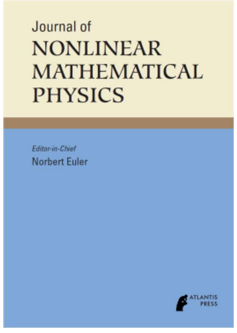

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous *N*-Body Problem

Oksana Bihun, Francesco Calogero, Ge Yi

To cite this article: Oksana Bihun, Francesco Calogero, Ge Yi (2013) Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous *N*-Body Problem, Journal of Nonlinear Mathematical Physics 20:1, 158–178, DOI: https://doi.org/10.1080/14029251.2013.792494

To link to this article: https://doi.org/10.1080/14029251.2013.792494

Published online: 04 January 2021

Journal of Nonlinear Mathematical Physics, Vol. 20, No. 1 (March 2013), 158-178

Diophantine Properties Associated to the Equilibrium Configurations of an Isochronous N-Body Problem

Oksana Bihun[1], Francesco Calogero[2], and Ge Yi[2]

[1] *Department of Mathematics, Concordia College at Moorhead, MN, USA*
[2] *Physics Department, University of Rome "La Sapienza" and Istituto Nazionale di Fisica Nucleare, Sezione di Roma*

obihun@cord.edu,
francesco.calogero@roma1.infn.it, francesco.calogero@uniroma1.it,
ge.yi@roma1.infn.it, yigeking@mail.ustc.edu.cn

Received 14 December 2012

Accepted 18 February 2013

Recently a *solvable* N-body problem featuring several free parameters has been investigated, and conditions on these parameters have been identified which guarantee that this system is *isochronous* (*all* its solutions are *periodic* with a *fixed* period) and that it possesses *equilibria*. The N coordinates $\bar{z}_n$ characterizing the equilibrium configurations are in some cases explicitly known, in others coincide with the N zeros of certain para-Jacobi polynomials or are arbitrary numbers. In the present paper the behavior of this N-body system in the immediate vicinity of its equilibria is studied, and *Diophantine* relations satisfied by the N coordinates $\bar{z}_n$ are thereby identified.

1. Introduction

Recently a *solvable* N-body problem featuring several free parameters has been investigated, and conditions on these parameters have been identified which guarantee that this system is *isochronous* (*all* its solutions are *periodic* with the same *fixed* period, independent of the initial data) and that it possesses *equilibria* [1]. In the present paper *Diophantine* findings are derived from these results.

The idea to obtain *Diophantine* relations for the coordinates $\bar{z}_n$ of the *equilibria* of an (autonomous) *isochronous* N-body model is rather simple. One focusses on the behavior of that system in the *immediate vicinity* of its *equilibria*. The standard solution of the *linearized* equations of motion characterizing this behavior entails that these motions are a *linear* superposition of *periodic* motions, the frequencies of which coincide with the N eigenvalues of certain specific $N \times N$ matrices constructed with the coordinates $\bar{z}_n$ of the *equilibria*. But if the system is *isochronous*, *all* its motions—including of course those in the *immediate vicinity* of its *equilibria*—are *completely periodic* with a *fixed* period. Hence the eigenvalues associated with the motions in the *immediate vicinity* of the *equilibria* must *all* be *integer multiples* of a common frequency. Thus, the outcome of this approach is to identify specific $N \times N$ matrices *all* eigenvalues of which must be *integer multiples* of a common factor: a *Diophantine* finding. This approach has been extensively used to arrive at *Diophantine* findings: see for instance Appendix C, entitled "Diophantine findings and conjectures", of monograph [2]. Let us also note that, while the fact that the N-body model providing the starting point for this approach is *isochronous* guarantees that *Diophantine* findings emerge via this

approach (of course, provided equilibria exist and can be found), such findings can also emerge from models which are *not* altogether *isochronous*: see below.

The present paper presents some new results of this kind. It is organized as follows. In the next Section 2 the findings of [1] are reviewed—largely *verbatim*, but merely to the extent needed to make the present paper understandable to readers who prefer not to go preliminarily through [1] (although such neglect is not recommended); and the strategy to arrive at the results of the present paper is detailed. This section can be omitted in a first reading by whoever is primarily interested in the new findings reported in the present paper, who might therefore immediately jump (albeit at the risk of missing some notational indications) to Section 3 where these results are reported. These findings reveal *Diophantine* properties associated with the N coordinates $\bar{z}_n$ characterizing the equilibrium configurations of the solvable N-body problem treated in [1], which in some cases (i.e. for certain assignments of the parameters of this model) are *explicitly* known, in other cases coincide with the N zeros of certain para-Jacobi polynomials [8] or are just a set of *arbitrary* numbers. These results are then proven in the subsequent Section 4 by investigating the behavior in the immediate vicinity of its equilibria of the N-body problem treated in [1]. A very terse Section 5, entitled "Outlook", concludes the paper by mentioning possible future developments. A number of useful identities are collected in Appendix A.

2. Preliminaries

The *solvable* N-body problem discussed in [1] is characterized by the Newtonian equations of motion

$$\begin{aligned}\ddot{z}_n &= -E\,\dot{z}_n - (N-1)\,A_1 + B\,z_n - 2\,(N-1)\,A_3\,z_n^2 \\ &+ \sum_{\ell=1,\ \ell\neq n}^{N} \Big\{ (z_n - z_\ell)^{-1}\ \big[2\,\dot{z}_n\,\dot{z}_\ell - (D_0 + D_1\,z_n)\ (\dot{z}_n + \dot{z}_\ell) \\ &- D_2\,z_n\ (\dot{z}_n\,z_\ell + \dot{z}_\ell\,z_n) + 2\ \big(A_1 + A_2\,z_n + A_3\,z_n^2\big)\ z_n\big]\Big\}\ . \end{aligned} \tag{2.1}$$

Notation 2.1. Here and hereafter N is an *arbitrary positive integer* (generally $N \geq 2$); the N coordinates $z_n(t)$ are, generally *complex*, variables depending on the *real* (independent) variable t ("time"); superimposed dots denote time-differentiations; the indices n, m, k, ℓ take all *integer* values from 1 to N, unless otherwise indicated; the 8 (generally *complex*) constants $A_1, A_2, A_3, B, D_0, D_1, D_2, E$ are *a priori arbitrary* (but see below). The N coordinates $z_n(t)$ are of course moving in the *complex* z-plane as the time t evolves. But they can be identified with the coordinates $\vec{r}_n(t)$ of N (unit mass, pointlike) particles moving in the "physical", horizontal plane spanned by the *real* 2-vector $\vec{r} \equiv (x,y)$ via the relation $\vec{r}_n \equiv (x_n, y_n)$ with the $2N$ Cartesian coordinates x_n and y_n corresponding to the *real* and *imaginary* parts of the *complex* number $z_n = x_n + i\,y_n$ (see for instance Chapter 4, entitled "Solvable and/or integrable many-body problems in the plane, obtained by complexification", of Ref. [4]). Here and hereafter i is the *imaginary unit*, $i^2 = -1$. In the following we generally work with *complex* variables, but we feel free to refer to the evolution of the N coordinates $z_n(t)$ as describing an N-body problem. ■

The N coordinates $z_n(t)$ coincide with the N zeros of a time-dependent polynomial of degree N in z which evolves in time according to the *linear*, *autonomous*, partial differential equation (PDE)

$$\begin{aligned} \psi_{tt} + \left(D_0 + D_1\, z + D_2\, z^2\right)\, \psi_{zt} + \left[E - (N-1)\, D_2\, z\right]\, \psi_t \\ + \left(A_1 + A_2\, z + A_3\, z^2\right)\, z\, \psi_{zz} - \left[(N-1)\, A_1 - B\, z + 2\,(N-1)\, A_3\, z^2\right]\, \psi_z \\ -N\, \left[(N-1)\, (A_2 - A_3\, z) + B\right]\, \psi = 0\,; \end{aligned} \tag{2.2a}$$

$$\psi(z,t) = \prod_{n=1}^{N} \left[z - z_n(t)\right] = z^N + \sum_{m=1}^{N} \left[c_m(t)\, z^{N-m}\right]\,. \tag{2.2b}$$

Notation 2.2. In (2.2a) appended variables denote partial differentiations; while (2.2b), besides displaying the identification of the N coordinates $z_n(t)$ evolving according to the Newtonian equations of motion (2.1) with the N zeros of the time-dependent (monic) polynomial $\psi(z,t)$ evolving according to the PDE (2.2a), introduces the N coefficients $c_m(t)$ of the (monic) polynomial $\psi(z,t)$. ■

The N coefficients $c_m(t)$ evolve according to the system of N linear autonomous ordinary differential equations (ODEs)

$$\begin{aligned} \ddot{c}_m + (N+1-m)\, D_0\, \dot{c}_{m-1} + \left[(N-m)\, D_1 + E\right]\, \dot{c}_m - m\, D_2\, \dot{c}_{m+1} \\ -(N+1-m)\, (m-1)\, A_1\, c_{m-1} - m\, \left[(2\,N-m-1)\, A_2 + B\right]\, c_m \\ +m\, (m+1)\, A_3\, c_{m+1} = 0\,. \end{aligned} \tag{2.3}$$

Hence the *solution* of this system is detailed by the following

Proposition 2.1:

$$c_m(t) = \sum_{j=1}^{2\,N} \gamma_j\, \tilde{c}_m^{(j)}\, \exp\left(\lambda_j\, t\right)\,, \tag{2.4}$$

where the $2N$ numbers γ_j are *a priori* arbitrary (or can be *a posteriori* fixed, in the context of the *initial-value* problem, by imposing consistency with the $2N$ initial data $c_m(0)\,, \dot{c}_m(0)$); while the $2N$ quantities λ_j, respectively the $2N^2$ components $\tilde{c}_m^{(j)}$ of the $2N$ N-vectors $\underline{\tilde{c}}^{(j)} \equiv \left(\tilde{c}_1^{(j)}, \tilde{c}_2^{(j)}, ..., \tilde{c}_n^{(j)}\right)$, are the $2N$ *eigenvalues*, respectively the $2N$ *eigenvectors*, of the generalized eigenvalue problem

$$\left(\lambda^2\, \underline{I} + \lambda\, \underline{D} + \underline{A}\right)\, \underline{\tilde{c}}^{(j)} = 0\,, \tag{2.5a}$$

with the two $(N \times N)$-matrices $\underline{A}$ and $\underline{D}$ defined componentwise as follows:

$$\begin{aligned} A_{m,n} = -(N+1-m)\, (m-1)\, A_1\, \delta_{m,n+1} \\ -m\, \left[(2\,N-m-1)\, A_2 + B\right]\, \delta_{m,n} + m\, (m+1)\, A_3\, \delta_{m,n-1}\,, \end{aligned} \tag{2.5b}$$

$$D_{m,n} = (N+1-m)\, D_0\, \delta_{m,n+1} + \left[(N-m)\, D_1 + E\right]\, \delta_{m,n} - m\, D_2\, \delta_{m,n-1}\,. \blacksquare \tag{2.5c}$$

Notation 2.3. N-vectors respectively $(N \times N)$-matrices are denoted by underlined lower-case respectively upper-case Latin letters; their components respectively their elements are denoted by the corresponding (not underlined) letters. $\underline{I}$ is the *unit* $(N \times N)$-matrix, $I_{mn} = \delta_{mn}$; in the following we will generally omit to write out this matrix. Of course above and hereafter $\delta_{mn} \equiv \delta_{m,n}$ is the standard Kronecker symbol, $\delta_{mn} = 1$ if $m = n$, $\delta_{mn} = 0$ if $m \neq n$. ■

Remark 2.1. Clearly formula (2.4) holds as written only if *all* the $2N$ eigenvalues λ_j are *different among themselves*. It continues to hold in a limiting sense if two or more eigenvalues coincide, in which case clearly the time evolution in the right-hand side of (2.4) also features terms depending on *integer powers* of the time variable t. ■

Remark 2.2. If the real parts of the $2N$ eigenvalues λ_j vanish and they are all different among themselves, it is plain from (2.2b) with (2.4) that, for *all* time, the coordinates $z_n(t)$ remain *confined* to a *finite* region of the *complex* z-plane. ■

Remark 2.3. If

$$A_1 = D_0 = 0 \quad \text{or} \quad A_3 = D_2 = 0 \tag{2.6}$$

the two matrices $\underline{A}$ and $\underline{D}$ are *triangular*; the $2N$ eigenvalues λ_j are then given by the explicit formulas

$$\lambda_m^{(\pm)} = -\frac{1}{2}\ [(N-m)\ D_1 + E \pm \Delta_m]\ , \tag{2.7a}$$

$$\Delta_m = \left\{ [(N-m)\ D_1 + E]^2 + 4\, m\ [(2\,N-m-1)\ A_2 + B] \right\}^{1/2} \tag{2.7b}$$

where, say, $\lambda_{2m-1} = \lambda_m^{(+)}$, $\lambda_{2m} = \lambda_m^{(-)}$, $m = 1,...,N$. Clearly in this case *all* these eigenvalues λ_j are *imaginary* if there hold the following two conditions on the parameters: (i) D_1 and E are *imaginary* (or *vanishing*); (ii) A_2 and B are *real* and have values which entail that Δ_m is *imaginary* (for $m = 1,2,...,N$). But *no* assignment of the parameters—other than the special one described in the following *Proposition 2.2*—yields $2N$ eigenvalues λ_j which are *all different among themselves* and are *all integer multiples* of the same *nonvanishing imaginary* number $i\,\omega$. ■

In the following we shall focus on models satisfying one of the two conditions (2.6), in order to take advantage of the *explicit* expressions (2.7) of the eigenvalues characterizing the time evolution of the dynamical system (2.3). Additional conditions which guarantee that this time evolution is actually *isochronous* are detailed by the following

Proposition 2.2. If one of the two conditions (2.6) holds (see *Remark 2.3*), and there moreover hold the following 4 additional conditions on the parameters: (i) the real parts of D_1 and E vanish,

$$D_1 = i\delta\ , \quad E = i\eta \tag{2.8a}$$

with δ and η two *real* parameters; (ii) the remaining parameters are *all real*; (iii) $A_2 > -\delta^2/4$ so that the quantity

$$\omega = \sqrt{A_2 + \delta^2/4} \tag{2.8b}$$

is as well *real* (and, by convention, *positive*, $\omega > 0$); (iv) there holds the following relation among the 5 parameters N, δ, η, A_2, B:

$$(N\,\delta + \eta)\ (\delta + 2\,\omega) + 2\ [(2\,N-1)\ A_2 + B] = 0\ ; \tag{2.8c}$$

then the expression (2.7) of the $2N$ eigenvalues becomes

$$\lambda_m^{(\pm)} = i\ \left[(N\,\delta + \eta)\ \left(\frac{\pm 1 - 1}{2} \right) + m\ \left(\frac{\delta}{2} \pm \omega \right) \right]\ . \tag{2.9}$$

If, moreover,

$$\delta = \frac{p_1}{q_1}\,\omega\,, \quad \eta = \frac{p_2}{q_2}\,\omega\,, \tag{2.10}$$

with q_1, q_2 two *arbitrary positive integers*, p_1, p_2 two *arbitrary integers*, and $p_1 \neq \pm 2q_1$ (so that $\delta \pm 2\omega \neq 0$), and the eigenvalues (which then read)

$$\lambda_m^{(\pm)} = i\,\left[(N\,p_1\,q_2 + p_2\,q_1)\;(\pm 1 - 1) + m\;(p_1 \pm 2q_1)\;q_2\right]\,\Omega\,, \tag{2.11a}$$

$$\Omega = \frac{\omega}{2\,q_1\,q_2} \tag{2.11b}$$

are all different among themselves, then clearly *all* the coefficients $c_m(t)$, hence as well the polynomial (2.2b), evolve in time *periodically* with the *same* period,

$$c_m(t \pm T) = c_m(t)\,, \quad \psi(z, t \pm T) = \psi(z,t)\,, \quad T = \frac{2\,\pi}{\Omega}\,. \tag{2.12}$$

Hence (see (2.2b)) in this special case *all* the coordinates $z_n(t)$ also evolve in time *periodically* with the *same* period T (or possibly, some of them, with a period which is a, generally small, *integer multiple* of T, due to the possibility that over the time evolution some of the zeros of the polynomial $\psi(z,t)$ exchange their roles; for a discussion of this possibility, including a justification of the assertion that the *integer multiple* in question is "generally small", see [5]). Anyway, in this special case the N-body problem (2.1) is *isochronous*. ■

Remark 2.4. Note, however, that the phenomenon described at the end of *Proposition 2.2* cannot happen for motions sufficiently close (in particular, infinitesimally close) to equilibrium configurations, such as those considered below. ■

The next development is to consider the N-body problem (2.1) in the immediate vicinity of its equilibrium configurations, which are clearly characterized by N coordinates $\bar{z}_n$ satisfying the N nonlinear algebraic equations

$$\begin{aligned} &(N-1)\,A_1 - B\,\bar{z}_n + 2\,(N-1)\,A_3\,\bar{z}_n^2 \\ &= 2\,\left(A_1 + A_2\,\bar{z}_n + A_3\,\bar{z}_n^2\right)\,\bar{z}_n \sum_{\ell=1,\,\ell\neq n}^{N} (\bar{z}_n - \bar{z}_\ell)^{-1}\,. \end{aligned} \tag{2.13}$$

Clearly the N equilibrium coordinates $\bar{z}_n$ can be equivalently characterized—see (2.2)—as the N zeros of the polynomial

$$\bar{\psi}(z) = \prod_{n=1}^{N} (z - \bar{z}_n) = z^N + \sum_{m=1}^{N} \left[\bar{c}_m\,z^{N-m}\right]\,, \tag{2.14a}$$

satisfying the time-independent version of the PDE (2.2a), i. e. the ODE

$$\begin{aligned} \left(A_1 + A_2\,z + A_3\,z^2\right)\,z\,\bar{\psi}'' - \left[(N-1)\,A_1 - B\,z + 2\,(N-1)\,A_3\,z^2\right]\,\bar{\psi}' \\ -N\,\left[(N-1)\;(A_2 - A_3\,z) + B\right]\,\bar{\psi} = 0\,. \end{aligned} \tag{2.14b}$$

Notation 2.4. Here and hereafter appended primes denote derivatives with respect to the argument of the function they are appended to. ■

To investigate the behavior of the N-body problem (2.1) in the *immediate vicinity* of its *equilibria* one of course sets

$$z_n(t) = \bar{z}_n + \varepsilon\ \zeta_n(t)\ , \tag{2.15}$$

with the N time-independent coordinates $\bar{z}_n$ characterizing the *equilibria* and ε *infinitesimal*. The equations of motion (2.1) get thereby linearized:

$$\underline{\ddot{\zeta}} + \underline{\bar{D}}\ \underline{\dot{\zeta}} + \underline{\bar{A}}\ \underline{\zeta} = 0\ , \tag{2.16}$$

with the two $N \times N$ matrices $\underline{\bar{D}}$ and $\underline{\bar{A}}$ obtained—in terms of the equilibrium coordinates $\bar{z}_n$—in a standard manner (see below) from the N-body problem (2.1).

The *general* solution of this system of N linear equations of motion—characterizing the behavior of the N-body problem (2.1) in the immediate neighborhood of its equilibria, see (2.15)—reads

$$\zeta_n(t) = \sum_{j=1}^{2N} \eta_j\ \bar{\zeta}_n^{(j)}\ \exp\left(\bar{\lambda}_j\ t\right) \tag{2.17a}$$

where the $2N$ numbers η_j are *a priori* arbitrary (or can be *a posteriori* fixed, in the context of the *initial-value* problem, by imposing consistency with the $2N$ initial data $\zeta_n(0)\,,\ \dot{\zeta}_n(0)$), while the $2N$ quantities $\bar{\lambda}_j$, respectively the $2N^2$ components $\bar{\zeta}_n^{(j)}$ of the $2N$ N-vectors $\underline{\bar{\zeta}}^{(j)} \equiv \left(\bar{\zeta}_1^{(j)}, \bar{\zeta}_2^{(j)}, ..., \bar{\zeta}_n^{(j)}\right)$, are the $2N$ *eigenvalues*, respectively the $2N$ *eigenvectors*, of the generalized eigenvalue problem

$$\left(\bar{\lambda}^2 + \bar{\lambda}\ \underline{\bar{D}} + \underline{\bar{A}}\right)\ \underline{\bar{\zeta}}^{(j)} = 0 \tag{2.17b}$$

(with analogous remarks to those made above after (2.5), see in particular *Remark 2.1*).

It is then plain that these $2N$ *eigenvalues* $\bar{\lambda}_j$—which characterize the behavior of the N-body problem (2.1) in the *immediate neighborhood* of its equilibria—coincide with the $2N$ *eigenvalues* λ_j of the generalized eigenvalue problem (2.5a), which, via (2.2b) and (2.4), characterize the *general* behavior of the solutions $z_n(t)$ of the N-body problem (2.1):

$$\bar{\lambda}_j = \lambda_j\ , \quad j = 1, 2, ..., 2N\ . \tag{2.18}$$

Hence—in all cases when the $2N$ eigenvalues λ_j can be explicitly computed, see above—one can similarly assert that the eigenvalues $\bar{\lambda}_j$ are known; and in the special cases when the N-body problem (2.1) is *isochronous*—hence the $2N$ eigenvalues λ_j are, up to a common integer factor, *integer* numbers, see (2.11)—the same *Diophantine* assertion can be made for the $2N$ eigenvalues $\bar{\lambda}_j$ of the generalized eigenvalue problem (2.17b).

These shall be the main findings of the present paper. To obtain them we must still find the two $N \times N$ matrices $\underline{\bar{D}}$ and $\underline{\bar{A}}$. To this end we now take again advantage of the results of [1], in particular those concerning the equilibria of the N-body problem (2.1).

We already saw that the numbers $\bar{z}_n$ providing the N coordinates characterizing the *equilibria* of the N-body problem (2.1) coincide with the N zeros of the (monic) polynomial $\bar{\psi}(z)$, see (2.14a), satisfying the ODE (2.14b). This ODE—featuring, in addition to the *arbitrary positive integer* N, the 4 *a priori* arbitrary parameters A_1, A_2, A_3, B (any single one of which can of course be divided away, unless it vanishes)—is generally *not* of hypergeometric type hence *not* reducible to the ODEs characterizing the classical polynomials (except in special cases, see below). Yet it can be rather

explicitly solved in terms of quadratures over elementary functions (see [1]), as detailed by the following

Proposition 2.3. The *general* solution of the ODE (2.14b) is given by the formula

$$\bar{\psi}(z) = z^N \, \varphi(z) \ , \tag{2.19}$$

with

$$\varphi(z) = \rho + \gamma \int^z dx \, x^\nu \ (x - z_+)^{\nu_+} \ (x - z_-)^{\nu_-} \ , \tag{2.20a}$$

or equivalently

$$\varphi(z) = \rho + \gamma \int^{1/z} dy \ (1 - z_+ \, y)^{\nu_+} \ (1 - z_- \, y)^{\nu_-} \ . \tag{2.20b}$$

The parameters ν, $\nu_\pm$ and $z_\pm$ are related to the *5* parameters N, A_1, A_2, A_3, B by the following formulas:

$$\nu = -(N+1) \ , \quad \nu_+ + \nu_- = N - 1 \ , \tag{2.21a}$$

$$b = -\frac{1}{2} \ [3 \ (N-1) \ a_2 + (\nu_+ - \nu_-) \ \Delta] \ , \tag{2.21b}$$

$$\Delta^2 = a_2^2 - 4 \, a_1 \ , \tag{2.21c}$$

$$z_\pm = \frac{-a_2 \pm \Delta}{2} \ , \tag{2.21d}$$

where

$$a_1 = \frac{A_1}{A_3} \ , \quad a_2 = \frac{A_2}{A_3} \ , \quad b = \frac{B}{A_3} \ . \tag{2.21e}$$

Of course this solution holds as written for *generic* values of the *5* parameters N, A_1, A_2, A_3, B; special cases are treated separately below. ■

Notation 2.5. In (2.20) and throughout ρ and γ denote two *arbitrary* (integration) constants, without implying that they have the same values in the different equations where they appear. ■

These findings imply that there holds (see [1]) the following

Proposition 2.4. In the *generic* case (implying in particular that $A_3 \neq 0, A_1 \neq 0$ and $A_2^2 - 4A_1A_3 \neq 0$) the single (*Diophantine*) restriction on the 5 parameters N, A_1, A_2, A_3, B reading

$$B = -\frac{1}{2} \ \left[3 \ (N-1) \ A_2 + (N+1-2 \, k) \ \left(A_2^2 - 4 \, A_1 \, A_3\right)^{1/2}\right] \tag{2.22}$$

with k an *arbitrary integer* in the range $1 \leq k \leq N$, is *necessary* and *sufficient* to guarantee that the *general* solution of the ODE (2.14b) be *polynomial*, indeed (up to an *arbitrary* overall multiplicative

constant, assigned here so as to make the polynomial $\bar{\psi}(z)$ *monic*) this *general* solution reads

$$\bar{\psi}(z) = z^N + \gamma \sum_{m=1}^{N} \left(\bar{c}_m \, z^{N-m}\right) \ , \tag{2.23a}$$

$$\bar{c}_m = \frac{(-1)^m}{m} \sum_{j=\max(0,\, m-k)}^{\min(N-k,\, m-1)} \binom{N-k}{j} \binom{k-1}{m-1-j} z_+^j \, z_-^{m-1-j} \ , \tag{2.23b}$$

where γ is an *arbitrary* (integration) constant. Here notation (2.21) is employed, and the standard notation for the binomial coefficient,

$$\binom{p}{q} \equiv \frac{p!}{q! \ (p-q)!} \ , \tag{2.23c}$$

where p is an *arbitrary nonnegative integer* and q is an *integer* in the range $0 \leq q \leq p$. ■

Remark 2.5. Note that the *generic* case considered in this *Proposition 2.4* violates the conditions (2.6) allowing an *explicit* evaluation of the eigenvalues λ_j, see above *Remark 2.3*. ■

We therefore hereafter focus on other, *less generic*, cases in which the *general* solution of the ODE (2.14b) is *polynomial*: see the following *Proposition 2.5*, reporting the findings of [1] concerning *all* the *equilibria* of the N-body problem (2.1), hence the solutions $\bar{z}_n$ of the set of N algebraic equations (2.13) which coincide with the N zeros $\bar{z}_n$ of the *polynomial* solutions, see (2.14a), of the ODE (2.14b).

Proposition 2.5. The following conditions on the 5 parameters N, A_1, A_2, A_3, B are *necessary and sufficient* to imply that the *general* solution of the ODE (2.14b) be a *polynomial* of degree N, hence they are as well *necessary and sufficient* to imply that the N-body problem (2.1) feature a (time-independent) *equilibrium* solution, the configuration of which is indeed provided by the N zeros $\bar{z}_n$ of this polynomial (and it is a *genuine equilibrium configuration*, featuring N coordinates $\bar{z}_n$ *all different among themselves*).

Case (i) (the most generic):

$$A_3 \neq 0, \quad A_1 \neq 0, \quad \text{and} \quad A_2^2 - 4\, A_1\, A_3 \neq 0 \ , \tag{2.24a}$$

$$B = -\frac{1}{2} \left[3\,(N-1)\, A_2 + (N-1-2\,k) \ \left(A_2^2 - 4\, A_1\, A_3\right)^{1/2} \right] \ , \tag{2.24b}$$

with k an *arbitrary integer* in the range $1 \leq k \leq N$ (see (2.22)). Then the equilibrium configuration is identified by the N zeros $\bar{z}_n$ of the polynomial (2.23), see *Proposition 2.4* (of course, for *generic* values of the *arbitrary* parameter γ; excluding *exceptional* values—such as $\gamma = 0$—which are *incompatible* with the assumption that the N zeros $\bar{z}_n$ of the polynomial (2.23) be *all different among themselves*). Note that this case violates the conditions (2.6) allowing to evaluate *explicitly* the eigenvalues λ_j (see *Remark 2.3*).

Case (ii):

$$A_3 \neq 0, \quad A_2 \neq 0 \ , \quad A_1 = 0 \ . \tag{2.25}$$

There are then two subcases.

Case (ii.a):

$$B = -(N-1)\, A_2 \,, \tag{2.26a}$$

with the explicit expression of the N equilibrium coordinates $\bar{z}_n$ reading

$$\bar{z}_n = -\frac{A_2}{A_3}\left[1+\gamma\,\exp\left(\frac{2\,\pi\,i\,n}{N}\right)\right]^{-1} . \tag{2.26b}$$

Case (ii.b):

$$B = -N\, A_2 \,, \tag{2.27a}$$

with the explicit expression of the N equilibrium coordinates $\bar{z}_n$ reading

$$\bar{z}_n = -\frac{A_2}{A_3}\left[1+\gamma\,\exp\left(\frac{2\,\pi\,i\,n}{N-1}\right)\right]^{-1} , \quad n = 1,2,...,N-1\,; \quad \bar{z}_n = 0\,. \tag{2.27b}$$

Let us repeat that, here and throughout, i is of course the *imaginary unit*, $i^2 = -1$; while here γ is a *constant parameter, arbitrary* except for the following restrictions: $\gamma \neq 0$, $\gamma \neq -1$, and moreover in *Case (ii.a)* $\gamma \neq 1$ if N is *even,* in *Case (ii.b)* $\gamma \neq 1$ if N is *odd*.

Case (iii):

$$A_3 \neq 0, \quad A_2 \neq 0\,, \quad A_1 = \frac{A_2^2}{4\,A_3} \neq 0\,, \tag{2.28a}$$

$$B = -\frac{3\,(N-1)\,A_2}{2}\,, \tag{2.28b}$$

with the explicit expression of the N equilibrium coordinates $\bar{z}_n$ reading

$$\bar{z}_n = -\frac{A_2}{2\,A_3}\left[1+\gamma\,\exp\left(\frac{2\,\pi\,i\,n}{N}\right)\right]^{-1} , \quad n = 1,2,...,N\,, \tag{2.28c}$$

where i is again the *imaginary unit* and γ a *constant parameter*, *arbitrary* except for the following restrictions: $\gamma \neq 0$, $\gamma \neq -1$, and $\gamma \neq 1$ if N is *even*. Note that this case violates the conditions (2.6) allowing to evaluate *explicitly* the eigenvalues λ_j (see *Remark 2.3*).

Case (iv):

$$A_3 = 0, \quad A_2 \neq 0\,, \quad A_1 \neq 0\,. \tag{2.29a}$$

In this case the parameter B must satisfy the Diophantine condition

$$B = (-2\,N+1+k)\,A_2\,. \tag{2.29b}$$

Here k is an *integer* in the range $1 \leq k \leq N$, and the N equilibrium coordinates $\bar{z}_n$ are the N zeros of the monic polynomial

$$\bar{\psi}(z) = z^N + \gamma \sum_{m=k}^{N} \left(\bar{c}_m\, z^{N-m}\right)\,, \tag{2.30a}$$

with γ an *arbitrary* (nonvanishing) *constant parameter* and

$$\bar{c}_m = \binom{N-k}{m-k} \frac{a^{m-k}}{m} \ , \quad a = \frac{A_1}{A_2} \ , \quad m = k, k+1, ..., N \ . \tag{2.30b}$$

Equivalently, these N zeros are those of the following function:

$$\varphi(z) = 1 + \gamma \, P_n^{(-k, \ -N-1+k)} \left(\frac{z + 2\,a}{z} \right) \ , \quad a = \frac{A_1}{A_2} \ , \tag{2.31}$$

with γ an *arbitrary* (nonvanishing) constant, $P_n^{(\alpha, \ \beta)}(x)$ the standard (conveniently renormalized, to make it monic) Jacobi polynomial (see for instance [6]) and k again an *arbitrary integer* in the range $1 \leq k \leq N$. Note that this is in fact a para-Jacobi polynomial [8]; and it also easily seen that the polynomial $\bar{\psi}(z)$ coincides itself with the following para-Jacobi polynomial [8]:

$$\bar{\psi}(z) = \left(\frac{a}{2} \right)^N \ p_n \left(0, k-1; \gamma; 1 + \frac{2\,z}{a} \right) \ . \tag{2.32}$$

Case (v):

$$A_3 = A_2 = 0 \ , \quad A_1 \neq 0 \ . \tag{2.33}$$

Then if B does not vanish, $B \neq 0$, the basic assumption that $\bar{\psi}(z)$, see (2.14a), be a polynomial of degree N in z is *not* compatible with the ODE (2.14b), hence in this case there is *no* equilibrium configuration of the N-body problem (2.1); while if B does vanish, $B = 0$, then (up to an *arbitrary* overall multiplicative *constant*, which has been fixed here to make the polynomial $\bar{\psi}(z)$ *monic*)

$$\bar{\psi}(z) = z^N + \gamma \tag{2.34a}$$

with γ an *arbitrary* (nonvanishing) constant, entailing for its N zeros the simple formula

$$\bar{z}_n = \gamma \, \exp \left(\frac{2\,\pi\,i\,n}{N} \right) \ . \tag{2.34b}$$

The *arbitrary constants* γ in (2.34a) and (2.34b) are of course different, being related to each other in an obvious manner.

Case (vi):

$$A_3 = A_2 = A_1 = 0 \ . \tag{2.35}$$

In this case if B does *not vanish*, $B \neq 0$, the N-body problem (2.1) has *no genuine equilibrium configuration*; while, if instead also B vanishes, $B = 0$, then *any* configuration $\bar{z}_n$ with the N coordinates $\bar{z}_n$ *all different among themselves* is a *genuine equilibrium configuration* of the N-body problem (2.1) (but it is easily seen that in this case the N-body problem (2.1) is *not* altogether *isochronous*, because N of the $2N$ eigenvalues λ_j vanish). ■

Remark 2.6. Let us re-emphasize that we characterize an *equilibrium configuration* as *genuine* if the N coordinates $\bar{z}_n$ are *all different among themselves*, $\bar{z}_n \neq \bar{z}_m$ if $n \neq m$; and that in *Proposition 2.5*—indeed, throughout this paper—we focus on *such* equilibria, to avoid the ambiguities otherwise caused by the denominators present in the right-hand sides of the equations (2.1) and (2.13). Readers interested in cases when this restriction does not hold are referred to Appendix A of [1]. ■

Let us conclude this Section 2 by listing the cases of interest in the present paper. They are those such that the N-body problem (2.1) possesses a *genuine equilibrium configuration* (see *Proposition 2.5*) and moreover allows to compute *explicitly* the eigenvalues λ_j (see *Remark 2.3*). We also highlight the conditions which are sufficient to guarantee the *isochrony* of the N-body problem (see *Proposition 2.2*, with $q = q_1$, $p = p_1$, $p \neq 2q$, $\tilde{\omega} = \omega/(2q)$).

Case 1.

$$\begin{aligned} &A_1 = D_0 = 0\,, \quad A_2 = \left(4\,q^2 - p^2\right)\,\tilde{\omega}^2\,, \quad A_3 \neq 0\,, \\ &B = -(N-1)\,A_2\,, \quad D_1 = 2\,i\,p\,\tilde{\omega}\,, \quad E = -N\,\left(2\,q + p\right)\,i\,\tilde{\omega}\,. \end{aligned} \tag{2.36a}$$

Note that D_2 remains unrestricted. Here p, q and $\tilde{\omega}$ are 3 *a priori* arbitrary parameters. The corresponding eigenvalues $\lambda_m^{(\pm)}$ are

$$\lambda_m^{(\pm)} = i\,\left[N\,\left(p - 2\,q\right)\,\left(\frac{\pm 1 - 1}{2}\right) + m\,\left(p \pm 2\,q\right)\right]\,\tilde{\omega}\,. \tag{2.36b}$$

And the corresponding equilibrium coordinates read (see (6.57))

$$\bar{z}_n = -\frac{A_2}{A_3\,\left(1 + \gamma\,u_n\right)} = -\frac{A_2\,w_n}{A_3}\,. \tag{2.36c}$$

These results follow via elementary algebra from the relevant results of *Proposition 2.2* and of *Case (ii.a)* of *Proposition 2.5*. The corresponding N-body model (2.1) is *isochronous* iff p and q are integers and $p \pm 2\,q \neq 0$.

Case 2.

$$\begin{aligned} &A_1 = D_0 = 0\,, \quad A_2 = \left(4\,q^2 - p^2\right)\,\tilde{\omega}^2\,, \quad A_3 \neq 0\,, \\ &B = -N\,A_2\,, \quad D_1 = 2\,i\,p\,\tilde{\omega}\,, \quad E = -\left[(N-1)\,2\,q + (N+1)\,p\right]\,i\,\tilde{\omega}\,. \end{aligned} \tag{2.37a}$$

Again, D_2 remains unrestricted and p, q, $\tilde{\omega}$ are 3 *a priori* arbitrary parameters. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$\lambda_m^{(\pm)} = i\,\left[N\,\left(p - 2\,q + 2\right)\,\left(\frac{\pm 1 - 1}{2}\right) + m\,\left(p \pm 2\,q\right)\right]\,\tilde{\omega}\,. \tag{2.37b}$$

And the corresponding equilibrium coordinates read (see (6.57) with $u_n = \exp\left(\frac{2\,\pi\,i\,n}{N-1}\right)$)

$$\bar{z}_n = -\frac{A_2}{A_3\,\left(1 + \gamma\,u_n\right)} = -\frac{A_2\,w_n}{A_3}\,, \quad n = 1, \ldots, N-1; \quad \bar{z}_n = 0\,. \tag{2.37c}$$

These results follow via elementary algebra from the relevant results of *Proposition 2.2* and of *Case (ii.b)* of *Proposition 2.5*. Again, the corresponding N-body model (2.1) is *isochronous* iff p and q are integers and $p \pm 2\,q \neq 0$.

Case 3.

$$A_1 \neq 0\ , \quad A_2 = \left(4\,q^2 - p^2\right)\ \tilde{\omega}^2\ , \quad A_3 = D_2 = 0\ , \quad D_1 = 2\,i\,p\,\tilde{\omega}\ ,$$
$$B = (-2N + k + 1)\ A_2\ , \quad E = -\left[k\ (2\,q - p) + 2\,N\,p\right]\ i\,\tilde{\omega}\ . \tag{2.38a}$$

Here k is any *integer* in the range from 1 to N, $k = 1, 2, ..., N$, while D_0 is unrestricted and p, q, $\tilde{\omega}$ are 3 *a priori* arbitrary parameters. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$\lambda_m^{(\pm)} = i\ \left[k\ (p - 2\,q)\ \left(\frac{\pm 1 - 1}{2}\right) + m\ (p \pm 2\,q)\right]\ \tilde{\omega}\ . \tag{2.38b}$$

And the corresponding equilibrium coordinates $\bar{z}_n$ are the N zeros of the para-Jacobi polynomial $p_n\left(0, k-1; \gamma; 1 + \frac{2\,z}{a}\right)$, see (2.32) and, more explicitly, (2.30). These results follow via elementary algebra from the relevant results of *Proposition 2.2* and of *Case (iv)* of *Proposition 2.5*. Again, the corresponding N-body model (2.1) is *isochronous* iff p and q are integers and $p \pm 2\,q \neq 0$.

Case 4.1.

$$A_1 \neq 0\ , \quad A_2 = A_3 = B = D_2 = 0\ ,$$
$$D_1 = 2\,i\,\omega\ , \quad E = -2\,N\,i\,\omega\ , \tag{2.39a}$$

and D_0 unrestricted. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$\lambda_m^{(+)} = 2\,i\,m\,\omega, \quad \lambda_m^{(-)} = 0\ . \tag{2.39b}$$

Case 4.2

$$A_1 \neq 0\ , \quad A_2 = A_3 = B = D_2 = 0\ ,$$
$$D_1 = -2\,i\,\omega\ , E = i\,p\,\omega\ . \tag{2.40a}$$

Here D_0 is unrestricted and p is an *arbitrary* rational number. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$\lambda_m^{(+)} = 0, \ \ \lambda_m^{(-)} = i\ (2\,N - p - 2\,m)\ \omega\ . \tag{2.40b}$$

As for the corresponding equilibrium coordinates, they read, in both these two *Cases 4.1* and *4.2*, as follows (see (6.57)):

$$\bar{z}_n = \gamma\,u_n\ . \tag{2.41}$$

Clearly in these two cases the corresponding N-body model is *not isochronous*. These results follow via elementary algebra from the relevant results of *Proposition 2.2* and of *Case (v)* of *Proposition 2.5*.

Case 5.1.

$$A_1 = A_2 = A_3 = B = 0 \,,$$
$$D_1 = 2\, i\, \omega \,, \quad E = -2\, N\, i\, \omega \,, \tag{2.42a}$$

here $D_0 = 0$ or $D_2 = 0$. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$\lambda_m^{(+)} = 2\, i\, m\, \omega, \quad \lambda_m^{(-)} = 0 \,. \tag{2.42b}$$

Case 5.2.

$$A_1 = A_2 = A_3 = B = 0 \,,$$
$$D_1 = -2\, i\, \omega \,, \quad E = i\, p\, \omega \,, \tag{2.43a}$$

here $D_0 = 0$ or $D_2 = 0$ and p is an *arbitrary* rational number. The corresponding eigenvalues $\lambda_m^{(\pm)}$ now read

$$\lambda_m^{(+)} = 0, \;\; \lambda_m^{(-)} = i\, (2\, N - p - 2\, m)\, \omega \,. \tag{2.43b}$$

And the corresponding equilibrium coordinates $\bar{z}_n$ are in both cases N, quite *arbitrary*, numbers (all different among themselves, for a genuine equilibrium).

These results follow via elementary algebra from the relevant results of *Proposition 2.2* and of *Case (vi)* of *Proposition 2.5*. Again, the corresponding N-body model (2.1) is not *isochronous*.

Finally let us recall that we excluded here the *Cases (i)* and *(iii)* of *Proposition 2.5* because they violate the conditions allowing to compute *explicitly* the eigenvalues λ_j (see *Remark 2.3*).

3. Main findings

In this section the main findings of the present paper are reported; and tersely commented upon at the end.

Proposition 3.1. Let the $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$ be defined, componentwise, as follows:

$$\bar{A}_{nn} = -\frac{(4\, q^2 - p^2)\, \tilde{\omega}^2}{6}\, \left(N^2 - 1\right)\, \left(1 + \frac{1}{\tilde{u}_n}\right) \,, \tag{3.44a}$$

$$\bar{A}_{nm} = -2\, (4\, q^2 - p^2)\, \tilde{\omega}^2\, \frac{\tilde{u}_n}{1 + \tilde{u}_n}\, \left(\frac{1 + \tilde{u}_m}{\tilde{u}_n - \tilde{u}_m}\right)^2 \,, \quad n \neq m \,; \tag{3.44b}$$

$$\bar{D}_{nn} = \frac{(N-1)\, (-2\, i\, p\, \tilde{\omega} + \alpha)}{2\, \tilde{u}_n} - (2\, N\, q + p)\, i\, \tilde{\omega} \,, \tag{3.44c}$$

$$\bar{D}_{nm} = \left(-2\, i\, p\, \tilde{\omega} + \frac{\alpha}{1 + \tilde{u}_n}\right)\, \frac{1 + \tilde{u}_m}{\tilde{u}_n - \tilde{u}_m} \,, \quad n \neq m \,. \tag{3.44d}$$

Here $\tilde{u}_n = \gamma\, u_n = \gamma \exp\left(2\pi i n / N\right)$ (see (6.57a)), with γ, $\tilde{\omega}$ (see (2.36b)) and $\alpha = D_2 a_2$ three *arbitrary* parameters and p and q two *arbitrary integers* (see (2.10)). Then the $2N$ roots of the determinantal

polynomial $P_{2N}(\lambda) \equiv \det\left(\lambda^2 + \lambda \underline{\bar{D}} + \underline{\bar{A}}\right)$ are $\lambda_j = i\tilde{\omega} s_j$ with the $2N$ numbers s_j *all integers*, indeed

$$P_{2N}(\lambda) \equiv \det(\lambda^2 + \lambda\, \underline{\bar{D}} + \underline{\bar{A}}) = \prod_{m=1}^{N} (\lambda - \lambda_m^{(+)})\, (\lambda - \lambda_m^{(-)}) \tag{3.45a}$$

with

$$\lambda_m^{(\pm)} = i\, \left[N\, (p - 2\, q)\, \left(\frac{\pm 1 - 1}{2}\right) + m\, (p \pm 2\, q)\right]\, \tilde{\omega}\, . \tag{3.45b}$$

Of course these formulas, (3.44) and (3.45), also hold for *arbitrary* (not integer) values of the two parameters p and q. ■

These results follow from *Case 1* (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

The results which follow from *Case 2* coincide with those of *Proposition 3.1*, up to a shift of the parameter N; hence they are not reported.

Proposition 3.2. Let the $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$ be defined, componentwise, as follows:

$$\bar{A}_{nn} = -\frac{N^2 - 1}{6}\, \frac{A_1}{\tilde{u}_n}\, ; \quad \bar{A}_{nm} = -2\, A_1\, \frac{\tilde{u}_n}{(\tilde{u}_n - \tilde{u}_m)^2}\, , \quad n \neq m\, ; \tag{3.46a}$$

$$\bar{D}_{nn} = E + \frac{N-1}{2}\, \left(D_1 + \frac{D_0}{\tilde{u}_n}\right) \tag{3.46b}$$

$$\bar{D}_{nm} = \frac{D_0 + D_1\, \tilde{u}_n}{\tilde{u}_n - \tilde{u}_m}\, , \quad n \neq m\, . \tag{3.46c}$$

Here again $\tilde{u}_n = \gamma u_n = \gamma \exp(2\pi i n/N)$ (see (6.57a)), with γ, D_0 and A_1 three *arbitrary* parameters. As for D_1 and E, either $D_1 = 2\, i\, \omega$, $E = -2\, N\, i\, \omega$ (corresponding to *Case 4.1* of Section 2), or $D_1 = -2\, i\, \omega$, $E = i\, p\, \omega$, with ω an *arbitrary* parameter and p an *arbitrary integer* (corresponding to *Case 4.2* of Section 2). Then (in both cases) the $2N$ roots of the determinantal polynomial $P_{2N}(\lambda) \equiv \det\left(\lambda^2 + \lambda \underline{\bar{D}} + \underline{\bar{A}}\right)$ are $\lambda_j = i\omega s_j$ with the $2N$ numbers s_j *all integers* (actually, N of them vanishing), indeed

$$P_{2N}(\lambda) \equiv \det(\lambda^2 + \lambda\, \underline{\bar{D}} + \underline{\bar{A}}) = \lambda^N\, \prod_{m=1}^{N} (\lambda - \lambda_m) \tag{3.47}$$

with

$$\lambda_m = \lambda_m^{(+)} = 2\, i\, m\, \omega\, , \tag{3.48a}$$

when

$$D_1 = 2\, i\, \omega,\, E = -2\, N\, i\, \omega\, , \tag{3.48b}$$

and

$$\lambda_m = \lambda_m^{(-)} = i\, (2\, N - p - 2m)\, \omega \tag{3.48c}$$

when

$$D_1 = -2\, i\, \omega,\, E = i\, p\, \omega. \tag{3.48d}$$

Of course the last two formulas, (3.48c) and (3.48d), also hold for *arbitrary* (not integer) values of the parameter p. ■

These results follow from *Case 4* (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

Proposition 3.3. Let the $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$ be defined, componentwise, as follows:

$$\bar{A}_{nn} = \frac{\left(4\,q^2 - p^2\right)\,\tilde{\omega}^2}{6}\,\left[\left(k^2-1\right)\,x_n^2 + 2\,(k-1)\,(2\,N-k+1)\,x_n\right.$$
$$\left.+6\,(2\,N-k-1) + 4\,(N-1)\,(N-k-1) + (k-1)^2\right]\,\left(1-x_n^2\right)^{-1}\,, \tag{3.49a}$$

$$\bar{A}_{nm} = 2\,\left(4\,q^2 - p^2\right)\,\tilde{\omega}^2\,\frac{1-x_n^2}{\left(x_n - x_m\right)^2}\,, \quad n \neq m\,; \tag{3.49b}$$

$$\bar{D}_{nn} = -\left[k\,(2\,q-p) + 2\,N\,p\right]\,i\,\tilde{\omega}$$
$$-\left[\alpha + 2\,i\,p\,\tilde{\omega}\,(x_n-1)\right]\,\frac{k-1+(2\,N-k-1)\,x_n}{2\,(1-x_n^2)}\,, \tag{3.49c}$$

$$\bar{D}_{nm} = \left[\alpha + 2\,i\,p\,\tilde{\omega}\,(x_n-1)\right]\,\left(x_n - x_m\right)^{-1}\,, \quad n \neq m\,. \tag{3.49d}$$

Here x_n are the N zeros of the para-Jacobi polynomial $p_n\,(0,k-1;\gamma;x)$ [8], with k an *arbitrary integer* in the range $1 \leq k \leq N$, and the parameter γ an *arbitrary* number; and as well *arbitrary* are the two parameters $\tilde{\omega}$ (see (2.36b)) and $\alpha = 2D_0/a$, and the two *integers* p and q. Then the $2N$ roots of the determinantal polynomial $P_{2N}(\lambda) \equiv \det\left(\lambda^2 + \lambda\underline{\bar{D}} + \underline{\bar{A}}\right)$ are $\lambda_j = i\tilde{\omega}s_j$ with the $2N$ numbers s_j *all integers*, indeed

$$P_{2N}(\lambda) \equiv \det(\lambda^2 + \lambda\,\underline{\bar{D}} + \underline{\bar{A}}) = \prod_{m=1}^{N}(\lambda - \lambda_m^{(+)})\,(\lambda - \lambda_m^{(-)}) \tag{3.50a}$$

with

$$\lambda_m^{(\pm)} = i\,\left[k\,(p-2\,q)\,\left(\frac{\pm 1 - 1}{2}\right) + m\,(p \pm 2\,q)\right]\,\tilde{\omega}\,. \tag{3.50b}$$

Of course these formulas, (3.49) and (3.50), also hold for *arbitrary* (not integer) values of the two parameters p and q. ■

These results follow from *Case 3* (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

Proposition 3.4. Let the $N \times N$ matrix $\underline{\bar{D}}$ be defined, componentwise, as follows:

$$\bar{D}_{nn} = E + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\left(\bar{z}_n - \bar{z}_\ell\right)^{-1}\,\left(D_0 + D_1\,\bar{z}_n + D_2\,\bar{z}_n\,\bar{z}_\ell\right)\right]\,, \tag{3.51a}$$

$$\bar{D}_{nm} = \left[\left(\bar{z}_n - \bar{z}_m\right)\right]^{-1}\left(D_0 + D_1\,\bar{z}_n + D_2\,\bar{z}_n^2\right)\,, \quad n \neq m\,, \tag{3.52a}$$

Here $D_0 = 0$ or $D_2 = 0$ and the N coordinates $\bar{z}_n$ are now as well *arbitrary* (but different among themselves). As for D_1 and E, either $D_1 = 2\,i\,\omega$, $E = -2\,N\,i\,\omega$ (corresponding to *Case 5.1* of

Section 2), or $D_1 = -2\, i\, \omega,\ E = i\, p\, \omega$, with ω an *arbitrary* parameter and p an *arbitrary integer* (corresponding to *Case 5.2* of Section 2). Then the determinantal polynomial $P_n(\lambda) \equiv \det(\lambda + \underline{\bar{D}})$ reads

$$P_n(\lambda) \equiv \det(\lambda + \underline{\bar{D}}) = \prod_{m=1}^{N} (\lambda - \lambda_m)\ , \tag{3.53}$$

with

$$\lambda_m = \lambda_m^{(+)} = 2\, i\, m\, \omega\ , \tag{3.54a}$$

when

$$D_1 = 2\, i\, \omega,\ E = -2\, N\, i\, \omega\ , \tag{3.54b}$$

and

$$\lambda_m = \lambda_m^{(-)} = i\, (2\, N - p - 2m)\, \omega \tag{3.54c}$$

when

$$D_1 = -2\, i\, \omega,\ E = i\, p\, \omega. \tag{3.54d}$$

Of course the last two formulas, (3.54c) and (3.54d), also hold for *arbitrary* (not integer) values of the parameter p. ■

These results follow from *Case 5* (as detailed in the last part of the preceding Section 2), as shown in the following Section 4.

Let us complete this section with the following comments on the results reported herein.

These findings are somewhat analogous to previous results: see in particular those reviewed in [7], in Appendix D (entitled "Remarkable matrices and related identities") of [4] and in Appendix C (entitled "Diophantine findings and conjectures") of [2], as well as those reported in the relevant papers quoted in the Preface of the updated paperback version of [2] and in the very recent papers [8] and [9]. The novelty is that the $N \times N$ matrices identified herein—whose eigenvalues are exhibited above and have a *Diophantine* character—depend on quite a few *arbitrary parameters*; more than the somewhat analogous $N \times N$ matrices previously identified as featuring analogous properties. On the other hand we expect that these results could also be proven by techniques analogous to those previously employed (see in particular [7] and [4]): indeed, it is generally the case—in the field of special functions and related topics—that alternative demonstrations are easily produced after some findings have been identified and proven to begin with...

4. Behavior near equilibria and consequential Diophantine findings

In this section we pursue the investigation of the behavior of the N-body problem (2.1) in the *immediate vicinity* of its *equilibria* (when they exist), see (2.15) and (2.16) yielding (2.17). Our task here is to compute explicitly the two $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$, see (2.17b). This is a standard task. We report here the result, which the diligent reader will easily verify by inserting the *ansatz* (2.15) in the equations of motion (2.1). Componentwise, these two $N \times N$ matrices read as follows:

$$\begin{aligned}\bar{A}_{nn} = -\{&B - 4\,(N-1)\,A_3\,\bar{z}_n \\ &+2\,\left(A_1 + 2\,A_2\,\bar{z}_n + 3\,A_3\,\bar{z}_n^2\right) \sum_{\ell=1,\ \ell\neq n}^{N} (\bar{z}_n - \bar{z}_\ell)^{-1} \\ &-2\,\left(A_1 + A_2\,\bar{z}_n + A_3\,\bar{z}_n^2\right)\,\bar{z}_n \sum_{\ell=1,\ \ell\neq n}^{N} (\bar{z}_n - \bar{z}_\ell)^{-2}\ \}\,,\end{aligned} \tag{4.55a}$$

$$\bar{A}_{nm} = -2\,\left(A_1 + A_2\,\bar{z}_n + A_3\,\bar{z}_n^2\right)\,\bar{z}_n\,(\bar{z}_n - \bar{z}_m)^{-2}\ ,\quad n \neq m\,; \tag{4.55b}$$

$$\bar{D}_{nn} = E + \sum_{\ell=1,\ \ell\neq n}^{N} \left[(\bar{z}_n - \bar{z}_\ell)^{-1}\ (D_0 + D_1\,\bar{z}_n + D_2\,\bar{z}_n\,\bar{z}_\ell)\right]\,, \tag{4.56a}$$

$$\bar{D}_{nm} = \left(D_0 + D_1\,\bar{z}_n + D_2\,\bar{z}_n^2\right)\,(\bar{z}_n - \bar{z}_m)^{-1}\ . \tag{4.56b}$$

The next task is to evaluate these two $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$ by inserting in their definitions (4.55) and (4.56) the explicit expressions of the N coordinates $\bar{z}_n$ which correspond to the various *equilibria* of the dynamical system (2.1). The relevant *Cases 1-5* are reviewed at the end of Section 2.

We consider firstly the assignments corresponding to *Case 1*. Then one gets (via the notation and the identities reported in the Appendix) the explicit expressions (3.44) of the two $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$. *Proposition 3.1* is thereby proven

Secondly, we consider the assignments corresponding to *Case 2*, proceeding as in the preceding case. But at the end of the relevant computations we conclude that this case yields the same findings as the previous one (up to appropriate redefinitions of the parameter N).

Thirdly, we consider the assignments corresponding to the two *Cases 4*. Then one gets (again, via the notation and the identities reported in the Appendix) the explicit expressions (3.46) of the two $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$. *Proposition 3.2* is thereby proven.

Fourthly, we consider the assignments corresponding to *Case 3*. Then one gets the explicit expressions (3.49) of the two $N \times N$ matrices $\underline{\bar{A}}$ and $\underline{\bar{D}}$. *Proposition 3.3* is thereby proven.

Finally, we consider the assignments corresponding to *Case 5*. Then one finds that the matrix $\underline{\bar{A}}$ vanishes identically, $\underline{\bar{A}} = 0$, while the matrix $\underline{\bar{D}}$ has the explicit expression (3.51). *Proposition 3.4* is thereby proven.

5. Outlook

We believe that the search for new *Diophantine* findings of the kind reported in this paper is far from over; as well as the search for new N-body problems amenable to exact treatments. Hence this search constitutes an open—and, we opine, an interesting—research task, that we plan ourselves, and hope others, will be able and willing to pursue.

6. Appendix A: Identities

In this Appendix we display a number of identities which are presumably known and can in any case be easily proven by going through them sequentially. The protagonists of these identities are the N-th roots of unity,

$$u_n = \exp\left(\frac{2\,\pi\,i\,n}{N}\right) , \tag{6.57a}$$

and the related quantities

$$w_n = (1+\gamma\,u_n)^{-1} \tag{6.57b}$$

with γ an arbitrary number (itself, however, not an N-th root of unity, so that the N quantities w_n are all *finite*).

$$(1-u_n)+\left(1-u_n^{-1}\right) = (1-u_n)\ \left(1-u_n^{-1}\right) = 2\ \left[1-\cos\left(\frac{2\,\pi\,n}{N}\right)\right] , \tag{6.58a}$$

$$(1-u_n)^{-1}+\left(1-u_n^{-1}\right)^{-1} = \frac{(1-u_n)+\left(1-u_n^{-1}\right)}{(1-u_n)\ \left(1-u_n^{-1}\right)} = 1 . \tag{6.58b}$$

$$\begin{aligned}\sum_{\ell=1,\ \ell\neq n}^{N}\left[f\left(\frac{u_n}{u_\ell}\right)\right] &= \sum_{\ell=1}^{N-1}\left[f\left(u_\ell\right)\right] = \sum_{\ell=1}^{N-1}\left[f\left(u_\ell^{-1}\right)\right] \\ &= -f(1)+\sum_{\ell=1}^{N}\left[f\left(u_\ell\right)\right] = -f(1)+\sum_{\ell=1}^{N}\left[f\left(u_\ell^{-1}\right)\right] .\end{aligned} \tag{6.59}$$

In these formulas, (6.59), $f(u)$ is an *arbitrary* function.

$$\sum_{\ell=1}^{N}\left(\frac{u_n}{u_\ell}\right)^p = \sum_{\ell=1}^{N} u_\ell^p = \sum_{\ell=1}^{N} u_\ell^{-p} = 0 , \quad p = 1,2,...,N-1 , \tag{6.60a}$$

$$\sum_{\ell=1,\ \ell\neq n}^{N}\left(\frac{u_n}{u_\ell}\right)^p = \sum_{\ell=1}^{N-1} u_\ell^p = \sum_{\ell=1}^{N-1} u_\ell^{-p} = -u_n^p = -1 , \quad p = 1,2,...,N-1 . \tag{6.60b}$$

$$\begin{aligned}\sum_{\ell=1,\ \ell\neq n}^{N}\left(1-\frac{u_n}{u_\ell}\right)^{-1} &= \sum_{\ell=1}^{N-1}\left(1-u_\ell\right)^{-1} = \sum_{\ell=1}^{N-1}\left(1-u_\ell^{-1}\right)^{-1} \\ &= \frac{1}{2}\sum_{\ell=1}^{N-1}\left[\left(1-u_\ell\right)^{-1}+\left(1-u_\ell^{-1}\right)^{-1}\right] = \frac{1}{2}\sum_{\ell=1}^{N-1}(1) = \frac{N-1}{2} ,\end{aligned} \tag{6.61a}$$

$$\begin{aligned}\sum_{\ell=1,\ \ell\neq n}^{N}\left[\left(1-\frac{u_n}{u_\ell}\right)^{-1}\ \frac{u_n}{u_\ell}\right] &= \sum_{\ell=1}^{N-1}\left[\left(1-u_\ell\right)^{-1}\ u_\ell\right] = \sum_{\ell=1}^{N-1}\left(u_\ell^{-1}-1\right)^{-1} \\ = \sum_{\ell=1}^{N-1}\left(u_\ell-1\right)^{-1} &= -\sum_{\ell=1}^{N-1}\left(1-u_\ell\right)^{-1} = -\frac{N-1}{2} ,\end{aligned} \tag{6.61b}$$

$$\sum_{\ell=1,\ \ell\neq n}^{N}\left(1-\frac{u_n}{u_\ell}\right)^{-1} u_\ell^{-1} = u_n^{-1}\sum_{\ell=1,\ \ell\neq n}^{N}\left(1-\frac{u_n}{u_\ell}\right)^{-1}\ \frac{u_n}{u_\ell} = -\frac{N-1}{2\,u_n}\ ; \tag{6.61c}$$

$$\sum_{\ell=1,\ \ell\neq n}^{N}\left[\left(1-\frac{u_n}{u_\ell}\right)^{-1}\ \frac{u_\ell}{u_n}\right] = \sum_{\ell=1}^{N-1}\left[(1-u_\ell)^{-1}\ u_\ell^{-1}\right]$$
$$= \sum_{\ell=1}^{N-1}\left[(u_\ell-1)^{-1}+u_\ell^{-1}\right] = \frac{N-3}{2}\ , \tag{6.62a}$$

$$\sum_{\ell=1,\ \ell\neq n}^{N}\left[\left(1-\frac{u_n}{u_\ell}\right)^{-1}\ u_\ell\right] = u_n\sum_{\ell=1,\ \ell\neq n}^{N}\left[\left(1-\frac{u_n}{u_\ell}\right)^{-1}\ \frac{u_\ell}{u_n}\right] = \frac{N-3}{2}\ u_n\ ; \tag{6.62b}$$

$$\sum_{\ell=1,\ \ell\neq n}^{N}\left[\left(1-\frac{u_n}{u_\ell}\right)^{-1}\ u_\ell^2\right] = u_n^2\sum_{\ell=1,\ \ell\neq n}^{N}\left[\left(\frac{u_n}{u_\ell}\right)^{-2}\ \left(1-\frac{u_n}{u_\ell}\right)^{-1}\right]$$
$$= u_n^2\sum_{\ell=1}^{N-1}\left[u_\ell^{-2}\ (1-u_\ell)^{-1}\right] = u_n^2\sum_{\ell=1}^{N-1}\left\{u_\ell^{-1}\left[u_\ell^{-1}+(1-u_\ell)^{-1}\right]\right\}$$
$$= u_n^2\left(-1+\frac{N-3}{2}\right) = \left(\frac{N-5}{2}\right)\ u_n^2\ . \tag{6.63}$$

$$\sum_{\ell=1,\ell\neq n}^{N}(1-\frac{u_\ell}{u_n})^{-1} = -u_n\sum_{\ell=1,\ell\neq n}^{N}u_\ell^{-1}(1-\frac{u_n}{u_\ell})^{-1} = \frac{N-1}{2}\ ; \tag{6.64a}$$

$$\sum_{\ell=1,\ell\neq n}^{N}u_\ell^{-2}(1-\frac{u_n}{u_\ell})^{-1} = \frac{1}{u_n^2}\sum_{\ell=1,\ell\neq n}^{N}(\frac{u_n}{u_\ell})^2(1-\frac{u_n}{u_\ell})^{-1}$$
$$= \frac{1}{u_n^2}\sum_{\ell=1,\ell\neq n}^{N}u_\ell^{-2}[1-(1-u_\ell)^{-1}] = -\frac{N-3}{2u_n^2}\ ; \tag{6.64b}$$

$$\sum_{\ell=1,\ell\neq n}^{N}u_\ell^{-1}(1-\frac{u_\ell}{u_n})^{-1} = -u_n\sum_{\ell=1,\ell\neq n}^{N}u_\ell^{-2}(1-\frac{u_n}{u_\ell})^{-1} = \frac{N-3}{2u_n}; \tag{6.64c}$$

$$\sum_{\ell=1,\ell\neq n}^{N}u_\ell(1-\frac{u_\ell}{u_n})^{-1} = -u_n\sum_{\ell=1,\ell\neq n}^{N}(1-\frac{u_\ell}{u_n})^{-1} = -\frac{N-1}{2}u_n; \tag{6.64d}$$

$$\sum_{\ell=1,\ell\neq n}^{N}u_\ell^2(1-\frac{u_\ell}{u_n})^{-1} = -u_n\sum_{\ell=1,\ell\neq n}^{N}u_\ell(1-\frac{u_n}{u_\ell})^{-1} = -\frac{N-3}{2}u_n^2\ . \tag{6.64e}$$

$$w_n - w_\ell = -\gamma\, w_n\, w_\ell\ (u_n-u_\ell)$$
$$= -\gamma\left(u_n^{-1}+\gamma+\gamma\,\frac{u_\ell}{u_n}+\gamma^2\,u_\ell\right)^{-1}\ \left(1-\frac{u_\ell}{u_n}\right)\ , \tag{6.65a}$$

$$(w_n - w_\ell)^{-1} = -\gamma^{-1} \left(u_n^{-1} + \gamma + \gamma \frac{u_\ell}{u_n} + \gamma^2 u_\ell \right) \left(1 - \frac{u_\ell}{u_n} \right)^{-1} ; \tag{6.65b}$$

$$\sigma_n^{(1)}(N;\gamma) \equiv \sum_{\ell=1,\ \ell\neq n}^{N} (w_n - w_\ell)^{-1} = -\frac{N-1}{2\,\gamma\, u_n} \left(1 - \gamma^2 u_n^2 \right) . \tag{6.66}$$

$$\sigma_n^{(2)}(N;\gamma) \equiv \sum_{\ell=1,\ \ell\neq n}^{N} (w_n - w_\ell)^{-2} . \tag{6.67}$$

$$\begin{aligned}
\left[\sum_{\ell=1,\ \ell\neq n}^{N} (w_n - w_\ell)^{-1} \right]^2 &= \left[\sigma_n^{(1)}(N;\gamma) \right]^2 \\
&= \sum_{\ell=1,\ \ell\neq n}^{N} (w_n - w_\ell)^{-1} \sum_{\ell'=1,\ \ell'\neq n}^{N} (w_n - w_{\ell'})^{-1} \\
&= \sigma_n^{(2)}(N;\gamma) + \sum_{\ell=1,\ \ell'=1,\ \ell\neq n,\ \ell'\neq n,\ \ \ell'\neq \ell,}^{N} (w_n - w_\ell)^{-1} (w_n - w_{\ell'})^{-1} \\
&= \sigma_n^{(2)}(N;\gamma) \\
&+ \sum_{\ell=1,\ \ell'=1,\ \ell\neq n,\ \ell'\neq n,\ \ \ell'\neq \ell,}^{N} \left\{ \left[(w_n - w_\ell)^{-1} - (w_n - w_{\ell'})^{-1} \right] \ (w_\ell - w_{\ell'})^{-1} \right\} \\
&= \sigma_n^{(2)}(N;\gamma) + 2 \sum_{\ell=1,\ \ell'=1,\ \ell\neq n,\ \ell'\neq n,\ \ell'\neq \ell,}^{N} \left[(w_n - w_\ell)^{-1} \ (w_\ell - w_{\ell'})^{-1} \right] \\
&= 3\ \sigma_n^{(2)}(N;\gamma) + 2 \sum_{\ell=1,\ \ell'=1,\ \ell\neq n,\ \ell'\neq \ell,}^{N} \left[(w_n - w_\ell)^{-1} \ (w_\ell - w_{\ell'})^{-1} \right] ;
\end{aligned} \tag{6.68a}$$

$$\sigma_n^{(2)}(N;\gamma) = \frac{1}{3} \left[\sigma_n^{(1)}(N;\gamma) \right]^2 - \frac{2}{3} \sum_{\ell=1,\ \ell\neq n}^{N} \left[(w_n - w_\ell)^{-1} \ \sigma_\ell^{(1)}(N;\gamma) \right] ; \tag{6.68b}$$

$$\sigma_n^{(2)}(N;\gamma) = \frac{1}{3} \left[\sigma_n^{(1)}(N;\gamma) \right]^2 + \frac{N-1}{3\,\gamma} \sum_{\ell=1,\ \ell\neq n}^{N} \left[(w_n - w_\ell)^{-1} \ \left(u_\ell^{-1} - \gamma^2 u_\ell \right) \right] ; \tag{6.68c}$$

$$\begin{aligned}
&(w_n - w_\ell)^{-1} \ \left(u_\ell^{-1} - \gamma^2 u_\ell \right) \\
&= -\gamma^{-1} \left(u_n^{-1} + \gamma + \gamma \frac{u_\ell}{u_n} + \gamma^2 u_\ell \right) \left(u_\ell^{-1} - \gamma^2 u_\ell \right) \left(1 - \frac{u_\ell}{u_n} \right)^{-1} \\
&= \left(u_n^{-1} + \gamma \right) \left[-1 - (\gamma\, u_\ell)^{-1} + \gamma\, u_\ell \ (1 + \gamma\, u_\ell) \right] \left(1 - \frac{u_\ell}{u_n} \right)^{-1}
\end{aligned} \tag{6.69}$$

$$\sigma_n^{(2)}(N;\gamma) = -\frac{(N-1)\ (1+\gamma\, u_n)^2}{12\ (\gamma\, u_n)^2} \cdot$$
$$\cdot \left[(N-5)\ \left(1+\gamma^2\, u_n^2\right) + 2\,(N+1)\ \gamma\, u_n\right]\ . \tag{6.70}$$

References

[1] F. Calogero, "A solvable many-body problem, its equilibria, and a second-order ordinary differential equation whose general solution is polynomial", J. Math. Phys. **54**, 012703 (13 pages) (2013).

[2] F. Calogero, *Isochronous Systems*, Oxford University Press, Oxford, 2008 (a paperback version with some misprints corrected and an update of relevant references has been published in September 2012).

[3] F. Calogero and Ge Yi, "Can the general solution of the second-order ODE characterizing Jacobi polynomials be polynomial?", J. Phys. A: Math. Theor. **45**, 095206 (2012) (4 pages). doi: 10.1088/1751-8113/45/9/095206.

[4] F. Calogero, *Classical many-body problems amenable to exact treatments*, Lecture Notes in Physics Monographs **m66**, Springer, Berlin, 2001.

[5] D. Gómez-Ullate and M. Sommacal, "Periods of the goldfish many-body problem", J. Nonlinear Math. Phys. **12** Suppl. **1**, 351–362 (2005).

[6] *Higher Transcendental Functions* (main editor: A. Erdélyi), vol. **2**, McGraw-Hill, New York, 1953.

[7] S. Ahmed, M. Bruschi, F. Calogero, M. A. Olshanetsky, A. M. Perelomov, "Properties of the zeros of the classical polynomials, and of the Bessel functions", Nuovo Cimento **49**, 173–199 (1979).

[8] F. Calogero and G. Yi, "Diophantine properties of the zeros of certain Laguerre and para-Jacobi polynomials", J. Phys. A: Math. Theor. **45**, 095207 (2012) (9 pages). doi: 10.1088/1751-8113/45/9/095207.

[9] F. Calogero and G. Yi, "Polynomials satisfying functional and differential equations and Diophantine properties of their zeros", Lett. Math. Phys. DOI 10.1007/s11005-013-0612-y.