



Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

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To cite this article: Guenbo Hwang, A. S. Fokas (2013) The Modified Korteweg-de Vries Equation on the Half-Line with a Sine-Wave as Dirichlet Datum, Journal of Nonlinear Mathematical Physics 20:1, 135–157, DOI:

<https://doi.org/10.1080/14029251.2013.792492>

To link to this article: <https://doi.org/10.1080/14029251.2013.792492>

Published online: 04 January 2021

The Modified Korteweg-de Vries Equation on the Half-Line with a Sine-Wave as Dirichlet Datum

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Received 14 January 2013

Accepted 15 February 2013

Boundary value problems for integrable nonlinear evolution PDEs, like the modified KdV equation, formulated on the half-line can be analyzed by the so-called unified transform method. For the modified KdV equation, this method yields the solution in terms of the solution of a matrix Riemann-Hilbert problem uniquely determined in terms of the initial datum $q(x, 0)$, as well as of the boundary values $\{q(0, t), q_x(0, t), q_{xx}(0, t)\}$. For the Dirichlet problem, it is necessary to characterize the unknown boundary values $q_x(0, t)$ and $q_{xx}(0, t)$ in terms of the given data $q(x, 0)$ and $q(0, t)$. It is shown here that in the particular case of a vanishing initial datum and of a sine wave as Dirichlet datum, $q_x(0, t)$ and $q_{xx}(0, t)$ can be computed explicitly at least up to third order in a perturbative expansion and that at least up to this order, these functions are asymptotically periodic for large t .

Keywords: Initial-boundary value problem; Generalized Dirichlet to Neumann map; modified Korteweg-de Vries equation.

2000 Mathematics Subject Classification: 47K15, 35Q55

1. Introduction

A unified method for analyzing boundary value problems for linear and integrable nonlinear PDEs was introduced in [9] and used extensively in the literature (reviews for the implementation of this method can be found in [10, 13, 17]). For *integrable nonlinear* evolution PDEs, the unified transform method yields novel integral representations formulated in the complex k -plane (the Fourier plane). These integrals, in addition to the exponentials which appear in the integrals of the linearized version of these nonlinear PDEs, also contain the entries of a matrix-valued function $M(x, t, k)$, which is the solution of a matrix Riemann-Hilbert (RH) problem. This RH problem involves a jump matrix with explicit exponential (x, t) dependence. In addition to explicit exponentials, the entries of the above jump matrix involve functions -called *spectral functions*- which depend only on k and which can be computed in terms of the initial condition and the associated boundary values. For example, for the modified KdV equation on the half-line, the spectral functions can be computed in terms of $q(x, 0)$, $q(0, t)$, $q_x(0, t)$ and $q_{xx}(0, t)$; however for the Dirichlet problem $q_x(0, t)$ and $q_{xx}(0, t)$ are

unknown. The most difficult step of the new method is the characterization of the unknown boundary values in terms of the given initial and boundary conditions. In a recent important development, this characterization was achieved via the analysis of the so-called *global relation* using two different approaches [15, 18]. Both these approaches involve the analysis of a system of quadratically non-linear equations; the formulation in [15] is based on the direct analysis of the eigenfunctions characterizing the relevant spectral functions, while the formulation in [18] is based on an extension of the approach of [1]. However, it should be noted that: (a) For a particular class of boundary conditions called *linearizable*, the global relation yields the spectral functions *directly* in terms of the given initial and boundary conditions bypassing the need to solve the above nonlinear equations. Thus, for this class of problems the new method is as effective as the classical inverse scattering transform method. (b) For the case that the boundary conditions vanish for large t , it is possible to obtain effective asymptotic results using the Deift-Zhou and the Deift-Zhou-Venakides techniques for the asymptotic analysis of these RH problems [5–8].

In contrast to the case that the boundary conditions vanish as $t \rightarrow \infty$, the analysis of the physically significant case of boundary conditions which are periodic in t , require the explicit characterization of the unknown boundary values in terms of the given data. Important results for such boundary value problems have been obtained in [2–4].

Here we concentrate on the simple example

$$q(x, 0) = 0, \quad 0 < x < \infty; \quad q(0, t) = \varepsilon \sin t + O(\varepsilon^4), \quad \varepsilon \rightarrow 0, \quad t > 0. \quad (1.1)$$

Following the technique developed in [18] we compute explicitly the perturbation expansion in ε of both $q_x(0, t)$ and $q_{xx}(0, t)$. These explicit formulae show that at least up to the third order in ε , both $q_x(0, t)$ and $q_{xx}(0, t)$ are asymptotically periodic as $t \rightarrow \infty$.

In a recent breakthrough [14, 16], it has been shown that in the case of the nonlinear Schrödinger equation, if

$$q(x, 0) = 0, \quad 0 < x < \infty, \quad q(0, t) = ae^{i\omega t} + o(1), \quad t \rightarrow \infty, \quad (1.2)$$

with a, ω real constants, then it is possible to solve the global relation exactly as $t \rightarrow \infty$ and to obtain an explicit formula for $q_x(0, t)$ as $t \rightarrow \infty$. The generalization of this result to the more complicated case of

$$q(0, t) = a \sin \omega t + o(t), \quad t \rightarrow \infty, \quad (1.3)$$

for both the nonlinear Schrödinger and the modified KdV equations is an open problem.

2. The spectral functions of the mKdV equation

We study the mKdV equation on the half-line

$$q_t + q_{xxx} - 6\lambda q^2 q_x = 0, \quad 0 < x < \infty, \quad 0 < t < T, \quad (2.1)$$

where $\lambda = \pm 1$, $T < \infty$ and $q(x, t)$ decays rapidly for all t as $x \rightarrow \infty$. We assume that the given initial condition $q_0(x) = q(x, 0)$ vanishes sufficiently fast for all t as $x \rightarrow \infty$. We denote the boundary values as

$$g_0(t) = q(0, t), \quad g_1(t) = q_x(0, t), \quad g_2(t) = q_{xx}(0, t), \quad (2.2)$$

which are assumed to be sufficiently smooth.

It is well-known that (2.1) admits the following Lax pair formulation:

$$\mu_x + if_1(k)\hat{\sigma}_3\mu = Q(x,t,k)\mu, \quad (2.3a)$$

$$\mu_t + if_2(k)\hat{\sigma}_3\mu = \tilde{Q}(x,t,k)\mu, \quad (2.3b)$$

where $k \in \mathbb{C}$ is a spectral parameter, μ is a 2×2 matrix-valued eigenfunction, $\hat{\sigma}_3 A = [\sigma_3, A]$ with $\sigma_3 = \text{diag}(1, -1)$, and

$$f_1(k) = k, \quad f_2(k) = 4k^3, \quad Q(x,t) = \begin{pmatrix} 0 & q \\ \lambda q & 0 \end{pmatrix}, \quad (2.4)$$

$$\tilde{Q}(x,t) = 2Q^3 - Q_{xx} - 2ik(Q^2 + Q_x)\sigma_3 + 4k^2Q. \quad (2.5)$$

Let the domains $\{D_j\}_1^4$ be given by (cf. figure 1)

$$D_1 = \{\text{Im } f_1 > 0 \text{ and } \text{Im } f_2 > 0\}, \quad D_2 = \{\text{Im } f_1 > 0 \text{ and } \text{Im } f_2 < 0\}, \quad (2.6)$$

$$D_3 = \{\text{Im } f_1 < 0 \text{ and } \text{Im } f_2 > 0\}, \quad D_4 = \{\text{Im } f_1 < 0 \text{ and } \text{Im } f_2 < 0\}. \quad (2.7)$$

We introduce the complex-valued functions $\Psi(x,k)$ and $\Phi(x,k)$ as the following solutions of the x -part and the t -part of the Lax pair (2.3) evaluated at $t = 0$ and $x = 0$, respectively:

$$\Psi_x + if_1(k)\hat{\sigma}_3\Psi = Q_0\Psi, \quad \lim_{x \rightarrow \infty} \Psi = I, \quad (2.8a)$$

$$\Phi_x + if_2(k)\hat{\sigma}_3\Phi = \tilde{Q}_0\Psi, \quad \Phi(0,t) = I, \quad (2.8b)$$

where $Q_0(x) = Q(x,0)$ and $\tilde{Q}_0(t,k) = \tilde{Q}(0,t,k)$. Equations (2.8) imply that the functions Ψ and Φ satisfy the following system of linear Volterra integral equations:

$$\Psi(x,k) = I - \int_x^\infty e^{-if_1(k)(x-\xi)\hat{\sigma}_3}(Q_0\Psi)(\xi,k)d\xi, \quad k \in (\mathbb{C}^-, \mathbb{C}^+), \quad (2.9)$$

$$\Phi(x,k) = I + \int_0^t e^{-if_2(k)(t-\tau)\hat{\sigma}_3}(\tilde{Q}_0\Phi)(\tau,k)d\tau, \quad k \in \mathbb{C}, \quad (2.10)$$

where the notation $k \in (\mathbb{C}^-, \mathbb{C}^+)$ in (2.9) denotes that the first and second column vectors of $\Psi(x,k)$ are defined for $\text{Im } k \leq 0$ and $\text{Im } k \geq 0$ respectively. Due to the symmetry of Q , the functions Ψ and Φ can be written in the form

$$\Psi(x,k) = \begin{pmatrix} \overline{\Psi_2(x,\bar{k})} & \Psi_1(x,k) \\ \lambda \overline{\Psi_1(x,\bar{k})} & \Psi_2(x,k) \end{pmatrix}, \quad k \in (\mathbb{C}^-, \mathbb{C}^+), \quad (2.11a)$$

$$\Phi(t,k) = \begin{pmatrix} \overline{\Phi_2(t,\bar{k})} & \Phi_1(t,k) \\ \lambda \overline{\Phi_1(t,\bar{k})} & \Phi_2(t,k) \end{pmatrix}, \quad k \in \mathbb{C}. \quad (2.11b)$$

The spectral functions $\{a(k), b(k), A(k), B(k)\}$ can be expressed in terms of the initial datum and of the boundary values. Given $q_0(x)$, the spectral functions $\{a(k), b(k)\}$ are defined by

$$a(k) = \Psi_2(0,k), \quad b(k) = \Psi_1(0,k), \quad k \in \mathbb{C}^+, \quad (2.12)$$

while the spectral functions $\{A(k), B(k)\}$ are defined by

$$A(k) = \overline{\Phi_2(T,\bar{k})}, \quad B(k) = -e^{2if_2(k)T\sigma_3}\Phi_1(T,k), \quad k \in \mathbb{C}, \quad (2.13)$$

in terms of $\{g_0, g_1, g_2\}$.

The spectral functions satisfy the *global relation*

$$\Phi_1(t, k) + \frac{b(k)}{a(k)} \overline{\Phi_2(t, \bar{k})} e^{-2if_2(k)t} = c(t, k), \quad \text{Im } k \geq 0, \quad (2.14)$$

where $c(t, k)$ is analytic and bounded in $\text{Im } k > 0$ and is of $O(1/k)$ as $k \rightarrow \infty$.

It has been shown in [12, 15] that the functions Φ_1 and Φ_2 satisfy the following system of nonlinear integral equations:

$$\Phi_1(t, k) = \int_0^t e^{-8ik^3(t-t')} [-2ik\lambda g_0^2 \Phi_1 + (2\lambda g_0^3 + 4k^2 g_0 + 2ikg_1 - g_2) \Phi_2](t', k) dt', \quad (2.15a)$$

$$\Phi_2(t, k) = 1 + \lambda \int_0^t [(2\lambda g_0^3 + 4k^2 g_0 - 2ikg_1 - g_2) \Phi_1 + 2ikg_0^2 \Phi_2](t', k) dt'. \quad (2.15b)$$

Furthermore, for the Dirichlet problem, the unknown boundary values $g_1(t)$ and $g_2(t)$ satisfy the following relations:

$$\begin{aligned} g_1(t) = & \frac{2g_0(t)}{\pi} \int_{\partial D_3} \chi_2(t, k) dk + \frac{2}{\pi i} \int_{\partial D_3} \left[k\chi_1(t, k) - \frac{3g_0(t)}{2i} \right] dk \\ & - \frac{2}{\pi i} \int_{\partial D_3} k e^{-8ik^3 t} \left[(\alpha^2 - \alpha) \frac{b(\alpha k)}{a(\alpha k)} \overline{\Phi_2(t, \alpha \bar{k})} + (\alpha - \alpha^2) \frac{b(\alpha^2 k)}{a(\alpha^2 k)} \overline{\Phi_2(t, \alpha^2 \bar{k})} \right] dk \\ & + 4 \left\{ (1 - \alpha^2) \sum_{k_j \in D'_1} + (1 - \alpha) \sum_{k_j \in D''_1} \right\} k_j e^{-8ik_j^3 t} \text{Res}_{k_j} \frac{b(k)}{a(k)} \overline{\Phi_2(t, \bar{k})} \end{aligned} \quad (2.16a)$$

and

$$\begin{aligned} g_2(t) = & \lambda g_0^3(t) - \frac{4}{\pi} \int_{\partial D_3} \left[k^2 \chi_1(t, k) - \frac{3kg_0(t)}{2i} \right] dk \\ & + \frac{4g_0(t)}{\pi i} \int_{\partial D_3} k \hat{\chi}_2(t, k) dk + \frac{2g_1(t)}{\pi} \int_{\partial D_3} \chi_2(t, k) dk \\ & + \frac{4}{\pi} \int_{\partial D_3} k^2 e^{-8ik^3 t} \left[(1 - \alpha) \frac{b(\alpha k)}{a(\alpha k)} \overline{\Phi_2(t, \alpha \bar{k})} + (1 - \alpha^2) \frac{b(\alpha^2 k)}{a(\alpha^2 k)} \overline{\Phi_2(t, \alpha^2 \bar{k})} \right] dk \\ & - 8i \left\{ (1 - \alpha) \sum_{k_j \in D'_1} + (1 - \alpha^2) \sum_{k_j \in D''_1} \right\} k_j^2 e^{-8ik_j^3 t} \text{Res}_{k_j} \frac{b(k)}{a(k)} \overline{\Phi_2(t, \bar{k})}, \end{aligned} \quad (2.16b)$$

where $\alpha = e^{2\pi i/3}$, $\{k_j\}_1^N$ is the set of the possible zeros of $a(k)$ (assumed to be a finite set of simple zeros), $D'_1 = D_1 \cap \{\text{Re } k > 0\}$ and $D''_1 = D_1 \cap \{\text{Re } k < 0\}$, and for $j = 1, 2$, the symmetric combinations χ_j , $\hat{\chi}_j$ and $\check{\chi}_j$ are defined by

$$\chi_j(t, k) = \Phi_j(t, k) + \alpha \Phi_j(t, \alpha k) + \alpha^2 \Phi_j(t, \alpha^2 k), \quad (2.17)$$

$$\hat{\chi}_j(t, k) = \Phi_j(t, k) + \alpha^2 \Phi_j(t, \alpha k) + \alpha \Phi_j(t, \alpha^2 k), \quad (2.18)$$

$$\check{\chi}_j(t, k) = \Phi_j(t, k) + \Phi_j(t, \alpha k) + \Phi_j(t, \alpha^2 k). \quad (2.19)$$

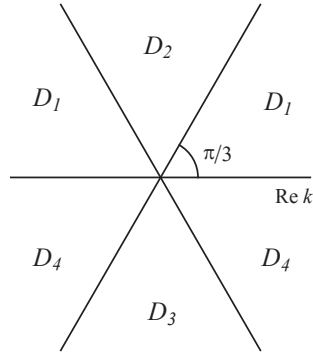


Fig. 1. The domains $\{D_j\}_1^4$.

3. A perturbation expansion of the unknown boundary values

We expand the functions Φ_j , q_0 and g_0 as

$$\Phi_j = \Phi_{j0} + \varepsilon \Phi_{j1} + \varepsilon^2 \Phi_{j2} + \cdots, \quad j = 1, 2, \quad (3.1a)$$

$$q_0 = \varepsilon q_{01} + \varepsilon^2 q_{02} + \cdots, \quad (3.1b)$$

$$g_0 = \varepsilon g_{01} + \varepsilon^2 g_{02} + \cdots, \quad (3.1c)$$

where ε is a small perturbation parameter. Substituting (3.1a) into (2.15a) and (2.15b), we find $\Phi_{10} = 0$, $\Phi_{20} = 1$ and $\Phi_{21} = 0$.

We next derive the first few terms of the above expansions. We assume a vanishing initial datum, that is, $q_0(x) = 0$. In this case, $a(k) = 1$ and $b(k) = 0$.

Proposition 3.1. Assume that $q(x, 0) = 0$, $x > 0$, and

$$q(0, t) = \varepsilon g_{01} + \varepsilon^2 g_{02} + \varepsilon^3 g_{03} + \cdots, \quad (3.2)$$

where $\varepsilon > 0$ is a small perturbation parameter and g_{0n} , $n \geq 1$, are sufficiently smooth and compatible with the zero initial datum. (a) The unknown boundary value $q_x(0, t)$ can be expressed in the form

$$q_x(0, t) = \varepsilon g_{11}(t) + \varepsilon^2 g_{12}(t) + \varepsilon^3 g_{13}(t) + \cdots, \quad (3.3)$$

where

$$g_{11}(t) = \frac{3c_1}{\pi} \int_0^t \frac{\dot{g}_{01}(t')}{(t-t')^{1/3}} dt', \quad g_{12}(t) = \frac{3c_1}{\pi} \int_0^t \frac{\dot{g}_{02}(t')}{(t-t')^{1/3}} dt' \quad (3.4)$$

and

$$\begin{aligned}
 g_{13}(t) = & -\frac{\lambda c_1}{\pi} \int_0^t g_{02}^2(t') \int_0^{t'} \frac{g_{11}(t'')}{(t-t'')^{4/3}} dt'' dt' + \frac{3c_1}{\pi} \int_0^t \frac{\dot{g}_{03}(t')}{(t-t')^{1/3}} dt' \\
 & + \frac{\lambda c_1}{\pi} \left\{ \int_0^t g_{01}(t') \left[\int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' - \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{\dot{g}_{01}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right. \right. \\
 & - \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \left. \right] dt' + \int_0^t g_{11}(t') \left[\int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{01}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right. \\
 & + \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{4/3}} dt''' dt'' - \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \left. \right] dt' \\
 & - \int_0^t g_{21}(t') \left[\int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{4/3}} dt''' dt'' - \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{g_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right. \\
 & \left. \left. + \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{01}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right] dt' \right\} + \lambda g_{01}(t) \int_0^t [g_{11}^2 - 2g_{01}g_{21}](t') dt', \quad (3.5)
 \end{aligned}$$

with $\tilde{t} = t - t' + t'' - t'''$ and

$$c_1 = -\frac{\sqrt{3}}{6} \Gamma(1/3). \quad (3.6)$$

(b) The unknown boundary value $q_{xx}(0, t)$ can be expressed in the form

$$q_{xx}(0, t) = \varepsilon g_{21}(t) + \varepsilon^2 g_{22}(t) + \varepsilon^3 g_{23}(t) + \cdots, \quad (3.7)$$

where

$$g_{21}(t) = \frac{6c_2}{i\pi} \int_0^t \frac{\dot{g}_{01}(t')}{(t-t')^{2/3}} dt', \quad g_{22}(t) = \frac{6c_2}{i\pi} \int_0^t \frac{\dot{g}_{02}(t')}{(t-t')^{2/3}} dt' \quad (3.8)$$

and

$$\begin{aligned}
 g_{23}(t) = & \frac{3\lambda}{2} g_{01}^3(t) - \frac{4\lambda c_2}{\pi i} \int_0^t g_{02}^2(t') \int_0^{t'} \frac{g_{11}(t'')}{(t-t'')^{5/3}} dt'' dt' + \frac{6c_2}{\pi i} \int_0^t \frac{\dot{g}_{03}(t')}{(t-t')^{2/3}} dt' \\
 & + \frac{4\lambda c_2}{\pi i} \left\{ \int_0^t g_{01}(t') \left[\int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{11}(t''')}{\tilde{t}^{5/3}} dt''' dt'' - \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{\dot{g}_{01}(t''')}{\tilde{t}^{5/3}} dt''' dt'' \right. \right. \\
 & - \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{5/3}} dt''' dt'' \left. \right] dt' + \int_0^t g_{11}(t') \left[\int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{01}(t''')}{\tilde{t}^{5/3}} dt''' dt'' \right. \\
 & + \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{5/3}} dt''' dt'' - \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{11}(t''')}{\tilde{t}^{5/3}} dt''' dt'' \left. \right] dt' \\
 & - \int_0^t g_{21}(t') \left[\int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{5/3}} dt''' dt'' - \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{g_{11}(t''')}{\tilde{t}^{5/3}} dt''' dt'' \right. \\
 & \left. \left. + \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{01}(t''')}{\tilde{t}^{5/3}} dt''' dt'' \right] dt' \right\} + \lambda g_{11}(t) \int_0^t [g_{11}^2 - 2g_{01}g_{21}](t') dt', \quad (3.9)
 \end{aligned}$$

with

$$c_2 = \frac{i\sqrt{3}}{12} \Gamma(2/3). \quad (3.10)$$

Proof. (a) Equations (2.16a) and (3.1) imply

$$g_{11}(t) = \frac{2}{\pi i} \int_{\partial D_3} \left[k \chi_{11}(t, k) - \frac{3g_{01}(t)}{2i} \right] dk. \quad (3.11)$$

The function $\Phi_{11}(t, k)$ is given by

$$\Phi_{11}(t, k) = \int_0^t e^{-8ik^3(t-t')} (4k^2 g_{01} + 2ik g_{11} - g_{21})(t') dt', \quad (3.12)$$

and hence,

$$\chi_{11}(t, k) = 12k^2 \int_0^t e^{-8ik^3(t-t')} g_{01}(t') dt'. \quad (3.13)$$

Substituting (3.13) into (3.11) and then integrating by parts with respect to dt' , (3.11) can be written in the form

$$g_{11}(t) = \frac{3}{\pi} \int_{\partial D_3} \int_0^t e^{-8ik^3(t-t')} \dot{g}_{01}(t') dt' dk. \quad (3.14)$$

In order to integrate with respect dk , we will use the identity

$$\int_{\partial D_3} e^{-8ik^3 t} dk = \frac{c_1}{t^{1/3}}, \quad c_1 = -\frac{\sqrt{3}}{6} \Gamma(1/3). \quad (3.15)$$

In order to derive (3.15) we let $\partial D_3 = C_1 \cup C_2$, where $C_1 = \partial D_3 \cap \mathbb{C}_{IV}$ and $C_2 = \partial D_3 \cap \mathbb{C}_{III}$ (see figure 2). Cauchy's theorem implies

$$\int_{C_1} e^{-8ik^3 t} dk = - \int_{\tilde{C}_1} e^{-8ik^3 t} dk = -e^{-i\pi/6} \int_0^\infty e^{-8\rho^3 t} d\rho, \quad (3.16)$$

where $\tilde{C}_1 = \{\arg k = -i\pi/6\}$ (cf. figure 2). Hence, evaluating the rhs of the above equation, we obtain

$$\int_{C_1} e^{-8ik^3 t} dk = -\frac{e^{-i\pi/6}}{6} \Gamma(1/3) \frac{1}{t^{1/3}}. \quad (3.17)$$

For the integral along C_2 , we change variables, $k \rightarrow \alpha^2 k$. Then the contour C_2 is deformed into $[0, \infty)$, and in analogy with (3.15) we now find

$$\int_{C_2} e^{-8ik^3 t} dk = \frac{e^{-5i\pi/6}}{6} \Gamma(1/3) \frac{1}{t^{1/3}}. \quad (3.18)$$

Hence (3.17) and (3.18) imply (3.15). Using (3.15) in (3.14), we obtain

$$g_{11}(t) = \frac{3c_1}{\pi} \int_0^t \frac{\dot{g}_{01}(t')}{(t-t')^{1/3}} dt'. \quad (3.19)$$

In a similar way we find that $g_{12}(t)$ is given by (3.4).

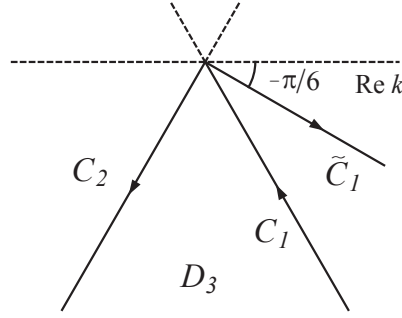


Fig. 2. The oriented contours C_1 , C_2 and \tilde{C}_1 .

In order to derive (3.5), we note that

$$g_{13}(t) = \frac{2}{\pi i} \int_{\partial D_3} \left[k \chi_{13}(t, k) - \frac{3g_{03}(t)}{2i} \right] dk + \frac{2g_{01}(t)}{\pi} \int_{\partial D_3} \chi_{22}(t, k) dk. \quad (3.20)$$

Using (2.15b) and (3.1), we find that the function Φ_{22} is given by

$$\Phi_{22}(t, k) = \lambda \int_0^t [(4k^2 g_{01} - 2ikg_{11} - g_{21})\Phi_{11} + 2ikg_{01}^2] dt'. \quad (3.21)$$

Thus,

$$\chi_{22}(t, k) = \lambda \int_0^t (4k^2 g_{01} \check{\chi}_{11} - 2ikg_{11} \hat{\chi}_{11} - g_{21} \chi_{11}) dt', \quad (3.22a)$$

$$\hat{\chi}_{22}(t, k) = \lambda \int_0^t (4k^2 g_{01} \chi_{11} - 2ikg_{11} \check{\chi}_{11} - g_{21} \hat{\chi}_{11} + 6ikg_{01}^2) dt' \quad (3.22b)$$

$$\check{\chi}_{22}(t, k) = \lambda \int_0^t (4k^2 g_{01} \hat{\chi}_{11} - 2ikg_{11} \chi_{11} - g_{21} \check{\chi}_{11}) dt', \quad (3.22c)$$

where the symmetric combinations $\hat{\chi}_{11}$ and $\check{\chi}_{11}$ are given by

$$\hat{\chi}_{11}(t, k) = 6ik \int_0^t e^{-8ik^3(t-t')} g_{11}(t') dt', \quad (3.23a)$$

$$\check{\chi}_{11}(t, k) = -3 \int_0^t e^{-8ik^3(t-t')} g_{21}(t') dt'. \quad (3.23b)$$

Furthermore, using (2.15a) we find

$$\begin{aligned} \Phi_{13}(t, k) = & \int_0^t e^{-8ik^3(t-t')} [-2ik\lambda g_{01}^2 \Phi_{11} + (2\lambda g_{01}^3 + 4k^2 g_{03} + 2ikg_{13} - g_{23}) \\ & + (4k^2 g_{01} + 2ikg_{11} - g_{21}) \Phi_{22}] dt', \end{aligned} \quad (3.24)$$

which implies

$$\chi_{13}(t, k) = \int_0^t e^{-8ik^3(t-t')} [-2ik\lambda g_{01}^2 \hat{\chi}_{11} + 12k^2 g_{03} + 4k^2 g_{01} \check{\chi}_{22} + 2ikg_{11} \hat{\chi}_{22} - g_{21} \chi_{22}] dt'. \quad (3.25)$$

We next evaluate the second integral of (3.20); using (3.22) and (3.23) we obtain

$$\begin{aligned} \int_{\partial D_3} \chi_{22}(t, k) dk &= \lambda \int_{\partial D_3} 12k^2 \left\{ \int_0^t \left[-g_{01}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} g_{21}(t'') dt'' \right. \right. \\ &\quad \left. \left. + g_{11}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} g_{11}(t'') dt'' - g_{21}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} g_{01}(t'') dt'' \right] dt' \right\} dk. \end{aligned} \quad (3.26)$$

Using the identity

$$\int_{\partial D_3} k^2 \int_0^{t'} e^{-8ik^3(t'-t'')} f(t'') dt'' dk = \frac{\pi}{24} f(t'), \quad (3.27)$$

we can integrate the rhs of (3.26) with respect to dk :

$$\int_{\partial D_3} \chi_{22}(t, k) dk = \frac{\lambda \pi}{2} \int_0^t g_{11}^2(t') dt' - \lambda \pi \int_0^t g_{01}(t') g_{21}(t') dt'. \quad (3.28)$$

Thus, the second integral of (3.20) yields

$$\frac{g_{01}(t)}{\pi} \int_{\partial D_3} \chi_{22}(t, k) dk = \lambda g_{01}(t) \int_0^t [g_{11}^2 - g_{01} g_{21}] (t') dt'. \quad (3.29)$$

Next we evaluate the first integral of (3.20); using (3.25) we find

$$\begin{aligned} \int_{\partial D_3} k \chi_{13}(t, k) dk &= \int_{\partial D_3} \int_0^t e^{-8ik^3(t-t')} \left[-2\lambda ik^2 g_{01}^2 \hat{\chi}_{11} + 12k^3 g_{03} + 4k^3 g_{01} \check{\chi}_{22} \right. \\ &\quad \left. + 2ik^2 g_{11} \hat{\chi}_{22} - k g_{21} \chi_{22} \right] (t', k) dt' dk. \end{aligned} \quad (3.30)$$

We will evaluate each integral of (3.30). For the first term of the rhs of (3.30), integrating by parts with respect to dt' , we find

$$\frac{3\lambda}{2i} \int_{\partial D_3} \int_0^t g_{02}^2(t') \left[e^{-8ik^3(t-t')} g_{11}(t') - \int_0^{t'} e^{-8ik^3(t-t'')} \dot{g}_{11}(t'') dt'' \right] dt' dk. \quad (3.31)$$

Using (3.15), (3.31) becomes

$$\frac{3\lambda c_1}{2i} \int_0^t g_{02}^2(t') \frac{g_{11}(t')}{(t-t')^{1/3}} dt' - \frac{3\lambda c_1}{2i} \int_0^t g_{02}^2(t') \int_0^{t'} \frac{\dot{g}_{11}(t'')}{(t-t'')^{1/3}} dt'' dt', \quad (3.32)$$

and then integrating by parts with respect to dt'' , we obtain

$$-2\lambda i \int_{\partial D_3} k^2 \int_0^t e^{-8ik^3(t-t')} g_{01}^2(t') \hat{\chi}_{11}(t', k) dt' dk = \frac{\lambda c_1}{2i} \int_0^t g_{02}^2(t') \int_0^{t'} \frac{g_{11}(t'')}{(t-t'')^{1/3}} dt'' dt'. \quad (3.33)$$

Similarly, the second term of rhs of (3.30) yields

$$\int_{\partial D_3} 12k^3 \int_0^t e^{-8ik^3(t-t')} g_{03}(t') dt' dk = \frac{3}{2i} \int_{\partial D_3} g_{03}(t) dk - \frac{3c_1}{2i} \int_0^t \frac{\dot{g}_{03}(t')}{(t-t')^{1/3}} dt'. \quad (3.34)$$

For the third term of the rhs of (3.30) we find

$$\begin{aligned}
 & \int_{\partial D_3} 4k^3 \int_0^t e^{-8ik^3(t-t')} g_{01}(t') \check{\chi}_{22}(t', k) dt' dk \\
 &= \lambda i \int_{\partial D_3} 96k^6 \int_0^t g_{01}(t') \int_0^{t'} g_{01}(t'') \int_0^{t''} e^{-8ik^3 \tilde{t}} g_{11}(t''') dt''' dt'' dt' \\
 & - \lambda i \int_{\partial D_3} 96k^6 \int_0^t g_{01}(t') \int_0^{t'} g_{11}(t'') \int_0^{t''} e^{-8ik^3 \tilde{t}} g_{01}(t''') dt''' dt'' dt' \\
 & + \lambda \int_{\partial D_3} 12k^3 \int_0^t g_{01}(t') \int_0^{t'} g_{21}(t'') \int_0^{t''} e^{-8ik^3 \tilde{t}} g_{21}(t''') dt''' dt'' dt', \quad (3.35)
 \end{aligned}$$

where $\tilde{t} = t - t' + t'' - t'''$. Integrating by parts with respect to dt''' , the first integral of (3.35) becomes

$$\begin{aligned}
 & \lambda \int_{\partial D_3} 12k^3 \int_0^t e^{-8ik^3(t-t')} g_{01}(t') \int_0^{t'} g_{01}(t'') g_{11}(t'') dt'' dt' dk \\
 & + \frac{3\lambda i}{2} \int_{\partial D_3} \int_0^t e^{-8ik^3(t-t')} g_{01}(t') \int_0^{t'} g_{01}(t'') \dot{g}_{11}(t'') dt'' dt' dk \\
 & - \frac{3\lambda i}{2} \int_{\partial D_3} \int_0^t g_{01}(t') \int_0^{t'} g_{01}(t'') \int_0^{t''} e^{-8ik^3 \tilde{t}} \ddot{g}_{11}(t'') dt'' dt' dk. \quad (3.36)
 \end{aligned}$$

Integrating by parts the first term of the above expression with respect to dt' and then employing (3.15), we find

$$\begin{aligned}
 & \lambda i \int_{\partial D_3} 96k^6 \int_0^t g_{01}(t') \int_0^{t'} g_{01}(t'') \int_0^{t''} e^{-8ik^3 \tilde{t}} g_{11}(t''') dt''' dt'' dt' \\
 &= \frac{3\lambda}{2i} \int_{\partial D_3} g_{01}(t) \int_0^t g_{01}(t') g_{11}(t') dt' - \frac{3\lambda c_1}{2i} \int_0^t \frac{\dot{g}_{01}(t')}{(t-t')^{1/3}} \int_0^{t'} g_{01}(t'') g_{11}(t'') dt'' \\
 & - \frac{3\lambda c_1}{2i} \int_0^t \frac{g_{01}^2(t') g_{11}(t')}{(t-t')^{1/3}} dt' - \frac{\lambda c_1}{2i} \int_0^t g_{01}(t') \int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' dt'. \quad (3.37)
 \end{aligned}$$

The other two terms of the rhs of (3.35) can be computed in a similar way; hence (3.35) becomes

$$\begin{aligned}
 & \int_{\partial D_3} 4k^3 \int_0^t e^{-8ik^3(t-t')} g_{01}(t') \check{\chi}_{22}(t', k) dt' dk \\
 &= -\frac{\lambda c_1}{2i} \int_0^t g_{01}(t') \left\{ \int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right. \\
 & \left. - \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{\dot{g}_{01}(t''')}{\tilde{t}^{4/3}} dt''' dt'' - \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right\}. \quad (3.38)
 \end{aligned}$$

Similarly, the fourth term of (3.30) yields

$$\begin{aligned}
 & 2i \int_{\partial D_3} k^2 \int_0^t e^{-8ik^3(t-t')} g_{11}(t') \hat{\chi}_{22}(t', k) dt' dk \\
 &= -\frac{\lambda c_1}{2i} \int_0^t g_{11}(t') \left\{ \int_0^{t'} g_{01}(t'') \int_0^{t''} \frac{\dot{g}_{01}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right. \\
 & \left. + \int_0^{t'} g_{11}(t'') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{4/3}} dt''' dt'' - \int_0^{t'} g_{21}(t'') \int_0^{t''} \frac{g_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' \right\}, \quad (3.39)
 \end{aligned}$$

and the last integral of (3.30) becomes

$$\begin{aligned} & - \int_{\partial D_3} k \int_0^t e^{-8ik^3(t-t')} g_{21}(t') \chi_{22}(t', k) dt' dk \\ &= \frac{\lambda c_1}{2i} \int_0^t g_{21}(t') \left\{ \int_0^{t'} g_{01}(t') \int_0^{t''} \frac{g_{21}(t''')}{\tilde{t}^{4/3}} dt''' dt'' dt' \right. \\ & \quad \left. - \int_0^{t'} g_{11}(t') \int_0^{t''} \frac{g_{11}(t''')}{\tilde{t}^{4/3}} dt''' dt'' dt' + \int_0^{t'} g_{21}(t') \int_0^{t''} \frac{g_{01}(t''')}{\tilde{t}^{4/3}} dt''' dt'' dt' \right\}. \end{aligned} \quad (3.40)$$

Combining (3.39) with (3.40), (3.33), (3.34) and (3.38), as well as (3.29), we obtain (3.5).

(b) Recall that $q_{xx}(0, t) = g_2(t)$ is given by

$$\begin{aligned} g_2(t) = & \lambda g_0^3(t) - \frac{4}{\pi} \int_{\partial D_3} \left[k^2 \chi_1(t, k) - \frac{3kg_0}{2i} \right] dk \\ & + \frac{4g_0(t)}{i\pi} \int_{\partial D_3} k \hat{\chi}_2(t, k) dk + \frac{2g_1(t)}{\pi} \int_{\partial D_3} \chi_2(t, k) dk. \end{aligned} \quad (3.41)$$

Hence, the first term of the expansion of g_2 is

$$g_{21}(t) = -\frac{4}{\pi} \int_{\partial D_3} \left[k^2 \chi_{11}(t, k) - \frac{3kg_{01}}{2i} \right] dk. \quad (3.42)$$

Substituting (3.13) into (3.42) and integrating by parts with respect to dt' we find

$$g_{21}(t) = \frac{6}{i\pi} \int_{\partial D_3} k \int_0^t e^{-8ik^3(t-t')} \dot{g}_{01}(t') dt' dk. \quad (3.43)$$

Following steps similar to those used in (a), we find

$$\int_{\partial D_3} k e^{-8ik^3 t} dk = \frac{c_2}{t^{2/3}}, \quad c_2 = \frac{i\sqrt{3}}{12} \Gamma(2/3). \quad (3.44)$$

Hence, employing (3.44) we obtain

$$g_{21}(t) = \frac{6c_2}{i\pi} \int_0^t \frac{\dot{g}_{01}(t')}{(t-t')^{2/3}} dt'. \quad (3.45)$$

The function $g_{22}(t)$ can be obtained in a similar manner.

Next, we derive the third order term of the expansion, which satisfies

$$\begin{aligned} g_{23}(t) = & \lambda g_{01}^3 - \frac{4}{\pi} \int_{\partial D_3} \left[k^2 \chi_{13}(t, k) - \frac{3kg_{01}}{2i} \right] dk \\ & + \frac{4g_{01}(t)}{i\pi} \int_{\partial D_3} k \hat{\chi}_{22}(t, k) dk + \frac{2g_{11}(t)}{\pi} \int_{\partial D_3} \chi_{22}(t, k) dk. \end{aligned} \quad (3.46)$$

In order to evaluate the second integral of the rhs of (3.46), we use (3.22b) together with (3.13) and (3.23):

$$\begin{aligned} \int_{\partial D_3} k \hat{\chi}_{22}(t, k) dk &= \lambda \int_{\partial D_3} 48k^5 \int_0^t g_{01}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} g_{01}(t'') dt'' dt' dk \\ &+ \lambda i \int_{\partial D_3} 6k^2 \int_0^t g_{11}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} g_{21}(t'') dt'' dt' dk \\ &- \lambda i \int_{\partial D_3} 6k^2 \int_0^t g_{21}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} g_{11}(t'') dt'' dt' dk + \lambda i \int_{\partial D_3} 6k^2 \int_0^t g_{01}^2(t') dt' dk. \end{aligned} \quad (3.47)$$

Using (3.27), we find that the second and third integrals of the rhs of (3.47) vanish. For the first integral of the rhs of (3.47), integrating by parts with respect to dt' we find

$$-6\lambda i \int_{\partial D_3} k^2 \int_0^t g_{01}^2(t') dt' dk + 6\lambda i \int_{\partial D_3} k^2 \int_0^t g_{01}(t') \int_0^{t'} e^{-8ik^3(t'-t'')} \dot{g}_{01}(t'') dt'' dt' dk. \quad (3.48)$$

The first term of (3.48) cancels with the last term of the rhs of (3.47). Also, using (3.27), the second term of (3.48) yields

$$\frac{\lambda i \pi}{4} \int_0^t g_{01}(t') \dot{g}_{01}(t') dt' = \frac{\lambda i \pi}{8} g_{01}^2(t). \quad (3.49)$$

Thus, we find

$$\frac{4g_{01}(t)}{i\pi} \int_{\partial D_3} k \hat{\chi}_{22}(t, k) dk = \frac{\lambda}{2} g_{01}^3(t). \quad (3.50)$$

The first and third integrals of (3.46) can be evaluated in a similar way as in (a) and hence (3.9) follows. \square

4. Periodic boundary condition

We next concentrate in the particular case that the Dirichlet datum is a sine wave.

Proposition 4.1. *Let*

$$q(0, t) = \varepsilon \sin t + O(\varepsilon^4), \quad \varepsilon \rightarrow 0, \quad t > 0; \quad q(x, 0) = 0, \quad x \in \mathbb{R}^+. \quad (4.1)$$

Then, $q_x(0, t)$ and $q_{xx}(0, t)$ are given by (3.3) and (3.7), where

$$g_{12}(t) = g_{22}(t) = 0, \quad (4.2)$$

$$g_{11}(t) = \frac{1}{2}(\alpha e^{it} + \alpha^2 e^{-it}) + \int_{\partial \tilde{D}_3} e^{-8ik^3 t} F(k^3) dk, \quad (4.3)$$

$$g_{21}(t) = \frac{i}{2}(\alpha^2 e^{it} - \alpha e^{-it}) - 2i \int_{\partial \tilde{D}_3} e^{-8ik^3 t} k F(k^3) dk, \quad (4.4)$$

with $\alpha = e^{2i\pi/3}$, $\partial \tilde{D}_3$ is depicted in figure 3 and

$$F(k^3) = \frac{12}{i\pi} \left(\frac{k^3}{8k^3 + 1} - \frac{k^3}{8k^3 - 1} \right). \quad (4.5)$$

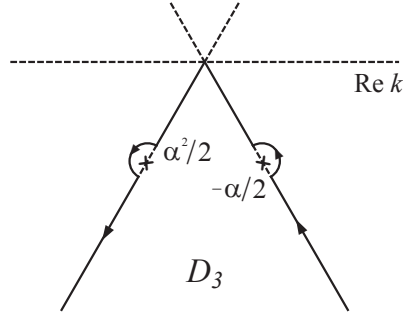


Fig. 3. The deformed contour $\partial\tilde{D}_3$, where $\alpha = e^{2i\pi/3}$.

Proof. Equation (3.13) implies that

$$k\chi_{11}(t, k) = -6ik^3 e^{-8ik^3 t} \int_0^t (e^{i\tau(8k^3+1)} - e^{i\tau(8k^3-1)}) d\tau. \quad (4.6)$$

Integrating the above equation with respect $d\tau$ and substituting the resulting expression into (3.11), we obtain

$$g_{11}(t) = \frac{2}{i\pi} \int_{\partial D_3} \left[-\frac{6k^3}{8k^3+1} (e^{it} - e^{-8ik^3 t}) + \frac{6k^3}{8k^3-1} (e^{-it} - e^{-8ik^3 t}) + \frac{3}{4} (e^{it} - e^{-it}) \right] dk. \quad (4.7)$$

Note that the integrand in (4.7) has removable singularities at $k = -\alpha/2$ and at $k = \alpha^2/2$. Hence, before splitting, the integral we deform the contour ∂D_3 to the contour $\partial\tilde{D}_3$ depicted in figure 3. Thus, (4.7) yields

$$g_{11}(t) = \frac{3}{2i\pi} \int_{\partial\tilde{D}_3} \left(\frac{e^{it}}{8k^3+1} + \frac{e^{-it}}{8k^3-1} \right) dk + \frac{12}{i\pi} \int_{\partial\tilde{D}_3} e^{-8ik^3 t} \left(\frac{k^3}{8k^3+1} - \frac{k^3}{8k^3-1} \right) dk. \quad (4.8)$$

The first integral in (4.8) can be evaluated by the residue theorem, and hence we find $g_{11}(t)$ as in (4.3).

In order to derive (4.4) we use the representation

$$g_{21}(t) = -\frac{3}{\pi} \int_{\partial\tilde{D}_3} \left(\frac{ke^{it}}{8k^3+1} + \frac{ke^{-it}}{8k^3-1} \right) dk - \frac{24}{\pi} \int_{\partial\tilde{D}_3} e^{-8ik^3 t} \left(\frac{k^4}{8k^3+1} - \frac{k^4}{8k^3-1} \right) dk. \quad (4.9)$$

□

Proposition 4.2. Let $q(x, 0)$ and $q(0, t)$ be given by (4.1). Then the function $\Phi_1(t, k)$ defined by (2.15a) satisfies $\Phi_1(t, k) = \varepsilon\Phi_{11}(t, k) + O(\varepsilon^2)$, $\varepsilon \rightarrow 0$, where

$$\begin{aligned} \Phi_{11}(t, k) = & \frac{e^{-8ik^3 t} - e^{it}}{2(2k + \alpha)} - \frac{e^{-8ik^3 t} - e^{-it}}{2(2k - \alpha^2)} + e^{-8ik^3 t} \left[\frac{\alpha(2k - \alpha)}{2(8k^3 + 1)} + \frac{\alpha^2(2k + \alpha^2)}{2(8k^3 - 1)} \right. \\ & \left. - \frac{4k^2(\sigma^2 + \sigma)}{(8k^3 + 1)(8k^3 - 1)} \right] + \frac{1}{4} \int_{\partial\tilde{D}_3^{\sigma k}} \frac{(k + k_1)F(k_1^3)}{k^3 - k_1^3} e^{-8ik_1^3 t} dk_1, \end{aligned} \quad (4.10)$$

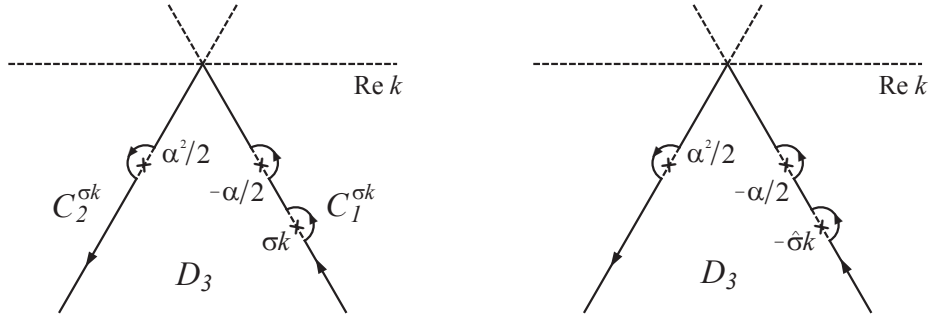


Fig. 4. The deformed contours $\partial\tilde{D}_3^{\sigma k}$ (left) and $\partial\tilde{D}_3^{\hat{\sigma}k}$ (right), where $\alpha = e^{2i\pi/3}$.

with $\partial\tilde{D}_3^{\sigma k}$ depicted in figure 4, and σ is given by

$$\sigma = \begin{cases} 1 & \text{if } k \in \partial D_3, \\ \alpha & \text{if } \alpha k \in \partial D_3, \\ \alpha^2 & \text{if } \alpha^2 k \in \partial D_3. \end{cases} \quad (4.11)$$

Proof. Substituting (4.3) and (4.4) into (3.12) and then integrating the resulting expression with respect to $d\tau$, we find

$$\begin{aligned} \Phi_{11}(t, k) &= \frac{e^{-8ik^3t} - e^{it}}{2(8k^3 + 1)} (4k^2 - 2\alpha k + \alpha^2) - \frac{e^{-8ik^3t} - e^{-it}}{2(8k^3 - 1)} (4k^2 + 2\alpha^2 k + \alpha) \\ &\quad + \frac{1}{4} \int_{\partial\tilde{D}_3} \frac{(k + k_1)F(k_1^3)}{k^3 - k_1^3} (e^{-8ik_1^3t} - e^{-8ik^3t}) dk_1. \end{aligned} \quad (4.12)$$

Employing the identities

$$4k^2 - 2\alpha k + \alpha^2 = (2k + 1)(2k + \alpha^2), \quad 4k^2 + 2\alpha^2 k + \alpha = (2k - 1)(2k - \alpha), \quad (4.13)$$

we obtain the first two terms in (4.10). On the other hand, the integrand in the second line in (4.12) has removable singularities at $k_1 = k$, αk , $\alpha^2 k$. Hence, before splitting the integrals we deform the contours $\partial\tilde{D}_3$ to the contour $\partial\tilde{D}_3^{\sigma k}$ which are depicted in figure 4.

Then, the residue theorem implies

$$\int_{\partial\tilde{D}_3^{\sigma k}} \frac{(k + k_1)F(k_1^3)}{k^3 - k_1^3} dk_1 = -2 \left[\frac{\alpha(2k - \alpha)}{8k^3 + 1} + \frac{\alpha^2(2k + \alpha^2)}{8k^3 - 1} - \frac{8k^2(\sigma^2 + \sigma)}{(8k^3 + 1)(8k^3 - 1)} \right], \quad (4.14)$$

where σ is given in (4.11). Using the above equation in (4.12), we find (4.10). \square

Proposition 4.3. Let $q(x, 0)$ and $q(0, t)$ be given by (4.1). Then, $\Phi_2(t, k)$ defined by (2.15b) satisfies $\Phi_2(t, k) = 1 + \varepsilon^2 \Phi_{22}(t, k) + O(\varepsilon^3)$, $\varepsilon \rightarrow 0$, where

$$\begin{aligned} \lambda \Phi_{22}(t, k) = & \frac{\alpha_1(k)e^{-it(8k^3-1)}}{2(2k-\alpha)} - \frac{\alpha_1(k)e^{-it(8k^3+1)}}{2(2k+\alpha^2)} + \frac{\alpha^2 e^{2it}}{8(2k+\alpha)} - \frac{\alpha e^{-2it}}{8(2k-\alpha^2)} + c_1(k) \\ & + \int_{\partial \tilde{D}_3^{\sigma k}} c_2(k, k_2) e^{-it(8k_2^3-1)} dk_2 + \int_{\partial \tilde{D}_3^{\sigma k}} c_3(k, k_2) e^{-it(8k_2^3+1)} dk_2 \\ & + \int_{\partial \tilde{D}_3^{\sigma k}} c_4(k, k_1) e^{-8it(k^3+k_1^3)} dk_1 + \int_{\partial \tilde{D}_3^{\sigma k}} c_5(k, k_1) e^{-it(8k_1^3-1)} dk_1 \\ & + \int_{\partial \tilde{D}_3^{\sigma k}} c_6(k, k_1) e^{-it(8k_1^3+1)} dk_1 + \int_{\partial \tilde{D}_3^{\sigma k}} \left(\int_{\partial \tilde{D}_3^{\sigma k}} c_7(k, k_1, k_2) e^{-8it(8k_1^3+k_2^3)} dk_2 \right) dk_1, \end{aligned} \quad (4.15)$$

with the contour $\partial \tilde{D}_3^{\hat{\sigma} k}$ depicted in figure 4, and the constant $\hat{\sigma}$ is given by

$$\hat{\sigma} = \begin{cases} 1 & \text{if } -k \in \partial D_3, \\ \alpha & \text{if } -\alpha k \in \partial D_3, \\ \alpha^2 & \text{if } -\alpha^2 k \in \partial D_3. \end{cases} \quad (4.16)$$

The functions $\alpha_1(k)$ and $\{c_j\}_1^7$ are defined by

$$\alpha_1(k) = -\frac{4k^2(\sigma^2 + \sigma + 1)}{(8k^3 + 1)(8k^3 - 1)}, \quad c_1(k) = c_1^{(1)}(k) + c_1^{(2)}(k) + c_1^{(3)}(k), \quad (4.17)$$

$$c_2(k, k_2) = \frac{(2k-1)(2k-\alpha^2)(k+k_2)}{8(k^3-k_2^3)(8k_2^3-1)} F(k_2^3), \quad (4.18)$$

$$c_3(k, k_2) = -\frac{(2k+1)(2k+\alpha)(k+k_2)}{8(k^3-k_2^3)(8k_2^3+1)} F(k_2^3), \quad (4.19)$$

$$c_4(k, k_1) = \frac{\alpha_1(k)(k-k_1)}{4(k^3+k_1^3)} F(k_1^3), \quad c_5(k, k_1) = -\frac{k-k_1}{(8k_1^3-1)(2k+\alpha)} F(k_1^3), \quad (4.20)$$

$$c_6(k, k_1) = \frac{k-k_1}{(8k_1^3+1)(2k-\alpha^2)} F(k_1^3), \quad c_7(k, k_1, k_2) = \frac{(k-k_1)(k+k_2)}{16(k_1^3+k_2^3)} F(k_1^3) F(k_2^3), \quad (4.21)$$

where

$$\begin{aligned} c_1^{(1)}(k) = & \frac{2k(2\alpha+1)-1}{8(4k^2+2k(2\alpha+1)-1)} + \frac{3k^2-k}{2(\alpha-1)(8k^3-1)} + \frac{\alpha(8k^3+5)}{24(8k^3-1)(2k-\alpha)} \\ & - \frac{\alpha^2(8k^3-5)}{24(8k^3+1)(2k+\alpha^2)} - \frac{\alpha}{4} \left(\frac{8k^3-1}{2(8k^3+1)} + \frac{\alpha k(8k^3-7)}{3(8k^3-1)(2k-\alpha)} - \frac{\alpha(2k+\alpha^2)}{6(2k+\alpha)} - \frac{1}{2} \right) \\ & - \frac{\alpha}{4} \left(\frac{\alpha(8k^3+1)}{2(8k^3-1)} + \frac{k(8k^3+7)}{3(8k^3+1)(2k+\alpha^2)} - \frac{2k-\alpha}{6(2k-\alpha^2)} - \frac{\alpha}{2} \right) \\ & - \frac{64k^5+8k^4+20k^2+7k}{12(8k^3+1)^2} - \frac{\alpha(3k^2+k)}{6(8k^3+1)} + \frac{112k^5-24k^4-26k^2+9k}{12(8k^3-1)^2} \\ & + \frac{64i\sqrt{3}k^7+48i\sqrt{3}k^5+16k^4+20i\sqrt{3}k^3+16k^2+1}{6(16k^4+4k^2+1)^2} + \frac{4ik^4(3\log(k^2)-6\log(k))}{\pi(64k^6-1)^2}, \end{aligned} \quad (4.22a)$$

$$c_1^{(2)}(k) = \frac{2k^2(\sigma^2 + \sigma)}{(64k^6 - 1)(2k + \alpha^2)(2k - \alpha)} - \frac{4\alpha_1(k)k^2(\hat{\sigma}^2 + \hat{\sigma} + 1)}{(8k^3 + 1)(8k^3 - 1)} - \frac{2k^2(\sigma^2 + \sigma)(16k^4 + 8k^3 - 2k + 1)}{(8k^3 - 1)^2(8k^3 + 1)^2} - \frac{8\alpha k^3(\sigma^2 + \sigma)}{(64k^6 - 1)(16k^4 + 4k^2 + 1)}, \quad (4.22b)$$

and

$$c_1^{(3)}(k) = \frac{\alpha(8k^3 - 5)}{24(8k^3 - 1)^2} - \frac{\alpha}{8(8k^3 + 1)} + \frac{2k^3 - 1}{3(8k^3 - 1)^2} - \frac{3}{8(\alpha - 1)(8k^3 - 1)}. \quad (4.22c)$$

Proof. Equation (3.21) implies that Φ_{22} satisfies

$$\lambda \Phi_{22}(t, k) = \int_0^t [4k^2 g_{01}(\tau) - 2ik g_{11}(\tau) - g_{21}(\tau)] \Phi_{11}(\tau, k) d\tau + 2ik \int_0^t g_{01}^2(\tau) d\tau. \quad (4.23)$$

The second integral in (4.23) can be evaluated exactly,

$$2ik \int_0^t g_{01}^2(\tau) d\tau = \frac{k}{4} (e^{-2it} - e^{2it}) + ikt. \quad (4.24)$$

We also note that

$$4k^2 g_{01}(\tau) - 2ik g_{11}(\tau) - g_{21}(\tau) = -\frac{i}{2} (2k - 1)(2k - \alpha^2) e^{i\tau} + \frac{i}{2} (2k + 1)(2k + \alpha) e^{-i\tau} - 2i \int_{\partial \bar{D}_3} (k - k_1) F(k_1^3) e^{-8ik_1^3 \tau} dk_1. \quad (4.25)$$

We denote $\Phi_{11}(t, k)$ by

$$\Phi_{11}(\tau, k) = \alpha_1(k) e^{-8ik^3 \tau} + \alpha_2(k) e^{i\tau} + \alpha_3(k) e^{-i\tau} + \frac{1}{4} \int_{\partial \bar{D}_3^k} \frac{(k + k_2) F(k_2^3)}{k^3 - k_2^3} e^{-8ik_2^3 \tau} dk_2, \quad (4.26)$$

where $\alpha_1(k)$ is given in (4.17) and $\alpha_2(k), \alpha_3(k)$ are defined by

$$\alpha_2(k) = -\frac{1}{2(2k + \alpha)}, \quad \alpha_3(k) = \frac{1}{2(2k - \alpha^2)}. \quad (4.27)$$

Substituting (4.25) and (4.26) into (4.23) and integrating the resulting expression with respect $d\tau$, we find

$$\begin{aligned} \int_0^t [4k^2 g_{01}(\tau) - 2ik g_{11}(\tau) - g_{21}(\tau)] \Phi_{11}(\tau, k) d\tau &= \frac{\alpha_1(k)}{2(2k - \alpha)} (e^{-it(8k^3 - 1)} - 1) \\ &\quad - \frac{\alpha_1(k)}{2(2k + \alpha^2)} (e^{-it(8k^3 + 1)} - 1) - \frac{\alpha_2(k)}{4} (2k - 1)(2k - \alpha^2) (e^{2it} - 1) \\ &\quad - \frac{\alpha_3(k)}{4} (2k + 1)(2k + \alpha) (e^{-2it} - 1) + \frac{i\alpha_2(k)}{2} (2k + 1)(2k + \alpha) t \\ &\quad - \frac{i\alpha_3(k)}{2} (2k - 1)(2k - \alpha^2) t + I(t, k), \end{aligned} \quad (4.28)$$

where

$$\begin{aligned}
 I(t, k) = & \frac{1}{8}(2k-1)(2k-\alpha^2) \int_{\partial \bar{D}_3^{\sigma k}} \frac{(k+k_2)(e^{-it(8k_2^3-1)}-1)}{(k^3-k_2^3)(8k_2^3-1)} F(k_2^3) dk_2 \\
 & - \frac{1}{8}(2k+1)(2k+\alpha) \int_{\partial \bar{D}_3^{\sigma k}} \frac{(k+k_2)(e^{-it(8k_2^3+1)}-1)}{(k^3-k_2^3)(8k_2^3+1)} F(k_2^3) dk_2 \\
 & + 2\alpha_2(k) \int_{\partial \bar{D}_3} \frac{(k-k_1)(e^{-it(8k_1^3-1)}-1)}{8k_1^3-1} F(k_1^3) dk_1 \\
 & + 2\alpha_3(k) \int_{\partial \bar{D}_3} \frac{(k-k_1)(e^{-it(8k_1^3+1)}-1)}{8k_1^3+1} F(k_1^3) dk_1 \\
 & + \frac{\alpha_1(k)}{4} \int_{\partial \bar{D}_3} \frac{(k-k_1)(e^{-8it(k^3+k_1^3)}-1)}{k^3+k_1^3} F(k_1^3) dk_1 \\
 & + \frac{1}{16} \int_{\partial \bar{D}_3} \left(\int_{\partial \bar{D}_3^{\sigma k}} \frac{(k-k_1)(k+k_2)(e^{-8it(k_1^3+k_2^3)}-1)}{(k^3-k_2^3)(k_1^3+k_2^3)} F(k_1^3) F(k_2^3) dk_2 \right) dk_1. \quad (4.29)
 \end{aligned}$$

Using the definitions for $\alpha_2(k)$ and $\alpha_3(k)$ in (4.27), we find

$$\frac{i\alpha_2(k)}{2}(2k+1)(2k+\alpha)t - \frac{i\alpha_3(k)}{2}(2k-1)(2k-\alpha^2)t = -ikt; \quad (4.30)$$

thus, the term in (4.24) involving t cancels and then employing (4.24), equation (4.23) becomes

$$\begin{aligned}
 \lambda \Phi_{22}(t, k) = & \frac{\alpha_1(k)}{2(2k-\alpha)} (e^{-it(8k^3-1)}-1) - \frac{\alpha_1(k)}{2(2k+\alpha^2)} (e^{-it(8k^3+1)}-1) \\
 & - \frac{\alpha^2 e^{2it}}{8(2k+\alpha)} - \frac{\alpha e^{-2it}}{8(2k-\alpha^2)} + \frac{2k(2\alpha+1)-1}{8(4k^2+2k(2\alpha+1)-1)} + I(t, k). \quad (4.31)
 \end{aligned}$$

In what follows we evaluate each integral appearing in $I(t, k)$. Using the residue theorem we find

$$\begin{aligned}
 \int_{\partial \bar{D}_3^{\sigma k}} \frac{(k+k_2)F(k_2^3)}{(k^3-k_2^3)(8k_2^3-1)} dk_2 = & \frac{\alpha(2k-\alpha)}{8k^3+1} + \frac{2\alpha^2 k(8k^3-7)}{3(8k^3-1)^2} - \frac{\alpha(8k^3+5)}{3(8k^3-1)^2} \\
 & + \frac{16k^2(\sigma^2+\sigma)}{(8k^3-1)^2(8k^3+1)} := \beta_1(k), \quad (4.32)
 \end{aligned}$$

$$\begin{aligned}
 \int_{\partial \bar{D}_3^{\sigma k}} \frac{(k+k_2)F(k_2^3)}{(k^3-k_2^3)(8k_2^3+1)} dk_2 = & -\frac{\alpha^2(2k+\alpha^2)}{8k^3-1} - \frac{2\alpha k(8k^3+7)}{3(8k^3+1)^2} - \frac{\alpha^2(8k^3-5)}{3(8k^3+1)^2} \\
 & + \frac{16k^2(\sigma^2+\sigma)}{(8k^3-1)(8k^3+1)^2} := \beta_2(k), \quad (4.33)
 \end{aligned}$$

$$\int_{\partial \bar{D}_3} \frac{(k-k_1)F(k_1^3)}{8k_1^3-1} dk_1 = \frac{\alpha(2k+\alpha)}{8} + \frac{\alpha^2(2k+\alpha^2)}{24}, \quad (4.34)$$

$$\int_{\partial \bar{D}_3} \frac{(k-k_1)F(k_1^3)}{8k_1^3+1} dk_1 = -\frac{\alpha^2(2k-\alpha^2)}{8} - \frac{\alpha(2k-\alpha)}{24}. \quad (4.35)$$

We note that the integrand in the fifth line in (4.29) has removable singularities at $k_1 = -k, -\alpha k, -\alpha^2 k$ and hence before splitting the integral we deform $\partial\tilde{D}_3$ to $\partial\tilde{D}_3^{\hat{\sigma}^k}$ depicted in figure 4. Using the residue theorem we find

$$\int_{\partial\tilde{D}_3} \frac{(k-k_1)F(k_1^3)}{k^3+k_1^3} dk_1 = -\frac{2\alpha(2k+\alpha)}{8k^3-1} - \frac{2\alpha^2(2k-\alpha^2)}{8k^3+1} + \frac{16k^2(\hat{\sigma}^2+\hat{\sigma})}{(8k^3+1)(8k^3-1)} := \beta_3(k), \quad (4.36)$$

where the constant $\hat{\sigma}$ is defined in (4.16).

Regarding the double integral, we deform $\partial\tilde{D}_3$ to $\partial\tilde{D}_3^{\tilde{\sigma}^{k_2}}$, where $\tilde{\sigma}$ is defined by

$$\tilde{\sigma} = \begin{cases} 1 & \text{if } -k_2 \in \partial D_3, \\ \alpha & \text{if } -\alpha k_2 \in \partial D_3, \\ \alpha^2 & \text{if } -\alpha^2 k_2 \in \partial D_3. \end{cases} \quad (4.37)$$

Then, using a residue calculation we find

$$\int_{\partial\tilde{D}_3^{\tilde{\sigma}^{k_2}}} \frac{(k-k_1)F(k_1^3)}{k_1^3+k_2^3} dk_1 = -\frac{2\alpha(2k+\alpha)}{8k_2^3-1} - \frac{2\alpha^2(2k-\alpha^2)}{8k_2^3+1} + \frac{16\tilde{\sigma}k_2(k+\tilde{\sigma}k_2)}{(8k_2^3+1)(8k_2^3-1)}. \quad (4.38)$$

Thus, the double integral denoted by $\beta_4(k)$ can be written as

$$\begin{aligned} \beta_4(k) &= -\frac{1}{16} \int_{\partial\tilde{D}_3^{\tilde{\sigma}^k}} \left(\int_{\partial\tilde{D}_3^{\tilde{\sigma}^{k_2}}} \frac{(k-k_1)(k+k_2)}{(k^3-k_2^3)(k_1^3+k_2^3)} F(k_1^3)F(k_2^3) dk_1 \right) dk_2 \\ &= \beta_4^{(1)}(k) + \beta_4^{(2)}(k) + \beta_4^{(3)}(k), \end{aligned} \quad (4.39)$$

with

$$\beta_4^{(1)}(k) = -\int_{C_1^{\sigma k}} \frac{\alpha k_2(k+k_2)(k+\alpha k_2)F(k_2^3)}{(k^3-k_2^3)(8k_2^3+1)(8k_2^3-1)} dk_2, \quad (4.40)$$

$$\beta_4^{(2)}(k) = -\int_{C_2^{\sigma k}} \frac{\alpha^2 k_2(k+k_2)(k+\alpha^2 k_2)F(k_2^3)}{(k^3-k_2^3)(8k_2^3+1)(8k_2^3-1)} dk_2, \quad (4.41)$$

and

$$\beta_4^{(3)}(k) = \frac{1}{8} \int_{\partial\tilde{D}_3^{\sigma k}} \frac{(k+k_2)F(k_2^3)}{k^3-k_2^3} \left[\frac{\alpha(2k+\alpha)}{8k_2^3-1} + \frac{\alpha^2(2k-\alpha^2)}{8k_2^3+1} \right] dk_2, \quad (4.42)$$

where $C_1^{\sigma k}$ and $C_2^{\sigma k}$ are depicted in figure 4. A residue calculation implies

$$\begin{aligned} \beta_4^{(3)}(k) &= -\frac{128k^5+16k^4-8k^3+40k^2+14k+5}{24(8k^3+1)^2} - \frac{\alpha(12k^2+4k+3)}{24(8k^3+1)^2} \\ &\quad + \frac{112k^5-24k^4+8k^3-26k^2+9k-4}{12(8k^3-1)^2} + \frac{12k^2-4k-3}{8(\alpha-1)(8k^3-1)} \\ &\quad - \frac{2k^2(\sigma^2+\sigma)(16k^4+8k^3-2k+1)}{(64k^6-1)^2} - \frac{8\alpha k^2(\sigma^2+\sigma)}{(64k^6-1)(16k^4+4k^2+1)}. \end{aligned} \quad (4.43)$$

In order to compute $\beta_4^{(1)}$ by Cauchy's theorem, we first deform C_1 to \tilde{C}_1 (cf. figure 2). Using $k_2 = rv$ with $v = e^{-i\pi/6}$, $\beta_4^{(1)}(k)$ can be written in the form

$$\beta_4^{(1)}(k) = \frac{12}{i\pi} \int_0^\infty \frac{\alpha r^4 v^5 (k+rv)(k+\alpha rv)}{(k^3-r^3 v^3)(8r^3 v^3+1)(8r^3 v^3-1)} \left(\frac{1}{8r^3 v^3+1} - \frac{1}{8r^3 v^3-1} \right) dr. \quad (4.44)$$

For $\beta_4^{(2)}(k)$, letting $k_2 \rightarrow \alpha^2 k_2$, in analogy with $\beta_4^{(1)}(k)$, we obtain

$$\beta_4^{(2)}(k) = -\frac{12}{i\pi} \int_0^\infty \frac{\alpha r^4 v^5 (k + \alpha r v)(k + \alpha^2 r v)}{(k^3 - r^3 v^3)(8r^3 v^3 + 1)(8r^3 v^3 - 1)} \left(\frac{1}{8r^3 v^3 + 1} - \frac{1}{8r^3 v^3 - 1} \right) dr. \quad (4.45)$$

Evaluating (4.44) and (4.45) directly, we find

$$\begin{aligned} \beta_4^{(1)}(k) + \beta_4^{(2)}(k) &= \frac{64i\sqrt{3}k^7 + 48i\sqrt{3}k^5 + 16k^4 + 20i\sqrt{3}k^3 + 16k^2 + 1}{6(2k + \alpha)^2(2k + \alpha^2)^2(2k - \alpha)^2(2k - \alpha^2)^2} \\ &\quad + \frac{4ik^4(3\log(k^2) - 6\log(k))}{\pi(64k^6 - 1)^2}. \end{aligned} \quad (4.46)$$

Combining the above results, we find (4.15), where $c_1(k)$ is given by

$$\begin{aligned} c_1(k) &= \frac{\alpha_1(k)}{2(2k + \alpha^2)(2k - \alpha)} + \frac{2k(2\alpha + 1) - 1}{8(4k^2 + 2k(2\alpha + 1) - 1)} + \frac{\alpha^2(2k + \alpha^2)}{24(2k + \alpha)} + \frac{\alpha(2k - \alpha)}{24(2k - \alpha^2)} - \frac{1}{8} \\ &\quad - \frac{1}{8}(2k - 1)(2k - \alpha^2)\beta_1(k) + \frac{1}{8}(2k + 1)(2k + \alpha)\beta_2(k) - \frac{\alpha_1(k)}{4}\beta_3(k) + \beta_4(k). \end{aligned} \quad (4.47)$$

Simplifying (4.47), we find that $c_1(k)$ is given by the second equation in (4.17), where $\{c_1^{(j)}\}_1^3$ are defined in (4.22). \square

Proposition 4.4. Let $q(x, 0)$ and $q(0, t)$ be given by (4.1). Then, $q_x(0, t)$ and $q_{xx}(0, t)$ are given by (3.3) and (3.7), where g_{11} , g_{12} , g_{21} , g_{22} are given in proposition 4.1 and g_{13} , g_{23} are given by the following formulas:

$$\begin{aligned} \lambda g_{13}(t) &= \left(\frac{\alpha^2}{16} - \frac{\alpha^2}{32}(3^{1/3} - 1) \right) e^{3it} + \left(\frac{\alpha}{16} - \frac{\alpha}{32}(3^{1/3} - 1) \right) e^{-3it} \\ &\quad + d_1 e^{it} + d_2 e^{-it} + o(1), \end{aligned} \quad t \rightarrow \infty, \quad (4.48)$$

and

$$\begin{aligned} \lambda g_{23}(t) &= \left(\frac{i}{4} - \frac{i}{32}(3^{2/3} - 1) \right) e^{3it} - \left(\frac{i}{4} - \frac{i}{32}(3^{2/3} - 1) \right) e^{-3it} \\ &\quad + d_3 e^{it} + d_4 e^{-it} + o(1), \end{aligned} \quad t \rightarrow \infty, \quad (4.49)$$

with d_1, \dots, d_4 constants.

Proof. We will derive equation (4.48); equation (4.49) can be derived in a similar way. Recalling (3.20), we denote $\lambda g_{13}(t)$ by

$$\lambda g_{13}(t) = \lambda g_{13}^{(1)}(t) + \lambda g_{13}^{(2)}(t), \quad (4.50)$$

where

$$\lambda g_{13}^{(1)}(t) = \frac{2}{i\pi} \int_{\partial D_3} k \lambda [\Phi_{13}(t, k) + \alpha \Phi_{13}(t, \alpha k) + \alpha^2 \Phi_{13}(t, \alpha^2 k)] dk, \quad (4.51)$$

$$\lambda g_{13}^{(2)}(t) = \frac{e^{it} - e^{-it}}{\pi} \int_{\partial D_3} \lambda [\Phi_{22}(t, k) + \alpha \Phi_{22}(t, \alpha k) + \alpha^2 \Phi_{22}(t, \alpha^2 k)] dk. \quad (4.52)$$

We next compute the coefficients of the terms involving e^{3it} and e^{-3it} .

Regarding $\lambda g_{13}^{(2)}(t)$, we use (4.15). Note that the term involving e^{3it} arises from the third term in (4.15),

$$\frac{\alpha^2 e^{2it}}{8(2k + \alpha)}; \quad (4.53)$$

the relevant contribution for $\lambda g_{13}^{(2)}(t)$ is given by

$$\frac{\alpha^2}{8i\pi} \int_{\partial \bar{D}_3} \left(\frac{1}{2k + \alpha} + \frac{\alpha}{2\alpha k + \alpha} + \frac{\alpha^2}{2\alpha^2 k + \alpha} \right) dk = \frac{3\alpha^2}{2i\pi} \int_{\partial \bar{D}_3} \frac{k^2}{8k^3 + 1} dk = \frac{\alpha^2}{16}, \quad (4.54)$$

where we have used the fact that the integrand has poles at $k = -\alpha/2$ and at $k = \infty$. Also, noting that the term involving e^{-3it} arises from the fourth term in (4.15),

$$\frac{\alpha e^{-2it}}{8(2k - \alpha^2)}, \quad (4.55)$$

the relevant contribution for $\lambda g_{13}^{(2)}(t)$ is given by

$$\frac{\alpha}{8i\pi} \int_{\partial \bar{D}_3} \left(\frac{1}{2k - \alpha^2} + \frac{\alpha}{2\alpha k - \alpha^2} + \frac{\alpha^2}{2\alpha^2 k - \alpha^2} \right) dk = \frac{3\alpha^2}{2i\pi} \int_{\partial \bar{D}_3} \frac{k^2}{8k^3 - 1} dk = \frac{\alpha}{16}, \quad (4.56)$$

where we have used the fact that the integrand has poles at $k = \alpha^2/2$ and at $k = \infty$.

Regarding $\lambda g_{13}^{(1)}(t)$, we use (3.24). According to the symmetric combination for $\lambda \Phi_{13}(t, k)$ in (4.51), we first need to evaluate the integrals

$$\begin{aligned} & -2ike^{-8ik^3 t} \int_0^t e^{8ik^3 \tau} g_{01}^2(\tau) \Phi_{11}(\tau, k) d\tau \\ & + e^{-8ik^3 t} \int_0^t e^{8ik^3 \tau} [4k^2 g_{01}(\tau) + 2ikg_{11}(\tau) - g_{21}(\tau)] \Phi_{22}(\tau, k) d\tau. \end{aligned} \quad (4.57)$$

For the first integral (involving $\Phi_{11}(\tau, k)$), we substitute (4.26) into (4.57) and then integrate the resulting expression with respect $d\tau$; in this way we find the following explicit expressions:

$$\begin{aligned} & \frac{2}{k} e^{-8ik^3 t} \left[-2i\alpha_1(k)t + \frac{\alpha_1(k)(e^{2it} - 1)}{2} - \frac{\alpha_1(k)(e^{-2it} - 1)}{2} + (\alpha_3(k) - 2\alpha_2(k)) \frac{e^{it(8k^3+1)} - 1}{8k^3 + 1} \right. \\ & \left. + (\alpha_2(k) - 2\alpha_3(k)) \frac{e^{it(8k^3-1)} - 1}{8k^3 - 1} + \frac{\alpha_2(k)(e^{it(8k^3+3)} - 1)}{8k^3 + 3} + \frac{\alpha_3(k)(e^{it(8k^3-3)} - 1)}{8k^3 - 3} \right]. \end{aligned} \quad (4.58)$$

Hence, the term involving e^{3it} arises from the term

$$\frac{k\alpha_2(k)}{2(8k^3 + 3)} = -\frac{k}{4(2k + \alpha)(8k^3 + 3)}. \quad (4.59)$$

The relevant contribution for $\lambda g_{13}^{(1)}(t)$ is given by

$$-\frac{1}{2i\pi} \int_{\partial \bar{D}_3} \frac{k}{8k^3 + 3} \left(\frac{k}{2k + \alpha} + \frac{\alpha^2 k}{2\alpha k + \alpha} + \frac{\alpha k}{2\alpha^2 k + \alpha} \right) dk = \frac{3\alpha}{i\pi} \int_{\partial \bar{D}_3} \frac{k^3}{(8k^3 + 3)(8k^3 + 1)} dk. \quad (4.60)$$

Note that the above integrand has poles at $k = -\alpha/2$ and $k = -\frac{3^{1/3}}{2}\alpha$; thus,

$$\frac{3\alpha}{i\pi} \int_{\partial\bar{D}_3} \frac{k^3}{(8k^3+3)(8k^3+1)} dk = \frac{\alpha^2}{16} (3^{1/3} - 1). \quad (4.61)$$

The term involving e^{-3it} arises from the term

$$\frac{k\alpha_3(k)}{2(8k^3-3)} = \frac{k}{4(2k-\alpha^2)(8k^3-3)}, \quad (4.62)$$

and the relevant contribution for $\lambda g_{13}^{(1)}(t)$ is given by

$$\begin{aligned} & \frac{1}{2i\pi} \int_{\partial\bar{D}_3} \frac{k}{8k^3-3} \left(\frac{k}{2k-\alpha^2} + \frac{\alpha^2 k}{2\alpha k-\alpha^2} + \frac{\alpha k}{2\alpha^2 k-\alpha^2} \right) dk \\ &= \frac{3\alpha}{i\pi} \int_{\partial\bar{D}_3} \frac{k^3}{(8k^3-3)(8k^3-1)} dk. \end{aligned} \quad (4.63)$$

The above integrand has poles at $k = \alpha^2/2$ and $k = \frac{3^{1/3}}{2}\alpha^2$, thus

$$\frac{3\alpha}{i\pi} \int_{\partial\bar{D}_3} \frac{k^3}{(8k^3+3)(8k^3+1)} dk = \frac{\alpha}{16} (3^{1/3} - 1). \quad (4.64)$$

For the second integral in (4.57) (involving $\Phi_{22}(\tau, k)$), we substitute (4.15) into (4.57). We need to evaluate the integral

$$\begin{aligned} & e^{-8ik^3t} \int_0^t \left[-\frac{i}{2} e^{i\tau} (4k^2 - 2\alpha k + \alpha^2) + \frac{i}{2} e^{-i\tau} (4k^2 + 2\alpha^2 k + \alpha) \right] \\ & \times \left[\frac{\alpha_1(k) e^{i\tau}}{2(2k-\alpha)} - \frac{\alpha_1(k) e^{-i\tau}}{2(2k+\alpha^2)} + \frac{\alpha^2 e^{i\tau(8k^3+2)}}{8(2k+\alpha)} - \frac{\alpha e^{i\tau(8k^3-2)}}{8(2k-\alpha^2)} + c_1(k) e^{8ik^3\tau} \right] d\tau. \end{aligned} \quad (4.65)$$

Letting

$$u_1(k) = 4k^2 - 2\alpha k + \alpha^2, \quad u_2(k) = 4k^2 + 2\alpha^2 k + \alpha, \quad (4.66)$$

and integrating the above expression with respect $d\tau$, equation (4.65) can be written in the form

$$\begin{aligned} & e^{-8ik^3t} \left[\frac{i\alpha t}{4} \left(\frac{u_1(k)}{2k+\alpha^2} + \frac{u_2(k)}{2k-\alpha} \right) - \frac{\alpha_1(k)u_1(k)(e^{2it}-1)}{8(2k-\alpha)} + \frac{\alpha_1(k)u_2(k)(e^{-2it}-1)}{8(2k+\alpha^2)} \right. \\ & - \frac{e^{it(8k^3+1)}-1}{2(8k^3+1)} \left(c_1(k)u_1(k) - \frac{\alpha^2 u_2(k)}{8(2k+\alpha)} \right) + \frac{e^{it(8k^3-1)}-1}{2(8k^3-1)} \left(c_1(k)u_2(k) + \frac{\alpha u_1(k)}{8(2k-\alpha^2)} \right) \\ & \left. - \frac{\alpha^2 u_1(k)(e^{it(8k^3+3)}-1)}{16(8k^3+3)(2k+\alpha)} - \frac{\alpha u_2(k)(e^{it(8k^3-3)}-1)}{16(8k^3-3)(2k-\alpha^2)} \right]. \end{aligned} \quad (4.67)$$

Thus, the term involving e^{3it} arises from the term

$$-\frac{\alpha^2 u_1(k)}{16(8k^3+3)(2k+\alpha)}. \quad (4.68)$$

The relevant contribution for $\lambda g_{13}^{(1)}(t)$ is given by

$$\begin{aligned} & -\frac{\alpha^2}{8i\pi} \int_{\partial\tilde{D}_3} \frac{k}{8k^3+3} \left(\frac{u_1(k)}{2k+\alpha} + \frac{\alpha u_1(\alpha k)}{2\alpha k+\alpha} + \frac{\alpha^2 u_1(\alpha^2 k)}{2\alpha^2 k+\alpha} \right) dk \\ & = -\frac{9\alpha}{2i\pi} \int_{\partial\tilde{D}_3} \frac{k^3}{(8k^3+3)(8k^3+1)} dk = -\frac{3\alpha^2}{32} (3^{1/3}-1), \end{aligned} \quad (4.69)$$

where we have used the fact that the integrand has poles at $k = -\alpha/2$ and $k = -\frac{3^{1/3}}{2}\alpha$. Similarly, we find that the term involving e^{-3it} arises from the term

$$-\frac{\alpha u_2(k)}{16(8k^3-3)(2k-\alpha^2)}. \quad (4.70)$$

The relevant contribution for $\lambda g_{13}^{(1)}(t)$ is given by

$$\begin{aligned} & -\frac{\alpha}{8i\pi} \int_{\partial\tilde{D}_3} \frac{k}{8k^3-3} \left(\frac{u_2(k)}{2k-\alpha^2} + \frac{\alpha u_2(\alpha k)}{2\alpha k-\alpha^2} + \frac{\alpha^2 u_2(\alpha^2 k)}{2\alpha^2 k-\alpha^2} \right) dk \\ & = -\frac{9\alpha^2}{2i\pi} \int_{\partial\tilde{D}_3} \frac{k^3}{(8k^3-3)(8k^3-1)} dk = -\frac{3\alpha}{32} (3^{1/3}-1), \end{aligned} \quad (4.71)$$

where we have used the fact that the integrand has poles at $k = \alpha^2/2$ and $k = \frac{3^{1/3}}{2}\alpha^2$.

Combining (4.54), (4.61) and (4.69) for the terms involving e^{3it} , as well as (4.56), (4.64) and (4.71) for the terms involving e^{-3it} , we find the following expressions for e^{3it} and e^{-3it} :

$$\left(\frac{\alpha^2}{16} - \frac{\alpha^2}{32} (3^{1/3}-1) \right) e^{3it} + \left(\frac{\alpha}{16} - \frac{\alpha}{32} (3^{1/3}-1) \right) e^{-3it}. \quad (4.72)$$

□

Acknowledgments

The authors are grateful to the EPSRC, UK, for partial support.

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