps 9-18 take at most $O\left(\frac{|1+M||M|^2}{2}|U|\right)$. Then the total time complexity is at most $O\left(|M|\left(\frac{|1+M||M|}{2}|U| + |U| + 1\right)\right)$.

Algorithm 1: Construction algorithm for object-oriented linguistic concept lattice

Input: Fuzzy linguistic formal context (U, A, S);
Output: Object-oriented linguistic concept lattice $LL_O(U, M, S^{\Diamond})$;
1: $J_1 = (U, A, S)$;
2: $J_2 = (U, M, S^{\Diamond})$; // Compare the linguistic values relations corresponding to each attribute according to the dominance relation between attributes
3: $C_O = (X_i, B_i)$; // For $X_i \subseteq U_S, B_i \subseteq M$, set threshold λ, generate object-oriented linguistic concepts
4: $C_1 = (\emptyset, \emptyset)$; // The minimum object-oriented linguistic concept of C_O
5: **while** $B_i \neq \emptyset$
6: **if** B_i only has one attribute
7: Find X_i according to $X_i = \{\frac{s}{x} | x\prime \subseteq B_i\}$;
8: **end if**
9: **if** B_i has more than one attribute elements
10: Find X_i according to $B_i^{\triangle} = X_i$;
11: **end if**
12: Delete (X_i, B_i) with less attributes which has the same extent for other concepts;
13: **if** $X_i^* = B_i$ and $B_i^{\triangle} = X_i$
14: Generate object-oriented linguistic concept (X_i, B_i);
15: **end if**
16: **end while**
17: $LL_O(U, M, S^{\Diamond}) = \{(X_i, B_i)\}$;
18: **return** $LL_O(U, M, S^{\Diamond})$;

Example 2. Consider the grades of four students in a class as presented in Example 1. Set $\lambda = s_2$ as the standard score, in order to simplify the calculation, the attributes a_{12} and a_{44} in Table 3 are removed. We obtain the object-oriented linguistic concepts and its lattice structure which are as follows by Definition 9:

$$1\#\left(\frac{s_3}{x_1} + \frac{s_2}{x_3} + \frac{s_3}{x_4}, a_{11}a_{13}a_{23}a_{41}\right),\ 2\#\left(\frac{s_3}{x_1}, a_{13}a_{23}\right),$$
$$3\#\left(\frac{s_2}{x_1}, a_{21}\right),\ 4\#\left(\frac{s_2}{x_1} + \frac{s_3}{x_2} + \frac{s_2}{x_4}, a_{21}a_{22}a_{23}a_{24}\right),\ 5\#\left(\frac{s_2}{x_1}, a_{23}\right),$$
$$6\#\left(\frac{s_2}{x_1} + \frac{s_3}{x_2}, a_{21}a_{23}a_{24}\right),\ 7\#\left(\frac{s_4}{x_2} + \frac{s_3}{x_3} + \frac{s_4}{x_4}, a_{21}a_{31}a_{41}\right),$$
$$8\#\left(\frac{s_2}{x_1} + \frac{s_4}{x_2} + \frac{s_3}{x_3} + \frac{s_4}{x_4}, a_{21}a_{22}a_{23}a_{24}a_{31}a_{33}a_{41}\right),$$
$$9\#\left(\frac{s_2}{x_3} + \frac{s_3}{x_4}, a_{41}\right), 10\#\left(\frac{s_3}{x_1} + \frac{s_3}{x_2} + \frac{s_2}{x_3} + \frac{s_3}{x_4}, a_{11}a_{13}a_{21}a_{22}a_{23}a_{24}a_{41}\right),$$
$$11\#\left(\frac{s_3}{x_1} + \frac{s_4}{x_2} + \frac{s_3}{x_3} + \frac{s_4}{x_4}, a_{11}a_{13}a_{21}a_{22}a_{23}a_{24}a_{31}a_{33}a_{41}\right),$$
$$12\#\left(\frac{s_3}{x_1} + \frac{s_3}{x_2}, a_{13}a_{21}a_{23}a_{24}\right), 13\#\left(\frac{s_3}{x_1} + \frac{s_3}{x_2} + \frac{s_2}{x_4}, a_{13}a_{21}a_{22}a_{23}a_{24}\right),$$

$14\#\left(\frac{s_3}{x_2}+\frac{s_2}{x_3}+\frac{s_3}{x_4}, a_{21}a_{41}\right)$,
$15\#\left(\frac{s_2}{x_1}+\frac{s_3}{x_2}+\frac{s_2}{x_3}+\frac{s_3}{x_4}, a_{21}a_{22}a_{23}a_{24}a_{41}\right)$,
$16\#\left(\frac{s_2}{x_1}+\frac{s_2}{x_3}+\frac{s_3}{x_4}, a_{23}a_{41}\right)$, $17\#(\emptyset,\emptyset)$.

The Hasse diagram of object-oriented linguistic concept lattice is shown as Figure 1.

The object-oriented linguistic concept $1\#$ $(\frac{s_3}{x_1}+\frac{s_2}{x_3}+\frac{s_3}{x_4}, a_{11}a_{13}a_{23}a_{41})$ in Figure 1 indicates that at least one subject in the attribute set $\{a_1, a_2, a_4\}$ has achieved the standard score for at least one student of the object set $\{x_1, x_3, x_4\}$, i.e., taking students x_1, x_3 and x_4 as a group, it is best for Mathematics, Physics and Biology can reach good, medium, good, respectively. From the perspective of attribute dominance relation, at least one of students x_1, x_3 and x_4 meets a situation where the score of Mathematics is higher than Chemistry, the score of Physics is higher than Chemistry and the score of Biology is higher than Mathematics among their highest overall scores.

4. GRANULARIZATION OF OBJECT-ORIENTED LINGUISTIC CONCEPT LATTICE

Influenced by rough set and dominance relation, this paper constructs an object-oriented linguistic concept lattice, introducing the idea of approximate operator in rough set into linguistic formal concept analysis. In terms of granular computing, the rough set model can be characterized by multiple granules according to the equivalence relation. In order to describe the relation between objects and attributes under a class of attributes, we further study the granularization of object-oriented linguistic concept lattice.

4.1. Object-Oriented Linguistic Granular Concept Lattice

Similarly, we can define the classification in linguistic formal context based on attribute dominance relation according to the equivalence relation of rough set.

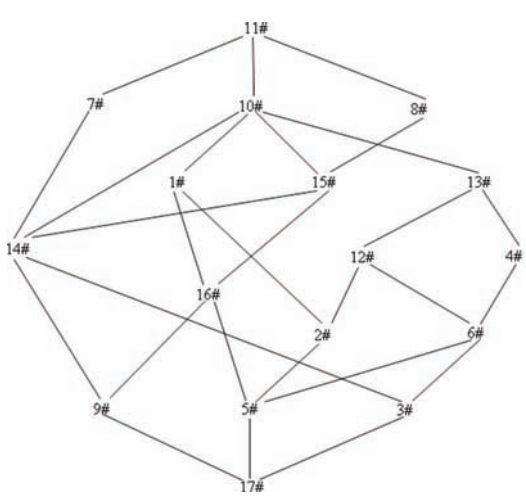

Figure 1 | Object-oriented linguistic concept lattice.

Definition 10. Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, for any $K \subseteq M$, the equivalence relation on object set U is defined as follows:

$$R_K = \{(x, v)|xS^{\diamondsuit}a = vS^{\diamondsuit}a, \forall x, v \in U, a \in K\} \tag{14}$$

the classification of equivalence relation R_K on object set U can be recorded as U/R_K, and $[x]_K = \{v|(x, v) \in R_K\}$ is an equivalence class on object set U.

Definition 11. Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, for any $X \subseteq U_S, K \subseteq M$, the approximation operator ^{*K} on attribute set K can be defined as follows:

$$X^{*K} = \{a \in M|a' \subseteq X \, and \, Y \subseteq U/R_K\} \tag{15}$$

where $Y = \{x \in U|(x, a) \in S^{\diamondsuit}\}$.

Proposition 3. *Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, $\forall X, X_1, X_2 \subseteq U_S, K \subseteq M$, there hold:*

1. $X_1 \subseteq X_2 \Rightarrow X_1^{*K} \subseteq X_2^{*K}$;
2. $X^{*K\triangle} \subseteq X$;
3. $X^{*K\triangle *K} = X^{*K}$;
4. $(X_1 \cap X_2)^{*K} = X_1^{*K} \cap X_2^{*K}$.

Proof. 1. $\forall a \in X_1^{*K}$, we have $a' \subseteq X_1$ and $Y \subseteq U/R_K$ where $Y = \{x \in U|(x, a) \in S^{\diamondsuit}\}$. If $X_1 \subseteq X_2$ is satisfied, there must be $a' \subseteq X_2$ and $Y \subseteq U/R_K$, therefore, $a \in X_2^{*K}$, i.e., $X_1 \subseteq X_2 \Rightarrow X_1^{*K} \subseteq X_2^{*K}$.

2. $\forall \frac{s}{x} \in X^{*K\triangle}$, we have $x' \bigcap X^{*K} \neq \emptyset$. Then there exists an attribute a such that $a \in x'$ and $a \in X^{*K}$, thus, $\frac{s}{x} \in X$, we can conclude $X^{*K\triangle} \subseteq X$.

3. Obviously, $X^{*K} \subseteq X^{*K\triangle *K}$. By (1)(2), it follows that $X^{*K\triangle *K} \subseteq X^{*K}$, hence, $X^{*K\triangle *K} = X^{*K}$.

4. Since(1), we obtain $(X_1 \cap X_2)^{*K} \subseteq X_1^{*K} \cap X_2^{*K}$. On the contrary, $\forall a \in X_1^{*K} \cap X_2^{*K}$, we note that $a' \subseteq X_1$ and $a' \subseteq X_2$, $Y \subseteq U/R_K$, then $a' \subseteq X_1 \cap X_2$, $Y \subseteq U/R_K$, i.e., $a \subseteq (X_1 \cap X_2)^{*K}$, Thus, $(X_1 \cap X_2)^{*K} = X_1^{*K} \cap X_2^{*K}$.

Definition 12. Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, for any $X \subseteq U_S, K, D \subseteq M$, if there satisfy $X^{*K} = D$ and $D^{\triangle} = X$, then (X, D) is called an object-oriented linguistic granular concept based on attribute set K, referred to as object-oriented linguistic granular concept.

The set of all the object-oriented linguistic granular concepts of $(U, M, S^{\diamondsuit})$ is denoted by $LL_{O/K}(U, M, S^{\diamondsuit})$. For $(X_1, D_1), (X_2, D_2) \in LL_{O/K}(U, M, S^{\diamondsuit})$, the partial order relation $\leqslant$ on $LL_{O/K}(U, M, S^{\diamondsuit})$ is defined by

$$(X_1, D_1) \leqslant (X_2, D_2) \Leftrightarrow X_1 \subseteq X_2 (\Leftrightarrow D_1 \subseteq D_2) \tag{16}$$

Obviously, $LL_{O/K}(U, M, S^{\diamondsuit})$ forms a complete lattice which is called the object-oriented linguistic granular concept lattice of $(U, M, S^{\diamondsuit})$.

Theorem 3. *Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, then $LL_{O/K}(U, M, S^{\diamondsuit})$ is a sub-lattice of $LL_O(U, M, S^{\diamondsuit})$.*

Proof. Obviously, the extent of $LL_{O/K}(U, M, S^{\diamondsuit})$ is the subset of the extent of $LL_O(U, M, S^{\diamondsuit})$. For any concepts $(X_1, D_1), (X_2, D_2) \in LL_{O/K}(U, M, S^{\diamondsuit})$, we have $(X_1, D_1) \wedge (X_2, D_2) = ((X_1 \cap X_2)^{*K\triangle}, D_1 \cap D_2)$. And $(X_1 \cap X_2)^{*K\triangle *K} = (X_1 \cap X_2)^{*K} = X_1^{*K} \cap X_2^{*K} = D_1 \cap D_2$; $(D_1 \cap D_2)^{\triangle} = (X_1^{*K} \cap X_2^{*K})^{\triangle} = (X_1 \cap X_2)^{*K\triangle}$.

Thus, $(X_1, D_1) \wedge (X_2, D_2) \in LL_{O/K}(U, M, S^{\diamondsuit})$. Similarly, $(X_1, D_1) \vee (X_2, D_2) \in LL_{O/K}(U, M, S^{\diamondsuit})$. Therefore, $LL_{O/K}(U, M, S^{\diamondsuit})$ is a sub-lattice of $LL_O(U, M, S^{\diamondsuit})$.

Theorem 4. *Let $(U, M, S^{\diamondsuit})$ be a linguistic formal context based on attribute dominance relation, for any $K_1, K_2 \subseteq M$, if $K_1 \neq K_2$ and $U/R_{K_1} = U/R_{K_2}$, then $LL_{O/K_1}(U, M, S^{\diamondsuit}) = LL_{O/K_2}(U, M, S^{\diamondsuit})$.*

Proof. For any $X \subseteq U_S, K_1, K_2 \subseteq M, U/R_{K_1} = U/R_{K_2}$, we have $X^{*K_1} = \{a \in M | a' \subseteq X \text{ and } Y \subseteq U/R_{K_1}\} = \{a \in M | a' \subseteq X \text{ and } Y \subseteq U/R_{K_2}\} = X^{*K_2}$ for $Y = \{x \in U | (x, a) \in S^{\diamondsuit}\}$ according to the Definition 11. To sum up, $LL_{O/K_1}(U, M, S^{\diamondsuit}) = LL_{O/K_2}(U, M, S^{\diamondsuit})$, this theorem is proved.

4.2. Construction Algorithm of Object-Oriented Linguistic Granular Concept Lattice

Based on the construction algorithm of object-oriented linguistic concept lattice mentioned in Algorithm 1, we can give the following construction algorithm for object-oriented linguistic granular concept lattice, combining the equivalence relations under a kind of attributes.

The main idea of the construction algorithm for object-oriented linguistic granular concept lattice is as follows: Set the threshold of linguistic information in the linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$ to obtain a new formal context for simplification. Find the equivalence relation on attribute set K and generate the object-oriented linguistic granular concepts according to the approximation operator. Then, the corresponding object-oriented linguistic granular concept lattice is performed as Algorithm 2.

On the basis of the established linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$, we make some analysis on the time complexity of construction algorithm for object-oriented linguistic granular concept lattice. In order to obtain the equivalence relation under the attribute set K, we need to take $O(|U||K|)$ from step 1 to step 4. Running steps 5-12 take at most $O(2^{|U/R_K|})$, then the total time complexity is at most $O(|U||K| + 2^{|U/R_K|})$.

Example 3. Consider the grades of four students in a class as described in Example 1, set attribute set to $K_1 = \{a_{21}, a_{23}, a_{24}\}$, the corresponding equivalence relation is $U/R_{K_1} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}\}$. Based on that, we can get 12 object-oriented linguistic granular concepts according to Definition 12, and the lattice structure is shown as Figure 2:

Likewise, we can get the object-oriented linguistic granular concept lattice with attribute set $K_2 = \{a_{13}, a_{23}\}$ which is shown in Figure 3.

Algorithm 2: Construction algorithm for object- oriented linguistic granular concept lattice

Input: A linguistic formal context based on attribute dominance relation $(U, M, S^{\diamondsuit})$;
Output: Object-oriented linguistic granular concept lattice $LL_{O/K}(U, M, S^{\diamondsuit})$;
1: $J_2 = (U, M, S^{\diamondsuit})$;
2: $C_{O'} = (X_i, D_i)$; // For $X_i \subseteq U_S, D_i \subseteq M$, set threshold λ', generate object-oriented linguistic granular concepts
3: **while** $K \subseteq M$ **do**
4: $R_K = \{(x, v) | xS^{\diamondsuit}a = vS^{\diamondsuit}a, \forall x, v \in U, a \in K\}$; // The equivalence relation on object set U
5: **while** $X_i \subseteq U_S, D_i \subseteq M$ **do**
6: **if** $X_i^{*K} = D_i, D_i^{\triangle} = X_i$ **then**
7: $C_{O'} = (X_i, D_i)$;
8: **end if**
9: **end while**
10: **end while**
11: $LL_{O/K}(U, M, S^{\diamondsuit}) = \{(X_i, D_i)\}$;
12: **return** $LL_{O/K}(U, M, S^{\diamondsuit})$;

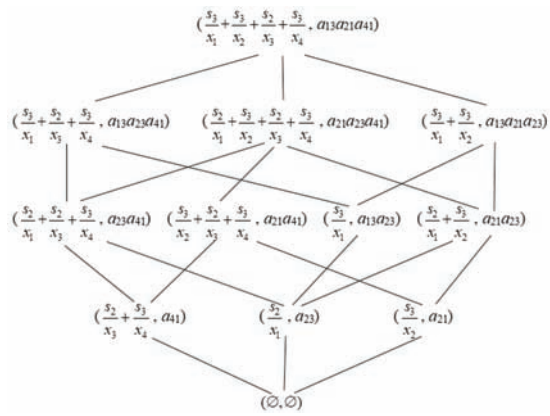

Figure 2 Object-oriented linguistic granular concept lattice with attribute set $K_1 = \{a_{21}, a_{23}, a_{24}\}$.

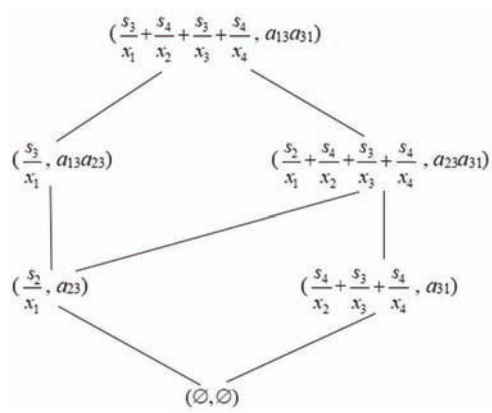

Figure 3 Object-oriented linguistic granular concept lattice with attribute set $K_2 = \{a_{13}, a_{23}\}$.

The object-oriented linguistic granular concept $\left(\frac{s_4}{x_2} + \frac{s_3}{x_3} + \frac{s_4}{x_4}, a_{31}\right)$ in Figure 3 represents that the score of Chemistry (a_3) for the student set $\{x_2, x_3, x_4\}$ has reached the standard with the meaning that the scores of Chemistry are excellent, good, excellent for students x_2 to x_4, respectively. And their scores

of Chemistry are higher than the Mathematics for these three students.

From the above discussions, we can conclude that the object-oriented linguistic granular concept lattice is the sub-lattice of the object-oriented linguistic concept lattice. The object-oriented linguistic concept lattice can express the highest level of linguistic evaluation for an objects group while the object-oriented linguistic granular concept lattice is to study linguistic concepts under an equivalence object set by classifying the objects under several attributes, so that we can quickly find out the characteristics of a class of objects in a targeted manner. Both of them can express the dominance relations between attributes, in addition, we can also obtain an object-oriented linguistic concept lattice by constructing object-oriented linguistic granular concepts.

5. THE CONSTRUCTION OF ATTRIBUTE-ORIENTED LINGUISTIC CONCEPT LATTICE

The above discussion is considered from the perspective of attribute dominance relations. Sometimes we also need to investigate the dominance relations of objects in the formal context. In order to describe the uncertain linguistic information under the object group with dominance relation, this section proposes the notion of attribute-oriented linguistic concept lattice based on object dominance relation.

5.1. Attribute-Oriented Linguistic Concept Lattice

For a fuzzy linguistic formal context (U, A, S), $\forall x_i, x_r \in U$, $a \in A$, an object partial order relation " $\leqslant_a$," in which $x_r \leqslant_a x_i$ indicates that the linguistic values corresponding to object x_i dominant x_r with respect to attribute a, i.e., $S(x_r, a) \leqslant S(x_i, a)$. For simplicity, the partial order relation on $x_r \leqslant_a x_i$ is abbreviated as x_{ir}.

Therefore, when there exists x_{ir} in a formal context, if $S(x_r, a) \leqslant S(x_i, a)$, we use $S(x_i, a)$ to represent the relation between x_{ir} and a. If $S(x_r, a) > S(x_i, a)$, we use " $\Diamond$" to represent that the linguistic values of object x_i don't dominant object x_r with respect to attribute a which we won't consider in this article. In this case, we can establish a linguistic formal context based on object dominance relation.

Definition 13. Let $(G, A, S^{\Diamond})$ be a linguistic formal context based on object dominance relation, $G = \{G_i | i \in 1, 2, \dots, n\}$, where $G_i = \{x_{ir} | r \in 1, 2, \dots, n\}$ denotes a set of objects with a partial order relation, $S^{\Diamond}$ is a relation between G and A, such that $S^{\Diamond}(x_{ir}, a) = \{s_l \in S$ or $\Diamond | x_{ir} \in G, a \in A\}$.

The linguistic formal context based on object dominance relation divides the objects into a number of small blocks $G_i (i = 1, 2, \dots, n)$ with dominance relation. Each small block represents the importance of the object x_i compared with other objects. Therefore, the objects of the same block are comparable while not all of the blocks have a dominance relation.

Theorem 5. *Let $(G, A, S^{\Diamond})$ be a linguistic formal context based on object dominance relation. For any $x_{ir} \in G$, if $i = r$, then the linguistic values under x_{ir} is the same as x_i such that $S^{\Diamond}(x_{ir}, a) = S(x_i, a)$.*

Table 4 Linguistic formal context based on object dominance relation $(G, A, S^{\Diamond})$.

G/A	a_1	a_2	a_3	a_4
x_{11}	s_3	s_2	s_2	s_1
x_{12}	s_3	$\Diamond$	$\Diamond$	s_1
x_{13}	s_3	s_2	$\Diamond$	$\Diamond$
x_{21}	$\Diamond$	s_3	s_4	$\Diamond$
x_{22}	s_1	s_3	s_4	s_0
x_{31}	$\Diamond$	$\Diamond$	s_3	s_2
x_{32}	s_2	$\Diamond$	$\Diamond$	s_2
x_{33}	s_2	s_0	s_3	s_2
x_{41}	s_3	s_2	s_4	s_3
x_{42}	s_3	$\Diamond$	s_4	s_3

Definition 14. Let $(G, A, S^{\Diamond})$ be a linguistic formal context based on object dominance relation, for any $x \in G$, the object dominant class $[x]_A^{\prec}$ can be defined as follows:

$$[x]_A^{\prec} = \{y \in G | S(x, a) \leqslant S(y, a), \forall a \in A\} \tag{17}$$

Example 4. Consider the fuzzy linguistic formal context (U, A, S) presented in Example 1. We can obtain a simplified linguistic formal context based on object dominance relation $(G, A, S^{\Diamond})$ as shown in Table 4, combining the dominance relation with respect to the objects, where $G = \{G_1, G_2, G_3, G_4\} = \{\{x_{11}, x_{12}, x_{13}\}, \{x_{21}, x_{22}\}, \{x_{31}, x_{32}, x_{33}\}, \{x_{41}, x_{42}\}\}$.

$S^{\Diamond}(x_{12}, a_1) = s_3$ represents that the Mathematics of student x_1 is good and his Mathematics is greater than or equal to student x_2 in linguistic formal context based on object dominance relation $(G, A, S^{\Diamond})$, while $S^{\Diamond}(x_{21}, a_1) = \Diamond$ means that the Mathematics of student x_2 is worse than student x_1. For the attribute set $H = \{a_1, a_3, a_4\}$, it follows $[x_{33}]_H^{\prec} = \{x_{33}, x_{41}, x_{42}\}$, i.e., the Mathematics, Chemistry, and Biology of student x_4 are higher than that of student x_3, and the three scores of student x_4 are higher than those of student x_1 and student x_2 as well. It is worth noting that the linguistic formal context based on object dominance relation $(G, A, S^{\Diamond})$ can not only express the information in fuzzy linguistic formal context (U, A, S) but also describe the partial order relation between objects.

For the linguistic formal context based on object dominance relation $(G, A, S^{\Diamond})$, we record A_S as the set which is the subset of A defined in S.

Definition 15. Let $(G, A, S^{\Diamond})$ be a linguistic formal context based on object dominance relation, $\gamma \in S$ be a credibility threshold. For any $\tilde{X} \subseteq G, \tilde{B} \subseteq A_S$, $a \in A, s \in S$, the operators $\sharp$, $*$ and $\triangle$ can be defined as follows:

$$x^{\sharp} = \{\frac{s}{a} \in A_S | \forall x \in G, S^{\Diamond}(x, a) \geqslant \gamma\}, \tag{18}$$

$$a^{\sharp} = \{x \in G | \forall a \in A, S^{\Diamond}(x, a) \geqslant \gamma\}, \tag{19}$$

$$\tilde{X}^{\triangle} = \vee\{\frac{s}{a} \in A_S | a^{\sharp} \cap \tilde{X} \neq \emptyset\}, \tag{20}$$

$$\tilde{B}^{*} = \{x \in G | x^{\sharp} \subseteq \tilde{B}\} \tag{21}$$

Remark 3. $\tilde{X}^{\triangle}$ denotes a set of maximum linguistic values between attributes and objects, which possesses at least one object in $\tilde{X}$, $\tilde{B}^*$ is the set of objects such that linguistic values corresponding to any attribute that satisfy one of them is necessarily in $\tilde{B}$.

Definition 16. Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation. For any $\tilde{X} \subseteq G, \tilde{B} \subseteq A_S$, if there exist $\tilde{X}^{\triangle} = \tilde{B}$ and $\tilde{X} = \tilde{B}^*$, then $(\tilde{X}, \tilde{B})$ is called attribute-oriented linguistic concept based on object dominance relation, abbreviated as attribute-oriented linguistic concept. $\tilde{X}$ and $\tilde{B}$ are called the extent and intent of the attribute-oriented linguistic concept, respectively.

The set of all the attribute-oriented linguistic concepts of $(G, A, S^{\diamondsuit})$ is denoted by $LL_A(G, A, S^{\diamondsuit})$. For $(\tilde{X}_1, \tilde{B}_1), (\tilde{X}_2, \tilde{B}_2) \in LL_A(G, A, S^{\diamondsuit})$, the partial order relation $\leqslant$ on $LL_A(G, A, S^{\diamondsuit})$ is defined by

$$(\tilde{X}_1, \tilde{B}_1) \leqslant (\tilde{X}_2, \tilde{B}_2) \Leftrightarrow \tilde{X}_1 \subseteq \tilde{X}_2 (\Leftrightarrow \tilde{B}_1 \subseteq \tilde{B}_2) \tag{22}$$

Obviously, $LL_A(G, A, S^{\diamondsuit})$ forms a complete lattice which is called the attribute-oriented linguistic concept lattice of $(G, A, S^{\diamondsuit})$.

Proposition 4. *Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation. For any $\tilde{X}, \tilde{X}_1, \tilde{X}_2 \subseteq G, \tilde{B}, \tilde{B}_1, \tilde{B}_2 \subseteq A_S$. Then*

1. $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_1^* \subseteq \tilde{B}_2^*, \tilde{X}_1 \subseteq \tilde{X}_2 \Rightarrow \tilde{X}_1^{\triangle} \subseteq \tilde{X}_2^{\triangle}$.
2. $\tilde{X} \subseteq \tilde{X}^{\triangle *}, \tilde{B}^{*\triangle} \subseteq \tilde{B}$.
3. $\tilde{X}^{\triangle *\triangle} = \tilde{X}^{\triangle}, \tilde{B}^{*\triangle *} = \tilde{B}^*$.
4. $(\tilde{X}_1 \cup \tilde{X}_2)^{\triangle} = \tilde{X}_1^{\triangle} \cup \tilde{X}_2^{\triangle}, (\tilde{B}_1 \cap \tilde{B}_2)^* = \tilde{B}_1^* \cap \tilde{B}_2^*$.

Proof. We only prove $\tilde{B}^{*\triangle *} = \tilde{B}^*$, the others are the same.

Suppose that $x \in \tilde{B}^{*\triangle *}$, then $x^{\sharp} \subseteq \tilde{B}^{*\triangle}$, i.e., when $\frac{s}{a} \in x^{\sharp}$, it satisfies $a^{\sharp} \cap \tilde{B}^* \neq \emptyset$. There exists an object y such that $y^{\sharp} \in a^{\sharp}$ and $y \in \tilde{B}^*$ if $\frac{s}{a} \in x^{\sharp}$, which means when $\frac{s}{a} \in x^{\sharp}$, we have $y \in a^{\sharp}$ and $y^{\sharp} \subseteq \tilde{B}$. Since $y \in a^{\sharp}$ if and only if $\frac{s}{a} \in y^{\sharp}$, we can find that $\frac{s}{a} \in \tilde{B}$ in the case of $\frac{s}{a} \in x^{\sharp}$ such that $x^{\sharp} \subseteq \tilde{B}$. Thus, $x \in \tilde{B}^*$, we can obtain $\tilde{B}^{*\triangle *} \subseteq \tilde{B}^*$.

On the contrary, suppose that $x \in \tilde{B}^*$, then $x^{\sharp} \subseteq \tilde{B}$. It means that for any $\frac{s}{a} \in x^{\sharp}, x \in \tilde{B}^*$, i.e., $x^{\sharp} \subseteq \{\frac{s}{a} | a^{\sharp} \cap \tilde{B}^* \neq \emptyset\} = \tilde{B}^{*\triangle}$, thus, $x \in \tilde{B}^{*\triangle *}$ such that $\tilde{B}^* \subseteq \tilde{B}^{*\triangle *}$.

Hence, we conclude that $\tilde{B}^{*\triangle *} = \tilde{B}^*$.

Proposition 5. *The attribute-oriented linguistic concept lattice $LL_A(G, A, S^{\diamondsuit})$ is a complete lattice. For any $(\tilde{X}_1, \tilde{B}_1), (\tilde{X}_2, \tilde{B}_2) \in LL_A(G, A, S^{\diamondsuit})$, the infimum and supermum are given as follows:*

$$\begin{aligned} (\tilde{X}_1, \tilde{B}_1) \vee (\tilde{X}_2, \tilde{B}_2) &= ((\tilde{X}_1 \cup \tilde{X}_2)^{\triangle *}, \tilde{B}_1 \cup \tilde{B}_2) \\ (\tilde{X}_1, \tilde{B}_1) \wedge (\tilde{X}_2, \tilde{B}_2) &= (\tilde{X}_1 \cap \tilde{X}_2, (\tilde{B}_1 \cap \tilde{B}_2)^{*\triangle}) \end{aligned} \tag{23}$$

Theorem 6. *Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation. $\forall x, y \in G, H \subseteq A, S(x, a) \leqslant S(y, a)$ if and only if $x^{\triangle H} \subseteq y^{\triangle H}$ for each $a \in A$.*

Corollary 2. *Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation. For any $x, y \in G, H \subseteq A, y \in [x]_H^{\leqslant}$ if and only if $x^{\triangle H} \subseteq y^{\triangle H}$.*

5.2. The Construction Algorithm of Attribute-Oriented Linguistic Concept Lattice

The construction algorithm for attribute-oriented linguistic concept lattice is similar to the construction algorithm for object-oriented linguistic concept lattice as follows:

Algorithm 3: Construction algorithm for attribute-oriented linguistic concept lattice

Input: Fuzzy linguistic formal context (U, A, S);
Output: Attribute-oriented linguistic concept lattice $LL_A(G, A, S^{\diamondsuit})$;
1: $J_1 = (U, A, S)$;
2: $J_2{}' = (G, A, S^{\diamondsuit})$; // Compare the linguistic values relations corresponding to each object according to dominance relation between objects
3: $C_A = (\tilde{X}_i, \tilde{B}_i)$; // For $\tilde{X}_i \subseteq G, \tilde{B}_i \subseteq A_S$, set threshold γ to generate attribute-oriented linguistic concepts
4: $C_1 = (\emptyset, \emptyset)$; // The minimum attribute-oriented linguistic concept of C_A
5: **while** $\tilde{X}_i \neq \emptyset$
6: **if** $\tilde{X}_i$ only has one object
7: Find $\tilde{B}_i$ according to $\tilde{B}_i = \{\frac{s}{a} | a^{\sharp} \subseteq \tilde{X}_i\}$
8: **end if**
9: **if** $\tilde{X}_i$ has more than one object elements
10: Find $\tilde{B}_i$ according to $\tilde{B}_i = \tilde{X}_i^{\triangle}$;
11: **end if**
12: Delete $(\tilde{X}_i, \tilde{B}_i)$ with less objects which has the same intent for other concepts;
13: **if** $\tilde{X}_i^{\triangle} = \tilde{B}_i$ and $\tilde{B}_i^* = \tilde{X}_i$
14: Generate attribute-oriented linguistic concept $(\tilde{X}_i, \tilde{B}_i)$;
15: **end if**
16: **end while**
17: $LL_A(G, A, S^{\diamondsuit}) = \{(\tilde{X}_i, \tilde{B}_i)\}$;
18: **return** $LL_A(G, A, S^{\diamondsuit})$;

Based on the construction algorithm of attribute-oriented linguistic concept lattice, we make some analysis on its time complexity. Comparing the linguistic values relation corresponding to each object according to the dominance relation between objects, the running steps 1-2 for getting a linguistic formal context based on object dominance relation take $O(|G|)$. Running steps 3-8 take at most $O(|G||A|)$, and running steps 9-18 take at most $O(\frac{|1+G||G|^2}{2}|A|)$. Then the total time complexity is at most $O(|G|(\frac{|1+G||G|}{2}|A| + |A| + 1))$.

Example 5. Consider the grades of four students in a class as presented in Example 1, we analysis the relations between students and subjects further from the perspective of attribute-oriented linguistic concepts. Set $\gamma = s_2$, the attribute-oriented linguistic concepts and the lattice structure are generated as follows after removing redundant information by Definition 16:

$$1\#\,(\emptyset, \emptyset)\,,\ 2\#\left(x_{11}x_{12}x_{13}, \frac{s_3}{a_1} + \frac{s_2}{a_2} + \frac{s_2}{a_3}\right),\ 3\#\left(x_{12}, \frac{s_3}{a_1}\right),$$
$$4\#\left(x_{12}x_{13}, \frac{s_3}{a_1} + \frac{s_2}{a_2}\right),\ 5\#\left(x_{21}, \frac{s_3}{a_2} + \frac{s_4}{a_3}\right),\ 6\#\left(x_{31}, \frac{s_3}{a_3} + \frac{s_2}{a_4}\right),$$
$$7\#\left(x_{32}, \frac{s_2}{a_1} + \frac{s_2}{a_4}\right),\ 8\#\left(x_{31}x_{32}x_{33}, \frac{s_2}{a_1} + \frac{s_3}{a_3} + \frac{s_2}{a_4}\right),$$

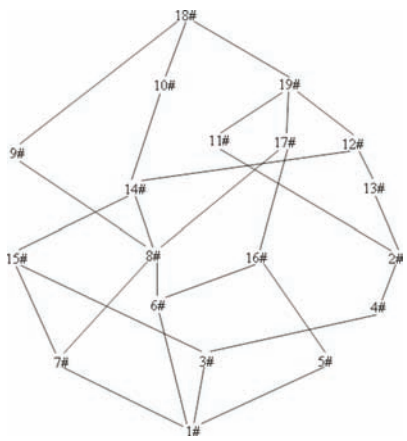

Figure 4 | Attribute-oriented linguistic concept lattice.

$$9\#\left(x_{11}x_{12}x_{13}x_{31}x_{32}x_{41}x_{42}, \frac{s_2}{a_1}+\frac{s_2}{a_2}+\frac{s_4}{a_3}+\frac{s_3}{a_4}\right),$$
$$10\#\left(x_{12}x_{31}x_{32}x_{33}x_{42}, \frac{s_3}{a_1}+\frac{s_4}{a_3}+\frac{s_3}{a_4}\right),$$
$$11\#\left(x_{11}x_{12}x_{13}x_{21}, \frac{s_3}{a_1}+\frac{s_3}{a_2}+\frac{s_4}{a_3}\right),$$
$$12\#\left(x_{11}x_{12}x_{13}x_{31}x_{32}x_{33}, \frac{s_3}{a_1}+\frac{s_2}{a_2}+\frac{s_3}{a_3}+\frac{s_2}{a_4}\right),$$
$$13\#\left(x_{11}x_{12}x_{13}x_{32}, \frac{s_3}{a_1}+\frac{s_2}{a_2}+\frac{s_2}{a_3}+\frac{s_2}{a_4}\right),$$
$$14\#\left(x_{12}x_{31}x_{32}x_{33}, \frac{s_3}{a_2}+\frac{s_3}{a_3}+\frac{s_2}{a_4}\right),$$
$$15\#\left(x_{12}x_{32}, \frac{s_3}{a_1}+\frac{s_2}{a_4}\right),\ 16\#\left(x_{21}x_{31}, \frac{s_3}{a_2}+\frac{s_4}{a_3}+\frac{s_2}{a_4}\right),$$
$$17\#\left(x_{21}x_{31}x_{32}x_{33}, \frac{s_2}{a_1}+\frac{s_3}{a_2}+\frac{s_4}{a_3}+\frac{s_2}{a_4}\right),$$
$$18\#\left(x_{11}x_{12}x_{13}x_{21}x_{31}x_{32}x_{33}x_{41}x_{42}, \frac{s_3}{a_1}+\frac{s_3}{a_2}+\frac{s_4}{a_3}+\frac{s_3}{a_4}\right),$$
$$19\#\left(x_{11}x_{12}x_{13}x_{21}x_{31}x_{32}x_{33}, \frac{s_3}{a_1}+\frac{s_3}{a_2}+\frac{s_4}{a_3}+\frac{s_2}{a_4}\right).$$

The Hasse diagram of attribute-oriented linguistic concept lattice is shown as Figure 4.

If the object elements $x_{ir}(i, r = 1, 2, ..., n)$ in the extent of attribute-oriented linguistic concept are under the same block G_i, take $\left(x_{11}x_{12}x_{13}, \frac{s_3}{a_1}+\frac{s_2}{a_2}+\frac{s_2}{a_3}\right)$ as an example, $x_{11}, x_{12}, x_{13} \in G_1$, which denotes that the Mathematics, Physics and Chemistry of student x_1 are good, medium and medium respectively. Meanwhile, at least one of the three subjects of x_1 is higher than or equal to those of student x_2 and student x_3. If the object elements x_{ir} in the extent of attribute-oriented linguistic concept are under different blocks, take $\left(x_{12}x_{32}, \frac{s_3}{a_1}+\frac{s_2}{a_4}\right)$ as an example, $x_{12} \in G_1, x_{32} \in G_3$, more specifically, it has two meanings, on the one hand, it means that at least one of student x_1 and student x_3 has achieved at least one of good Mathematics and medium Biology. On the other hand, it represents that dividing student x_1 and student x_3 into a group, their Mathematics and Biology can achieve the best results of good and medium. From the perspective of object dominance relation, at least one of the Mathematics and Biology of student x_1 is higher than those of student x_2, so are students x_3 and x_2 for the attribute-oriented linguistic concept $\left(x_{12}x_{32}, \frac{s_3}{a_1}+\frac{s_2}{a_4}\right)$.

6. GRANULARIZATION OF ATTRIBUTE-ORIENTED LINGUISTIC CONCEPT LATTICE

6.1. Attribute-Oriented Linguistic Granular Concept Lattice

In this section, in order to meet the needs of different people and construct an attribute-oriented linguistic concept lattice more purposefully, we propose the equivalence relation of attribute sets to study the granularization of attribute-oriented linguistic concept lattices, considering the dominance relation of objects in fuzzy linguistic formal context.

Definition 17. Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation, for any $F \subseteq G$, the equivalence relation on attribute set A is defined as follows:

$$R_F = \{(b, h)|xS^{\diamondsuit}b = xS^{\diamondsuit}h, \forall x \in F, b, h \in A\} \tag{24}$$

the classification of equivalence relation R_F on attribute set A can be recorded as A/R_F, and $[b]_F = \{h|(b, h) \in R_F\}$ is an equivalence class on attribute set A.

Definition 18. Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation, for any $F \subseteq G$, $\tilde{B} \subseteq A_S$, the approximation operator ^{*F} on object set F can be defined as follows:

$$\tilde{B}^{*F} = \{x \in G|x^{\sharp} \subseteq \tilde{B} \ \ and \ \ Z \subseteq A/R_F\} \tag{25}$$

where $Z = \{a \in A|(x, a) \in S^{\diamondsuit}\}$.

Proposition 6. *Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation, $\forall \tilde{B}, \tilde{B}_1, \tilde{B}_2 \subseteq A_S, F \subseteq G$, there hold:*

1. $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_1^{*F} \subseteq \tilde{B}_2^{*F}$;
2. $\tilde{B}^{*F\triangle} \subseteq \tilde{B}$;
3. $\tilde{B}^{*F\triangle *F} = \tilde{B}^{*F}$;
4. $(\tilde{B}_1 \cap \tilde{B}_2)^{*F} = \tilde{B}_1^{*F} \cap \tilde{B}_2^{*F}$.

Proof. 1. $\forall y \in \tilde{B}_1^{*F}$, we have $y^{\sharp} \subseteq \tilde{B}_1$ and $Z \subseteq A/R_F$ where $Z = \{a \in A|(x, a) \in S^{\diamondsuit}\}$. If $\tilde{B}_1 \subseteq \tilde{B}_2$ is satisfied, there must be $y^{\sharp} \subseteq \tilde{B}_2$ and $Z \subseteq A/R_F$, therefore, $y \in \tilde{B}_2^{*F}$, i.e., $\tilde{B}_1 \subseteq \tilde{B}_2 \Rightarrow \tilde{B}_1^{*F} \subseteq \tilde{B}_2^{*F}$.

2. $\forall \frac{s}{a} \in \tilde{B}^{*F\triangle}$, we have $a^{\sharp} \bigcap \tilde{B}_1^{*F} \neq \emptyset$. Then there exist $y \in a^{\sharp}$ and $y \in \tilde{B}^{*F}$, thus, $\frac{s}{a} \in \tilde{B}$, we can conclude $\tilde{B}^{*F\triangle} \subseteq \tilde{B}$.

3. Obviously, $\tilde{B}^{*F} \subseteq \tilde{B}^{*F\triangle *F}$. By (1)(2), it follows that $\tilde{B}^{*F\triangle *F} \subseteq \tilde{B}^{*F}$, hence, $\tilde{B}^{*F\triangle *F} = \tilde{B}^{*F}$.

4. Since(1), we obtain $(\tilde{B}_1 \cap \tilde{B}_2)^{*F} \subseteq \tilde{B}_1^{*F} \cap \tilde{B}_2^{*F}$. On the contrary, $\forall y \in \tilde{B}_1^{*F} \cap \tilde{B}_2^{*F}$, we note that $y^{\sharp} \subseteq \tilde{B}_1$ and $y^{\sharp} \subseteq \tilde{B}_2$, $Z \subseteq A/R_F$,

then $y^{\sharp} \subseteq \tilde{B}_1 \cap \tilde{B}_2$, $Z \subseteq A/R_F$, i.e., $y \subseteq (\tilde{B}_1 \cap \tilde{B}_2)^{*F}$. Thus, $(\tilde{B}_1 \cap \tilde{B}_2)^{*F} = \tilde{B}_1^{*F} \cap \tilde{B}_2^{*F}$.

Definition 19. Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation, for any $F, Q \subseteq G$, $\tilde{B} \subseteq A_S$, if there satisfy $\tilde{B}^{*F} = Q$ and $Q^{\triangle} = \tilde{B}$, then $(Q, \tilde{B})$ is called an attribute-oriented linguistic granular concept based on object set F, referred to as attribute-oriented linguistic granular concept.

The set of all the attribute-oriented linguistic granular concepts of $(G, A, S^{\diamondsuit})$ is denoted by $LL_{A/F}(G, A, S^{\diamondsuit})$. For $(Q_1, \tilde{B}_1), (Q_2, \tilde{B}_2) \in LL_{A/F}(G, A, S^{\diamondsuit})$, the partial order relation $\leqslant$ on $LL_{A/F}(G, A, S^{\diamondsuit})$ is defined by

$$(Q_1, \tilde{B}_1) \leqslant (Q_2, \tilde{B}_2) \Leftrightarrow Q_1 \subseteq Q_2 (\Leftrightarrow \tilde{B}_1 \subseteq \tilde{B}_2) \tag{26}$$

Obviously, $LL_{A/F}(G, A, S^{\diamondsuit})$ forms a complete lattice which is called the attribute-oriented linguistic granular concept lattice of $(G, A, S^{\diamondsuit})$.

Theorem 7. *Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation, then $LL_{A/F}(G, A, S^{\diamondsuit})$ is a sub-lattice of $LL_A(G, A, S^{\diamondsuit})$.*

Proof. Similar to Theorem 3 and the details are omitted.

Theorem 8. *Let $(G, A, S^{\diamondsuit})$ be a linguistic formal context based on object dominance relation, for any $F_1, F_2 \subseteq G$, if $F_1 \neq F_2$ and $A/R_{F_1} = A/R_{F_2}$, then $LL_{A/F_1}(G, A, S^{\diamondsuit}) = LL_{A/F_2}(G, A, S^{\diamondsuit})$.*

Proof. For any $\tilde{B} \subseteq A_S, F_1, F_2 \subseteq G, A/R_{F_1} = A/R_{F_2}$, we have $\tilde{B}^{*F_1} = \{x \in G | x^{\sharp} \subseteq \tilde{B} \text{ and } Z \subseteq A/R_{F_1}\} = \{x \in G | x^{\sharp} \subseteq \tilde{B} \text{ and } Z \subseteq A/R_{F_2}\} = \tilde{B}^{*F_2}$ for $Z = \{a \in A | (x, a) \in S^{\diamondsuit}\}$ according to the Definition 18. To sum up, $LL_{A/F_1}(G, A, S^{\diamondsuit}) = LL_{A/F_2}(G, A, S^{\diamondsuit})$, this theorem is proved.

6.2. Construction Algorithm of Attribute-Oriented Linguistic Granular Concept Lattice

Based on the construction algorithm of attribute-oriented linguistic concept lattice mentioned in Algorithm 3, we can give the following construction algorithm for attribute-oriented linguistic granular concept lattice, combining the equivalence relation under a kind of objects.

In linguistic formal context based on object dominance relation $(G, A, S^{\diamondsuit})$, we make some analysis on the time complexity of construction algorithm for attribute-oriented linguistic granular concept lattice. In order to obtain the equivalence relation under object set F, we need to take $O(|A||F|)$ from step 1 to step 4. Running steps 5-12 take at most $O(2^{|A/R_F|})$, then the total time complexity is at most $O(|A||F| + 2^{|A/R_F|})$.

Example 6. We consider the fuzzy linguistic formal context (U, A, S) presented in Example 1. Set object set to $F_1 = \{x_{41}, x_{42}\}$, then the corresponding equivalence relation is $A/R_{F_1} = \{\{a_1, a_4\}, \{a_2\}, \{a_3\}\}$. Based on that, we can get 4 attribute-oriented linguistic granular concepts according to Definition 19, and the lattice structure is shown as follows:

Algorithm 4: Construction algorithm for attribute-oriented linguistic granular concept lattice

Input: A linguistic formal context based on object dominance relation $(G, A, S^{\diamondsuit})$;
Output: Attribute-oriented linguistic granular concept lattice $LL_{A/F}(G, A, S^{\diamondsuit})$;
1: $J_2{\prime} = (G, A, S^{\diamondsuit})$;
2: $C_A{\prime} = (Q_i, \tilde{B}_i)$; // For $Q_i \subseteq G, \tilde{B}_i \subseteq A_S$, set threshold $\gamma{\prime}$, generate attribute-oriented linguistic granular concepts
3: **while** $F \subseteq G$
4: $R_F = \{(b, h) | xS^{\diamondsuit}b = xS^{\diamondsuit}h, \forall b, h \in A, x \in F\}$; // The equivalence relation on attribute set A
5: **while** $Q_i \subseteq G, \tilde{B}_i \subseteq A_S$
6: **if** $\tilde{B}_i^{*F} = Q_i$ and $Q_i^{\triangle} = \tilde{B}_i$
7: $C_A{\prime} = (Q_i, \tilde{B}_i)$;
8: **end if**
9: **end while**
10: **end while**
11: $LL_{A/F}(G, A, S^{\diamondsuit}) = \{(Q_i, \tilde{B}_i)\}$;
12: **return** $LL_{A/F}(G, A, S^{\diamondsuit})$;

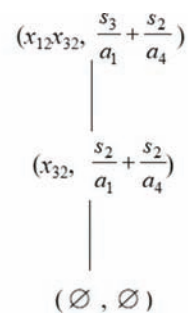

Figure 5 Attribute-oriented linguistic granular concept lattice with respect to $F_1 = \{x_{41}, x_{42}\}$.

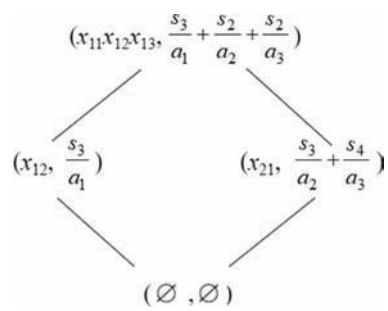

Figure 6 Attribute-oriented linguistic granular concept lattice with respect to $F_2 = \{x_{12}, x_{32}\}$.

Similarly, we can get the attribute-oriented linguistic granular concept lattice when the object set is $F_2 = \{x_{12}, x_{32}\}$ which is shown in Figure 6.

Through the above analysis, it is obvious that attribute-oriented linguistic granular concept lattice is the sub-lattice of attribute-oriented linguistic concept lattice. Constructing attribute-oriented linguistic granular concept lattices from the equivalence relations of attributes can obtain attribute-oriented linguistic concept lattice

with dominance relation. For a type of objects set, the linguistic relations between attributes and objects as well as the dominance relations of the objects are analyzed at the finer object granularity. In this way, we can get results faster according to the different needs of different people in a targeted manner.

7. COMPARATIVE ANALYSIS

In this section, we conduct some comparative analysis to assess the effectiveness of the proposed methods in this paper. Compared with [24] and [46], we summarize the following characteristics:

1. Shao *et al.* [24] proposed a method for constructing object(attribute)-oriented multi-granularity concept lattices, which realized the two-way transformation of attributes from fine-granularity to coarse-granularity and coarse-granularity to fine-granularity, effectively improving the construction speed of object (attribute)-oriented concept lattices. But sometimes 0 and 1 are not enough to accurately express the relation between objects and attributes, the object (attribute)-oriented linguistic concept lattice studied in this paper can directly describe the relations between objects and attributes with qualitative linguistic values. Meanwhile, the object (attribute)-oriented linguistic concept lattice can avoid the loss of information in the process of converting linguistic values into numeral values. And setting different linguistic thresholds can better meet different requirements of different people.
2. Liu *et al.* [46] discussed fuzzy linguistic formal context in uncertain environment, and studied the fuzzy linguistic concept lattice through the common relations between the objects and attributes. In real life, in addition to the importance of the commonality between objects and attributes, the problem of grouping is also very important. The object (attribute)-oriented linguistic concept lattice proposed in this paper takes into account the importance of groups on objects and attributes. Therefore, the object (attribute)-oriented linguistic concept lattice proposed in this paper further studies specific group problems and can solve different practical applications.
3. From the perspective of object dominance relation and attribute dominance relation, object (attribute)-oriented linguistic concept can reflect the relation between individuals, more importantly, it can express the comprehensive ability of the group to provide a new idea for group issues. In addition, the use of the dominance relation between objects and attributes to establish linguistic granular concept lattice can obtain the object (attribute)-oriented linguistic concept lattices, which finds more dominant objects and attributes quickly.

Finally, it should be pointed out that our object (attribute)-oriented linguistic concept lattices with dominance relation are proposed on the basis of the object (attribute)-oriented concept lattice. Its time complexity is a bit high. We will take it as the future discussion direction so as to propose better construction method.

8. CONCLUSION

Granular computing is an important idea in the cross fusion of rough set and concept lattice. It can divide large amounts of complex information into several simple parts according to its characteristics, which is more conducive to studying concept lattices. This paper has discussed the group problems by constructing object-oriented linguistic concept lattice and attribute-oriented linguistic concept lattice based on dominance relation in fuzzy linguistic formal context. For object-oriented linguistic concept lattice, we have put forward a construction algorithm of object-oriented linguistic granular concept lattice by using an equivalence relation with respect to objects, so that we can more purposeful get a specific concept lattice. The attribute-oriented linguistic granular concept lattice has been presented to construct attribute-oriented linguistic concept lattice under the dominance relation of objects. The method proposed in this paper further enriches the concept lattice theory, which describes the importance from the perspective of objects and attributes, thereby extracting the effective information in an uncertain environment.

The object-oriented linguistic granular concepts and attribute-oriented linguistic granular concepts proposed in this paper are constructed based on the equivalence relationship and only consider one granularity, which cannot meet the problem of multiple granularity choices in real life. In the future, we plan to further study the optimal granularity selection problem and construct object-oriented and attribute-oriented linguistic concept lattices by multiple granularities with dominance relation, so as to provide a new way of thinking for the research of formal concept analysis.

CONFLICTS OF INTEREST

Authors have no conflict of interest.

AUTHORS' CONTRIBUTIONS

Among the authors in the list, Hui Cui proposed the innovative points of the paper and mainly took charge of writing and researching. Ansheng Deng played a guiding role and gave a lot of advice and contributions on the paper during the review process. Chunmei Chang carried out algorithm design and data analysis during the revision of the paper. Hongyue Diao was mainly responsible for proofreading and revision issues. Li Zou was as the cooperation teacher who was responsible for checking and guiding of this paper.

ACKNOWLEDGMENTS

This work is partially supported by the National Natural Science Foundation of China (Nos. 61502068, 61772250) and the Scientific Research Project of Department of Education of Liaoning Province (No. LJ2020007).

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