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Nonlinear Self-Adjointness and Conservation Laws for the Hyperbolic Geometric Flow Equation

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We study the nonlinear self-adjointness of a class of quasilinear 2D second order evolution equations by applying the method of Ibragimov. Which enables one to establish the conservation laws for any differential equation. We first obtain conditions determining the self-adjointness for a sub-class in the general case. Then, we establish the conservation laws for hyperbolic geometric flow equation on Riemann surfaces.

Keywords: Nonlinear self-adjointness; conservation laws; hyperbolic geometric flow equation.

2000 Mathematics Subject Classification: 22E46, 53C35, 57S20

1. Introduction

Ibragimov recently introduced the concept of nonlinear self-adjointness and developed a technique for the determination of conservation laws for differential equations, [5–11]. Motivated by such work we propose to the study of self-adjointness and conservation laws for the following quasilinear 2D generalized second evolution equation

$$u_{tt} = Au_{xy} + Bu_xu_y + Cu_{xx} + Du_{yy} + Eu_y + Fu_x + Pu_x^2 + Qu_y^2 + G + Hu_t + Iu_t^2, \quad (1.1)$$

where the coefficients $A, B, C, D, E, F, P, Q, G, H, I$ are functions of the independent variables t, x, y and the dependent variable $u = u(t, x, y)$. Among such equations we consider the hyperbolic geometric flow equation in isothermic coordinates x, y on a Riemann surfaces, which has been introduced by Kong and Liu [12] to study the wave character of the Riemannian metric and curvature, the Ricci and the scalar curvatures, as singularities, existence and regularities of the flow solutions. In particular, this equation can be seen as the Einstein's hyperbolic geometric flow in vacuo. Here we consider an equation that generalizes the 2D hyperbolic flow equation. There are lot of nonlinear equations of type (1.1) arising in physics, chemistry and biology, [11].

In a recent paper we apply the method the Ibragimov to a class of evolution equations of type

$$Ru_t = Au_{xy} + Bu_xu_y + Cu_{xx} + Du_{yy} + Eu_y + Fu_x + Pu_x^2 + Qu_y^2 + G,$$

where the coefficients $A, B, C, D, E, F, P, Q, G$ and $R \neq 0$ are functions of the independent variables t, x, y and the dependent variable $u = u(t, x, y)$. We have established conditions determining the self-adjointness and the conservation laws for important Ricci flow equation, the modified Ricci flow

equation and the nonlinear heat equation 2D, [1]. So the present paper is a natural generalization of that.

Firstly we investigate the nonlinear self-adjointness of the general evolution equation (1.1) which is the most important point of applying the Ibragimov's method. Secondly we establish the conservation laws for the nonlinear self-adjoint (modified) hyperbolic geometric flow equation.

This paper is organized as follows. In Section 2 we present the Ibragimov's method, the important concept of nonlinear self-adjointness and conservation laws for a scalar equation of second order. In Section 3 we state the main result in this work. In Section 4 we establish the conservation laws for (modified) hyperbolic geometric flow equation. Finally, the proof of the main theorem is presented in Appendix.

2. The Ibragimov's Method

In this section we develop the Ibragimov's theory for a single second order partial differential equation in the independent variables t, x and y , dependent variable $u = u(t, x, y)$. For a general equation the basic literature recommended are the papers [4–11].

We define a differential function as [5, 6, 8, 11].

Definition 2.1. Let t, x, y be the independent variables, $u = u(t, x, y)$ the dependent variable, $u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$, etc., its partial derivatives. A function \mathcal{F} in the variable $t, x, y, u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$, etc., is called a differential function if it is locally analytic, i.e., if it admit locally a Taylor series expansion.

The concept of formal Lagrangian is introduced according to [5, 10, 11].

Definition 2.2. Let \mathcal{F} be a differential function in the variable $t, x, y, u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$, etc., and $v = v(t, x, y)$ is the new dependent variable, known as the adjoint variable or nonlocal variable [5], the formal Lagrangian function for the \mathcal{F} is the differential function in the variable $t, x, y, u, v, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$, etc., defined by

$$\mathcal{L} := v\mathcal{F}. \quad (2.1)$$

Definition 2.3. Let \mathcal{F} be a differential function and the differential equation

$$\mathcal{F}(t, x, y, u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0, \quad (2.2)$$

denoted by $\mathcal{F}[u] = 0$, we define the adjoint differential function to \mathcal{F} by

$$\mathcal{F}^* := \frac{\delta \mathcal{L}}{\delta u}, \quad (2.3)$$

and the adjoint differential equation by

$$\mathcal{F}^*(t, x, y, u, v, u_t, v_t, u_x, v_x, u_y, v_y, u_{xx}, v_{xx}, u_{xy}, v_{xy}, u_{yy}, v_{yy}, \dots) = 0, \quad (2.4)$$

denoted by $\mathcal{F}^*[u, v] = 0$, where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \mathcal{D}_t \frac{\partial}{\partial u_t} - \mathcal{D}_x \frac{\partial}{\partial u_x} - \mathcal{D}_y \frac{\partial}{\partial u_y} + \mathcal{D}_x \mathcal{D}_y \frac{\partial}{\partial u_{xy}} + \mathcal{D}_t^2 \frac{\partial}{\partial u_{tt}} + \mathcal{D}_x^2 \frac{\partial}{\partial u_{xx}} + \mathcal{D}_y^2 \frac{\partial}{\partial u_{yy}} - \dots \quad (2.5)$$

is the Euler-Lagrange operator, and

$$\begin{aligned}\mathcal{D}_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + v_{tt} \frac{\partial}{\partial v_t} + u_{tx} \frac{\partial}{\partial u_x} + v_{tx} \frac{\partial}{\partial v_x} + u_{ty} \frac{\partial}{\partial u_y} + \dots, \\ \mathcal{D}_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v} + u_{tx} \frac{\partial}{\partial u_t} + v_{tx} \frac{\partial}{\partial v_t} + u_{xx} \frac{\partial}{\partial u_x} + v_{xx} \frac{\partial}{\partial v_x} + u_{xy} \frac{\partial}{\partial u_y} + \dots, \\ \mathcal{D}_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial v} + u_{ty} \frac{\partial}{\partial u_t} + v_{ty} \frac{\partial}{\partial v_t} + u_{xy} \frac{\partial}{\partial u_x} + v_{xy} \frac{\partial}{\partial v_x} + u_{yy} \frac{\partial}{\partial u_y} + \dots,\end{aligned}$$

are the total derivative operator with respect to t , x and y , respectively.

Definition 2.4. The differential equation (2.2) is said to be *nonlinearly self-adjoint* if there exists a function

$$v = \varphi(t, x, y, u) \neq 0 \tag{2.6}$$

such that it satisfy

$$\mathcal{F}^*|_{v=\varphi(t,x,y,u)} = \lambda \mathcal{F} \tag{2.7}$$

for some undetermined coefficient $\lambda = \lambda(t, x, y, u, \dots)$. If $v = \varphi(u)$ in (2.6) and (2.7), the equation (2.1) is called *quasi self-adjoint*. If $v = u$, we say that the equation (2.1) is *strictly self-adjoint*.

Definition 2.5. A conservation law for equation $\mathcal{F}[u] = 0$ is the total divergence

$$Div(C) = \mathcal{D}_t C^1 + \mathcal{D}_x C^2 + \mathcal{D}_y C^3,$$

for some vectorial differential function $C[u] = (C^1, C^2, C^3)$, so-called conserved vector, such that the divergence vanishes for all solutions $u = f(t, x, y)$ for equation $\mathcal{F}[u] = 0$. A conservation law is said to be nonlocal conservation law if the conserved vector depends on the solutions v of the adjoint equation (2.4).

Now we state the New Conservation Theorem (or Theorem on nonlocal conservation laws, [8]) for the differential equation (2.4).

Theorem 2.1. *Every Lie point and Lie-Bäcklund symmetry*

$$X = \tau(t, x, y, u, \dots) \frac{\partial}{\partial t} + \xi(t, x, y, u, \dots) \frac{\partial}{\partial x} + \eta(t, x, y, u, \dots) \frac{\partial}{\partial y} + \phi(t, x, y, u, \dots) \frac{\partial}{\partial u},$$

as well as nonlocal symmetry, of the differential equation $\mathcal{F}[u] = 0$, provides a nonlocal conservation law

$$Div(C) = 0 \quad \text{for} \quad \mathcal{F}[u] = 0 \quad \text{and} \quad \mathcal{F}^*[u, v] = 0.$$

Let $W = \phi - \tau u_t - \xi u_x - \eta u_y$ be the Lie characteristic, then the components of the conserved vector, $C[u] = (C^1, C^2, C^3)$, are given by

$$\begin{aligned}C^1 - \tau \mathcal{L} &= W \left(\frac{\partial \mathcal{L}}{\partial u_t} - \mathcal{D}_t \frac{\partial \mathcal{L}}{\partial u_{tt}} \right) + \mathcal{D}_t W \frac{\partial \mathcal{L}}{\partial u_{tt}}, \\ C^2 - \xi \mathcal{L} &= W \left(\frac{\partial \mathcal{L}}{\partial u_x} - \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xx}} - \mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_{xy}} \right) + \mathcal{D}_x W \frac{\partial \mathcal{L}}{\partial u_{xx}} + \mathcal{D}_y W \frac{\partial \mathcal{L}}{\partial u_{xy}}, \\ C^3 - \eta \mathcal{L} &= W \left(\frac{\partial \mathcal{L}}{\partial u_y} - \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{yx}} - \mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_{yy}} \right) + \mathcal{D}_x W \frac{\partial \mathcal{L}}{\partial u_{yx}} + \mathcal{D}_y W \frac{\partial \mathcal{L}}{\partial u_{yy}}.\end{aligned}$$

We note that \mathcal{L} vanishes on the solutions of the equation (2.4), then the term $\tau\mathcal{L}$, $\xi\mathcal{L}$, and $\eta\mathcal{L}$ may be omitted in the conserved vector.

3. The Main 2D Generalized Quasilinear Equation

In this section we apply to the equation (1.1) the Ibragimov's method. For this purpose we write this equation in the form (2.1), where

$$\mathcal{F} := u_{tt} - Au_{xy} - Bu_xu_y - Cu_{xx} - Du_{yy} - Eu_y - Fu_x - Pu_x^2 - Qu_y^2 - G - Hu_t - Iu_t^2. \quad (3.1)$$

Then the corresponding formal Lagrangian (2.2) is given by

$$\mathcal{L} = v(u_{tt} - Au_{xy} - Bu_xu_y - Cu_{xx} - Du_{yy} - Eu_y - Fu_x - Pu_x^2 - Qu_y^2 - G - Hu_t - Iu_t^2) \quad (3.2)$$

and the Euler-Lagrange operator (2.5) assumes the following form:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \mathcal{D}_t \frac{\partial}{\partial u_t} - \mathcal{D}_x \frac{\partial}{\partial u_x} - \mathcal{D}_y \frac{\partial}{\partial u_y} + \mathcal{D}_t^2 \frac{\partial}{\partial u_{tt}} + \mathcal{D}_x^2 \frac{\partial}{\partial u_{xx}} + \mathcal{D}_x \mathcal{D}_y \frac{\partial}{\partial u_{xy}} + \mathcal{D}_y^2 \frac{\partial}{\partial u_{yy}} + \dots \quad (3.3)$$

Now we apply the Euler-Lagrange operator (3.3) to \mathcal{L} determined by (3.2) and after some work we obtain that the adjoint equation (2.4) to (1.1) (see Proposition 5.2 in Appendix) reads:

$$\begin{aligned} \mathcal{F}^* = & v_{tt} - Av_{xy} - Cv_{xx} - Dv_{yy} + (H + 2Iu_t)v_t \\ & + [(B - Au)u_y + 2(P - Cu)u_x + F - Ay - 2Cx]v_x \\ & + [(B - Au)u_x + 2(Q - Du)u_y + E - Ax - 2Dy]v_y \\ & + [2Iu_{tt} + 2(B - Au)u_{xy} + (Bu - A_{uu})u_xu_y + 2(P - Cu)u_{xx} \\ & + 2(Q - Du)u_{yy} + (B_x - A_{xu} + 2(Q_y - D_{yu}))u_y \\ & + (B_y - A_{yu} + 2(P_x - C_{xu}))u_x + (P_u - C_{uu})u_x^2 \\ & + (Q_u - D_{uu})u_y^2 - A_{xy} - C_{xx} - D_{yy} \\ & + E_y + F_x + H_t - G_u + 2I_tu_t + I_uu_t^2]v = 0. \end{aligned} \quad (3.4)$$

Therefore we get the main result in the present paper, which can be stated as follows.

Theorem 3.1. *The eq. (1.1) is nonlinearly self-adjoint if and only if there exists a two time differentiable function $\varphi = \varphi(t, x, y, u) \neq 0$ such that its coefficients satisfies the following relations:*

$$\varphi_u + \varphi I = 0, \quad (3.5)$$

if at least the function $A_1, B_1, C_1, D_1, E_1, F_1$, defined below, is nonzero,

$$\varphi_{tu} + (\varphi I)_t + H(\varphi_u + \varphi I) = 0, \quad (3.6)$$

$$((\varphi A)_u - \varphi B)_y + 2((\varphi C)_u - \varphi P)_x - 2F(\varphi_u + \varphi I) = 0, \quad (3.7)$$

$$((\varphi A)_u - \varphi B)_x + 2((\varphi D)_u - \varphi Q)_y - 2E(\varphi_u + \varphi I) = 0, \quad (3.8)$$

$$\begin{aligned} \varphi_{tt} - (\varphi A)_{xy} - (\varphi C)_{xx} - (\varphi D)_{yy} + (\varphi E)_y + (\varphi F)_x + (\varphi H)_t + \varphi_u G = \\ \varphi(G_u - 2GI), \end{aligned} \quad (3.9)$$

where $A, B, C, D, E, F, P, Q, G, H$ and I are function of t, x, y, u , and the functions $A_1, B_1, C_1, D_1, E_1, F_1$ are given by

$$A_1 := A_u, B_1 := DC_u - CD_u, C_1 := C_u,$$

$$D_1 := D_u, E_1 := AC_u - CA_u, F_1 := AD_u - DA_u.$$

Further, the coefficient functions satisfy the following conditions:

$$A_1 = AI + B, \tag{3.10}$$

$$B_1 = DP - CQ, \tag{3.11}$$

$$C_1 = CI + P, \tag{3.12}$$

$$D_1 = DI + Q, \tag{3.13}$$

$$E_1 = AP - CB, \tag{3.14}$$

$$F_1 = AQ - DB. \tag{3.15}$$

The quasi-self-adjointness and the strict self-adjointness of the studied equations are direct consequence of Theorem 3.1. Here we will apply the Theorem 3.1 to modified hyperbolic geometric flow equation, in particular, to hyperbolic geometric flow equation.

4. The modified hyperbolic geometric flow equation

In this section we apply the Theorem 3.1 to the equation

$$u_{tt} = e^{\lambda u} (u_{xx} + u_{yy}) + \lambda u_t^2, \tag{4.1}$$

where λ is a nonzero constant, and establish the corresponding conservation laws.

This equation is a natural generalization of the hyperbolic geometric flow equation, where $\lambda = -1$, and was suggested by professors Yuri Bozhkov and Igor Freire in [2].

The eq. (4.1) is of type (1.1) with

$$C = D = e^{\lambda u}, \quad I = \lambda, \quad A = B = E = F = G = P = Q = H = 0.$$

A first corollary of the Theorem 3.1 is the following:

Corollary 4.1. *The equation*

$$u_{tt} = f(u) (u_{xx} + u_{yy}) + \lambda u_t^2,$$

where $f = f(u)$ and λ do not vanish, is nonlinearly self-adjoint if and only if $f(u) = a e^{\lambda u}$ where a is nonzero constant.

Proof. According the Theorem 3.1 we have

$$\begin{aligned} A_1 &:= RA_u - AR_u = 0, \\ B_1 &:= DC_u - CD_u = 0, \\ C_1 &:= RC_u - CR_u = f'(u), \\ D_1 &:= RD_u - DR_u = C_1, \\ E_1 &:= AC_u - CA_u = 0, \\ F_1 &:= AD_u - DA_u = 0. \end{aligned}$$

Further, the coefficient functions must satisfy the following relations:

$$AI + RB = 0 = A_1, \quad DP - CQ = 0 = B_1, \quad CI + RP = \lambda f(u) = C_1,$$

$$AP - CB = 0 = E_1, \quad AQ - DB = 0 = F_1.$$

So we have $f'(u) = \lambda f(u)$, i.e., $f(u) = ae^{\lambda u}$ where a is a constant. As $C_1 \neq 0$ from eq. (3.5) we obtain

$$\varphi_u = -\lambda \varphi, \quad \text{i.e.,} \quad \varphi = \varphi^1 e^{-\lambda u},$$

where $\varphi^1 = \varphi^1(t, x, y)$ is any nonvanishing function.

The eqs. (3.6, 3.7, 3.8) are satisfied trivially by φ . From eq. (3.9) we have

$$0 = \varphi_{tt} - \varphi_{xx} f(u) - \varphi_{yy} f(u) = \varphi_{tt}^1 e^{-\lambda u} - a(\varphi_{xx}^1 + \varphi_{yy}^1),$$

this is,

$$0 = \varphi_{tt}^1 = \varphi_{xx}^1 + \varphi_{yy}^1.$$

Thus, $\varphi = (\alpha t + \beta) e^{-\lambda u}$ where $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are any harmonic functions both nonvanishing.

Therefore, the eq. $u_{tt} = f(u)(u_{xx} + u_{yy}) + \lambda u_t^2$ is nonlinearly self-adjoint if and only if $f(u) = ae^{\lambda u}$ where a is nonzero constant. \square

Then we calculate some conservation laws for eq. (4.1). Its Lie symmetry are computed in Proposition 5.3 of Appendix, in terms of two functions satisfying the Cauchy-Riemann equations and the general infinitesimal generator of symmetries is given by

$$X = (c_1 + c_4 t) \frac{\partial}{\partial t} + \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \frac{2}{\lambda} (\xi_x - c_4) \frac{\partial}{\partial u},$$

where c_1, c_4 are arbitrary constants, $\xi_x = \eta_y$ and $\eta_x = -\xi_y$.

We consider the following sub-algebra with infinitesimal generators of symmetries given by:

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{\partial}{\partial y}, \quad V_4 = t \frac{\partial}{\partial t} - \frac{2}{\lambda} \frac{\partial}{\partial u},$$

$$V_5 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{2}{\lambda} \frac{\partial}{\partial u}, \quad V_6 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

Let be the general generator

$$V = (c_1 + c_4 t) \frac{\partial}{\partial t} + (c_2 + c_5 x + c_6 y) \frac{\partial}{\partial x} + (c_3 - c_6 x + c_5 y) \frac{\partial}{\partial y} + \frac{2}{\lambda} (c_5 - c_4) \frac{\partial}{\partial u},$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

Consider the adjoint variable $v = (\alpha t + \beta) e^{-\lambda u}$, where $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ are any harmonic functions both nonvanishing, the formal Lagrangian is given by

$$\mathcal{L} = (\alpha t + \beta) e^{-\lambda u} (u_{tt} - e^{\lambda u} (u_{xx} + u_{yy}) - \lambda u_t^2),$$

that is,

$$\mathcal{L} = (\alpha t + \beta) (e^{-\lambda u} (u_{tt} - \lambda u_t^2) - u_{xx} - u_{yy}).$$

Now we apply to the conserved vectors given by New Conservation Theorem (Theorem 2.1), $C = (C^1, C^2, C^3)$, the simplifying operation described on pp. 50-51 of [11], and we obtain the following components:

$$C^1 = \left[\frac{2}{\lambda} (c_4 \alpha + \alpha_x (c_2 + c_5 x + c_6 y)) - [\alpha_x (c_2 + c_5 x + c_6 y) + \alpha_y (c_3 - c_6 x + c_5 y)] u \right] e^{-\lambda u}$$

$$+ e^{-\lambda u} \{ -2 \alpha c_5 u + [(\alpha_x t + \beta_x) (c_2 + c_5 x + c_6 y) + (\alpha_y t + \beta_y) (c_3 - c_6 x + c_5 y)]$$

$$+ \alpha (c_1 + c_4 t) + (\alpha t + \beta) c_4 \} u_t + (\lambda u - 2) \alpha (c_2 + c_5 x + c_6 y) u_x$$

$$+ \lambda u \alpha (c_3 - c_6 x + c_5 y) u_y \},$$

$$C^2 = \left(\frac{2}{\lambda} - u \right) a (\alpha_{yy} t + \beta_{yy}) (c_2 + c_5 x + c_6 y) + a (\alpha_x t + \beta_x) \left(\frac{2}{\lambda} (c_5 - c_4) + (c_5 + c_4) u \right)$$

$$+ a [(c_1 + c_4 t) \alpha_x + (\alpha_{xy} t + \beta_{xy}) (c_3 - c_6 x + c_5 y) - (\alpha_y t + \beta_y) c_6] u$$

$$- a [(\alpha_x t + \beta_x) (c_2 + c_5 x + c_6 y) + (\alpha_y t + \beta_y) (c_3 - c_6 x + c_5 y) + \alpha (c_1 + c_4 t)$$

$$+ (\alpha t + \beta) c_4] u_x - (\lambda u - 2) \alpha (c_2 + c_5 x + c_6 y) e^{-\lambda u} u_t,$$

$$C^3 = \left(u - \frac{2}{\lambda} \right) a (\alpha_{xy} t + \beta_{xy}) (c_2 + c_5 x + c_6 y) + a (\alpha_y t + \beta_y) \left(-\frac{2}{\lambda} c_4 + (c_5 + c_4) u \right)$$

$$+ \left(u + \frac{2}{\lambda} \right) a (\alpha_x t + \beta_x) c_6 + a [(c_1 + c_4 t) \alpha_y - (\alpha_{xx} t + \beta_{xx}) (c_3 - c_6 x + c_5 y)] u$$

$$- a [(\alpha_x t + \beta_x) (c_2 + c_5 x + c_6 y) + (\alpha_y t + \beta_y) (c_3 - c_6 x + c_5 y) + \alpha (c_1 + c_4 t)$$

$$+ (\alpha t + \beta) c_4] u_y - \lambda u \alpha (c_3 - c_6 x + c_5 y) e^{-\lambda u} u_t.$$

Therefore we prove the first corollary of Theorem 2.1:

Corollary 4.2. Let V be an infinitesimal generator of Lie symmetry for the eq.(4.1), give by

$$V = (c_1 + c_4 t) \frac{\partial}{\partial t} + (c_2 + c_5 x + c_6 y) \frac{\partial}{\partial x} + (c_3 - c_6 x + c_5 y) \frac{\partial}{\partial y} + \frac{2}{\lambda} (c_5 - c_4) \frac{\partial}{\partial u},$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants, λ is a nonzero constant, and let $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ be harmonic functions both nonvanishing, then the corresponding nonlocal conserved vectors are the following:

- For generator V_1 , the components of the conserved vector $C = (C^1, C^2, C^3)$, are given by

$$C^1 = e^{-\lambda u} \alpha u_t, \quad C^2 = \alpha_x u - \alpha u_x, \quad C^3 = \alpha_y u - \alpha u_y.$$

- For generator V_2 , the components of the conserved vector $C = (C^1, C^2, C^3)$, are given by

$$\begin{aligned} C^1 &= \left[\frac{2}{\lambda} \alpha_x - \alpha_x u + (\alpha_x t + \beta_x) u_t + (\lambda u - 2) \alpha u_x \right] e^{-\lambda u}, \\ C^2 &= \left(\frac{2}{\lambda} - u \right) (\alpha_{yy} t + \beta_{yy}) - (\alpha_x t + \beta_x) u_x - (\lambda u - 2) \alpha e^{-\lambda u} u_t, \\ C^3 &= \left(u - \frac{2}{\lambda} \right) (\alpha_{xy} t + \beta_{xy}) - (\alpha_x t + \beta_x) u_y. \end{aligned}$$

- For generator V_3 , the components of the conserved vector $C = (C^1, C^2, C^3)$, are given by

$$\begin{aligned} C^1 &= [-\alpha_y u + (\alpha_y t + \beta_y) u_t + \lambda u \alpha u_y] e^{-\lambda u}, \\ C^2 &= (\alpha_{xy} t + \beta_{xy}) u - (\alpha_y t + \beta_y) u_x, \\ C^3 &= -(\alpha_{xx} t + \beta_{xx}) u - (\alpha_y t + \beta_y) u_y - \lambda u \alpha e^{-\lambda u} u_t. \end{aligned}$$

- For generator V_4 , the components of the conserved vector $C = (C^1, C^2, C^3)$, are given by

$$\begin{aligned} C^1 &= \left[\frac{2}{\lambda} \alpha u + (2 \alpha t + \beta) u_t \right] e^{-\lambda u}, \\ C^2 &= \left(u - \frac{2}{\lambda} \right) (\alpha_x t + \beta_x) + t \alpha_x u - (2 \alpha t + \beta) u_x, \\ C^3 &= \left(u - \frac{2}{\lambda} \right) (\alpha_y t + \beta_y) + t \alpha_y u - (2 \alpha t + \beta) u_y. \end{aligned}$$

- For generator V_5 , the components of the conserved vector $C = (C^1, C^2, C^3)$, are given by

$$\begin{aligned} C^1 &= \left[\frac{2}{\lambda} \alpha_x x - (\alpha_x x + \alpha_y y + 2 \alpha) u + [(\alpha_x t + \beta_x) x + (\alpha_y t + \beta_y) y] u_t \right. \\ &\quad \left. + (\lambda u - 2) \alpha_x u_x + \lambda u \alpha_y u_y \right] e^{-\lambda u}, \\ C^2 &= \left(\frac{2}{\lambda} - u \right) (\alpha_{yy} t + \beta_{yy}) x + \left(\frac{2}{\lambda} + u \right) (\alpha_x t + \beta_x) \\ &\quad + (\alpha_{xy} t + \beta_{xy}) y u - [(\alpha_x t + \beta_x) x + (\alpha_y t + \beta_y) y] u_x - (\lambda u - 2) \alpha_x e^{-\lambda u} u_t, \\ C^3 &= \left(u - \frac{2}{\lambda} \right) (\alpha_{xy} t + \beta_{xy}) x + [\alpha_y t + \beta_y - (\alpha_{xx} t + \beta_{xx}) y] u \\ &\quad - [(\alpha_x t + \beta_x) x + (\alpha_y t + \beta_y) y] u_y - \lambda u \alpha_y e^{-\lambda u} u_t. \end{aligned}$$

- For generator V_6 , the components of the conserved vector $C = (C^1, C^2, C^3)$, are given by

$$\begin{aligned} C^1 &= \left[\frac{2}{\lambda} \alpha_x y - (\alpha_x y - \alpha_y x) u + [(\alpha_x t + \beta_x) y - (\alpha_y t + \beta_y) x] u_t \right. \\ &\quad \left. + (\lambda u - 2) \alpha_y u_x - \lambda u \alpha_x u_y \right] e^{-\lambda u}, \\ C^2 &= \left(\frac{2}{\lambda} - u \right) (\alpha_{yy} t + \beta_{yy}) y - [(\alpha_{xy} t + \beta_{xy}) x + \alpha_y t + \beta_y] u \\ &\quad - [(\alpha_x t + \beta_x) y - (\alpha_y t + \beta_y) x] u_x - (\lambda u - 2) \alpha_y e^{-\lambda u} u_t, \\ C^3 &= \left(u - \frac{2}{\lambda} \right) (\alpha_{xy} t + \beta_{xy}) y + \left(\frac{2}{\lambda} + u \right) (\alpha_x t + \beta_x) \\ &\quad + (\alpha_{xx} t + \beta_{xx}) x u - [(\alpha_x t + \beta_x) y - (\alpha_y t + \beta_y) x] u_y + \lambda u \alpha_x e^{-\lambda u} u_t. \end{aligned}$$

We note that local conservation laws can be obtained taking $\alpha = 0$ and β a nonzero constant. A second corollary of the Theorem 3.1 is the follows:

Corollary 4.3. *The hyperbolic geometric flow equation, $u_{tt} = e^{-u} (u_{xx} + u_{yy}) - u_t^2$, is quasi-self-adjoint.*

Proof. Follow from Corollary 4.1 with $f(u) = e^{-u}$, $\lambda = -1$, $\alpha = 0$ and β a nonzero constant. \square

A second corollary of the Theorem 2.1 is the follows:

Corollary 4.4. *Let X be the general infinitesimal generator of Lie symmetry for the hyperbolic geometric flow equation, give by*

$$X = (c_1 + c_4 t) \frac{\partial}{\partial t} + \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} - 2(\xi_x - c_4) \frac{\partial}{\partial u},$$

where c_1, c_4 are arbitrary constants, $\xi_x = \eta_y$, $\eta_x = -\xi_y$, then the corresponding nonlocal conserved vector $C = (C^1, C^2, C^3)$ is give by

$$C^1 = e^u u_t, \quad C^2 = -u_x, \quad C^3 = -u_y.$$

Proof. The general infinitesimal generator X is obtained in Proposition 5.3. Consider the adjoint variable $v = \beta e^u$, where β is a nonzero constant, then we obtain the following components for the conserved vectors:

$$\begin{aligned} C^1 &= \beta e^u \{ [2(c_4 - \xi_x) - (c_1 + c_4 t) u_t - \xi u_x - \eta u_y] u_t \\ &\quad - c_4 u_t - (c_1 + c_4 t) (e^{-u} (u_{xx} + u_{yy}) - u_t^2) - \xi u_{tx} - \eta u_{ty} \}, \\ C^2 &= \beta [2 \xi_{xx} + (c_1 + c_4 t) u_{xt} + \xi_x u_x + \eta_x u_y + \xi u_{xx} + \eta u_{xy}], \\ C^3 &= \beta [2 \xi_{yy} + (c_1 + c_4 t) u_{yt} + \xi_y u_x + \eta_y u_y + \xi u_{yx} + \eta u_{yy}]. \end{aligned}$$

Now apply to the conserved vectors $C = (C^1, C^2, C^3)$ the simplifying operation described on pp. 50-51 of [11] for obtain the following conserved vector, which is generated by

$$X = t \frac{\partial}{\partial t} + 2 \frac{\partial}{\partial u},$$

and its components are given by

$$C^1 = e^u u_t, \quad C^2 = -u_x, \quad C^3 = -u_y.$$

□

We note that this corollary can be obtained from Corollary 4.2 considering $\lambda = -1$, and the adjoint variable $v = \beta e^u$, with $\alpha = 0$ and β is a nonzero constant, we obtain a only non-trivial conserved vector which is generated by V_4 .

5. Appendix

Proposition 5.1. *The adjoint equation to $\mathcal{F} = 0$, where \mathcal{F} is determined by (3.1), is given by (3.4).*

Proof. Recall the formal Lagrangian is given by

$$\mathcal{L} = v(u_{tt} - A u_{xy} - B u_x u_y - C u_{xx} - D u_{yy} - E u_y - F u_x - P u_x^2 - Q u_y^2 - G - H u_t - I u_t^2).$$

See (3.2). Then the corresponding partial derivatives of \mathcal{L} are given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} &= v(-A u_{xy} - B u_x u_y - C u_{xx} - D u_{yy} - E u_y - F u_x - P u_x^2 - Q u_y^2 - G \\ &\quad - G_u - H u_t - I u_t^2), \\ \frac{\partial \mathcal{L}}{\partial u_t} &= -v(H + 2I u_t), \quad \frac{\partial \mathcal{L}}{\partial u_x} = -v(B u_y + 2P u_x + F), \quad \frac{\partial \mathcal{L}}{\partial u_y} = -v(2Q u_y + B u_x + E), \\ \frac{\partial \mathcal{L}}{\partial u_{tt}} &= v, \quad \frac{\partial \mathcal{L}}{\partial u_{xx}} = -vC, \quad \frac{\partial \mathcal{L}}{\partial u_{yy}} = -vD, \quad \frac{\partial \mathcal{L}}{\partial u_{xy}} = -vA. \end{aligned}$$

The total derivatives involved in the Euler operator are given by

$$\begin{aligned}
 -\mathcal{D}_t \frac{\partial \mathcal{L}}{\partial u_t} &= (2I u_t + H) v_t + (2I u_{tt} + (H_u + 2I_t) u_t + 2I_u u_t^2 + H_t) v, \\
 -\mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_x} &= (B u_y + 2P u_x + F) v_x + (B u_{xy} + B_u u_x u_y + 2P u_{xx} + B_x u_y + (F_u + 2P_x) u_x \\
 &\quad + 2P_u u_x^2 + F_x) v, \\
 -\mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_y} &= (2Q u_y + B u_x + E) v_y + (B u_{xy} + B_u u_x u_y + 2Q u_{yy} + (E_u + 2Q_y) u_y + B_y u_x \\
 &\quad + 2Q_u u_y^2 + E_y) v, \\
 \mathcal{D}_t \frac{\partial \mathcal{L}}{\partial u_{tt}} &= v_t, \quad \mathcal{D}_t^2 \frac{\partial \mathcal{L}}{\partial u_{tt}} = v_{tt}, \\
 \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xy}} &= -A v_x - (A_x + A_u u_x) v, \\
 \mathcal{D}_y \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xy}} &= -A v_{xy} - (A_x + A_u u_x) v_y - (A_y + A_u u_y) v_x - (A_u u_{xy} + A_{uu} u_x u_y + A_{xu} u_y \\
 &\quad + A_{yu} u_x + A_{xy}) v, \\
 \mathcal{D}_x \frac{\partial \mathcal{L}}{\partial u_{xx}} &= -C v_x - (C_x + C_u u_x) v, \\
 \mathcal{D}_x^2 \frac{\partial \mathcal{L}}{\partial u_{xx}} &= -C v_{xx} - 2(C_x + C_u u_x) v_x - (C_u u_{xx} + C_{uu} u_x^2 + 2C_{xu} u_x + C_{xx}) v, \\
 \mathcal{D}_y \frac{\partial \mathcal{L}}{\partial u_{yy}} &= -D v_y - (D_y + D_u u_y) v, \\
 \mathcal{D}_y^2 \frac{\partial \mathcal{L}}{\partial u_{yy}} &= -D v_{yy} - 2(D_y + D_u u_y) v_y - (D_u u_{yy} + 2D_{yu} u_y + D_{uu} u_y^2 + D_{yy}) v,
 \end{aligned}$$

Substituting these formulae in $\mathcal{F}^* = \frac{\delta \mathcal{L}}{\delta u}$, we obtain the adjoint equation (3.4). □

Proposition 5.2. *The self-adjointness determining equations of the equation (1.1) are given by:*

$$u_{tt} : \quad \varphi_u + 2I \varphi = \lambda, \tag{5.1}$$

$$u_{xy} : \quad -A \varphi_u + 2(B - A_u) \varphi = -\lambda A, \tag{5.2}$$

$$u_x u_y : \quad -A \varphi_{uu} + 2(B - A_u) \varphi_u + (B_u - A_{uu}) \varphi = -\lambda B, \tag{5.3}$$

$$u_{xx} : \quad -C \varphi_u + 2(P - C_u) \varphi = -\lambda C, \tag{5.4}$$

$$u_{yy} : \quad -D \varphi_u + 2(Q - D_u) \varphi = -\lambda D, \tag{5.5}$$

$$\begin{aligned}
 u_y : \quad &-A \varphi_{xu} - 2D \varphi_{yu} + (B - A_u) \varphi_x + 2(Q - D_u) \varphi_y \\
 &+ (E - A_x - 2D_y) \varphi_u + (B_x - A_{xu} + 2(Q_y - D_{yu})) \varphi = -\lambda E,
 \end{aligned} \tag{5.6}$$

$$\begin{aligned}
 u_x : \quad &-A \varphi_{yu} - 2C \varphi_{xu} + (B - A_u) \varphi_y + 2(P - C_u) \varphi_x \\
 &+ (F - A_y - 2C_x) \varphi_u + (B_y - A_{yu} + 2(P_x - C_{xu})) \varphi = -\lambda F,
 \end{aligned} \tag{5.7}$$

$$u_t : \quad 2\varphi_{tu} + 2I \varphi_t + H \varphi_u + 2I_t \varphi = -\lambda H, \tag{5.8}$$

$$u_x^2 : \quad -C \varphi_{uu} + 2(P - C_u) \varphi_u + (P_u - C_{uu}) \varphi = -\lambda P, \tag{5.9}$$

$$u_y^2 : \quad -D \varphi_{uu} + 2(Q - D_u) \varphi_u + (Q_u - D_{uu}) \varphi = -\lambda Q, \tag{5.10}$$

$$u_t^2 : \quad \varphi_{uu} + 2I\varphi_u + I_u\varphi = -\lambda I, \quad (5.11)$$

$$\begin{aligned} 1 : \quad & \varphi_{tt} - A\varphi_{xy} - C\varphi_{xx} - D\varphi_{yy} + (E - A_x - 2D_y)\varphi_y \\ & + (F - A_y - 2C_x)\varphi_x + H\varphi_t \\ & - (A_{xy} + C_{xx} + D_{yy} - E_y - F_x - H_t + G_u)\varphi = -\lambda G, \end{aligned} \quad (5.12)$$

for some differentiable function $\varphi = \varphi(t, x, y, u) \neq 0$ and undetermined coefficient $\lambda = \lambda(t, x, y, u)$.

Proof. Substituting $v = \varphi(t, x, y, u)$ and its partial derivatives $v_t, v_x, v_y, v_{xx}, v_{xy}$, and v_{yy} into \mathcal{F}^* given by Proposition 5.1 we have

$$\begin{aligned} \mathcal{F}^*|_{v=\varphi(t,x,y,u)} = & \varphi_{tt} + 2\varphi_{tu}u_t + \varphi_{uu}u_t^2 + \varphi_u u_{tt} \\ & - A(\varphi_{xy} + \varphi_{xu}u_y + \varphi_{yu}u_x + \varphi_{uu}u_xu_y + \varphi_u u_{xy}) \\ & - C(\varphi_{xx} + 2\varphi_{xu}u_x + \varphi_{uu}u_x^2 + \varphi_u u_{xx}) \\ & - D(\varphi_{yy} + 2\varphi_{yu}u_y + \varphi_{uu}u_y^2 + \varphi_u u_{yy}) \\ & + ((B - A_u)u_y + 2(P - C_u)u_x + F - A_y - 2C_x)(\varphi_x + \varphi_u u_x) \\ & + ((B - A_u)u_x + 2(Q - D_u)u_y + E - A_x - 2D_y)(\varphi_y + \varphi_u u_y) \\ & + (H + 2Iu_t)(\varphi_t + \varphi_u u_t) \\ & + [2Iu_{tt} + 2(B - A_u)u_{xy} + (B_u - A_{uu})u_xu_y + 2(P - C_u)u_{xx} \\ & + 2(Q - D_u)u_{yy} + (B_x - A_{xu} + 2(Q_y - D_{yu}))u_y \\ & + (B_y - A_{yu} + 2(P_x - C_{xu}))u_x + (P_u - C_{uu})u_x^2 \\ & + (Q_u - D_{uu})u_y^2 - A_{xy} - C_{xx} - D_{yy} \\ & + E_y + F_x + H_t - G_u + 2I_tu_t + I_uu_t^2] \varphi. \end{aligned}$$

That is,

$$\begin{aligned} \mathcal{F}^*|_{v=\varphi(t,x,y,u)} = & (\varphi_u + 2I\varphi)u_{tt} - (A\varphi_u - 2(B - A_u)\varphi)u_{xy} \\ & - (A\varphi_{uu} - 2(B - A_u)\varphi_u - (B_u - A_{uu})\varphi)u_xu_y \\ & - (C\varphi_u - 2(P - C_u)\varphi)u_{xx} + (D\varphi_u - 2(Q - D_u)\varphi)u_{yy} \\ & - [A\varphi_{xu} + 2D\varphi_{yu} - (B - A_u)\varphi_x - 2(Q - D_u)\varphi_y \\ & - (E - A_x - 2D_y)\varphi_u - (B_x - A_{xu} + 2(Q_y - D_{yu}))\varphi]u_y \\ & - [A\varphi_{yu} + 2C\varphi_{xu} - (B - A_u)\varphi_y - 2(P - C_u)\varphi_x \\ & - (F - A_y - 2C_x)\varphi_u - (B_y - A_{yu} + 2(P_x - C_{xu}))\varphi]u_x \\ & + (2\varphi_{tu} + 2I\varphi_t + H\varphi_u + 2I_t\varphi)u_t \\ & - [C\varphi_{uu} - 2(P - C_u)\varphi_u - (P_u - C_{uu})\varphi]u_x^2 \\ & - [D\varphi_{uu} - 2(Q - D_u)\varphi_u - (Q_u - D_{uu})\varphi]u_y^2 \\ & + (\varphi_{uu} + 2I\varphi_u + I_u\varphi)u_t^2 \\ & + \varphi_{tt} - A\varphi_{xy} - C\varphi_{xx} - D\varphi_{yy} + (E - A_x - 2D_y)\varphi_y \\ & + (F - A_y - 2C_x)\varphi_x + H\varphi_t \\ & - (A_{xy} + C_{xx} + D_{yy} - E_y - F_x - H_t + G_u)\varphi. \end{aligned}$$

Now we equalize the coefficients of the monomials $u_{tt}, u_{xy}, u_x u_y, u_{xx}, u_{yy}, u_y, u_x, u_x^2, u_y^2, u_t, u_t^2$ and $\mathbf{1}$ in both sides the eq. (2.7) for we obtain the self-adjointness determining equations. \square

Proof. of Theorem 3.1. The eq. (3.10) is obtained adding to eq. (5.2) the eq. (5.1) multiplied by A .

The eq. (3.11) is obtained adding to eq. (5.4) multiplied by D the eq. (5.5) multiplied by $-C$.

The eq. (3.12) is obtained adding to eq. (5.4) the eq. (5.1) multiplied by C .

The eq. (3.13) is obtained adding to eq. (5.5) the eq. (5.1) multiplied by D .

The eq. (3.14) is obtained adding to eq. (5.2) multiplied by C the eq. (5.4) multiplied by $-A$.

The eq. (3.15) is obtained adding to eq. (5.2) multiplied by D the eq. (5.5) multiplied by $-A$.

If $A_1 \neq 0$ the eq. (3.5) is obtained adding to eq. (5.3) the eq. (5.11) multiplied by A , after we substitute the eqs. (3.10, 5.1) in the resultant equation obtaining the follows: $A_u (\varphi_u + I\varphi) = 0$, this is, $\varphi_u + I\varphi = 0$.

If $B_1 \neq 0$ the eq. (3.5) is obtained adding to eq. (5.9) multiplied by D the eq. (5.10) multiplied by $-C$, after we substitute the eqs. (3.11, 3.12, 3.13, 5.1) in the resultant equation obtaining the follows: $(DP - CQ) (\varphi_u + I\varphi) = 0$, this is, $\varphi_u + I\varphi = 0$.

If $C_1 \neq 0$ the eq. (3.5) is obtained adding to eq. (5.9) the eq. (5.11) multiplied by C , after we substitute the eqs. (3.12, 5.1) in the resultant equation obtaining the follows: $C_u (\varphi_u + I\varphi) = 0$, this is, $\varphi_u + I\varphi = 0$.

If $D_1 \neq 0$ the eq. (3.5) is obtained adding to eq. (5.10) the eq. (5.11) multiplied by D , after we substitute the eqs. (3.13, 5.1) in the resultant equation obtaining the follows: $D_u (\varphi_u + I\varphi) = 0$, ou seja, $\varphi_u + I\varphi = 0$.

If $E_1 \neq 0$ the eq. (3.5) is obtained adding to eq. (5.3) multiplied by C the eq. (5.9) multiplied by $-A$, after we substitute the eqs. (3.10, 3.12, 3.14, 5.1) in the resultant equation obtaining the follows: $(AP - CB) (\varphi_u + I\varphi) = 0$, this is, $\varphi_u + I\varphi = 0$.

If $F_1 \neq 0$ the eq. (3.5) is obtained adding to eq. (5.3) multiplied by D the eq. (5.10) multiplied by $-A$, after we substitute the eqs. (3.10, 3.13, 3.15, 5.1) in the resultant equation obtaining the follows: $(AQ - DB) (\varphi_u + I\varphi) = 0$, this is, $\varphi_u + I\varphi = 0$.

The eq. (3.6) is obtained substituting the eq. (5.1) into (5.8).

The eq. (3.7) is obtained substituting the eq. (5.1) in (5.7).

The eq. (3.8) is obtained substituting the eq. (5.1) in (5.6).

Finally, the eq. (3.9) is obtained substituting the eq. (5.1) in (5.12). \square

Proposition 5.3. *The Lie symmetries of the modified hyperbolic geometric flow equation (4.1) are generate by following infinitesimal generator:*

$$X = (c_1 + c_4 t) \frac{\partial}{\partial t} + \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} + \frac{2}{\lambda} (\xi_x - c_4) \frac{\partial}{\partial u},$$

where c_1, c_4 are arbitrary constants, and the functions $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ satisfies the Cauchy-Riemann equations:

$$\xi_x = \eta_y, \quad \eta_x = -\xi_y.$$

Proof. We will follow the notation of [13], **Theorem 2.36**, pg. 110. Let be the infinitesimal generator

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u},$$

where $\tau = \tau(t, x, y, u)$, $\xi = \xi(t, x, y, u)$, $\eta = \eta(t, x, y, u)$ and $\phi = \phi(t, x, y, u)$, its infinitesimal generator of the second prolongation is

$$\mathbf{pr}^{(2)}X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u} + \phi^t \frac{\partial}{\partial u_t} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \dots,$$

where

$$\begin{aligned} \phi^t &= \mathcal{D}_t \phi - (\mathcal{D}_t \tau) u_t - (\mathcal{D}_t \xi) u_x - (\mathcal{D}_t \eta) u_y \\ &= \phi_t + (\phi_u - \tau_t) u_t - \tau_u u_t^2 - \xi_t u_x - \eta_t u_y - \xi_u u_t u_x - \eta_u u_t u_y, \\ \phi^{tt} &= \mathcal{D}_t \phi^t - (\mathcal{D}_t \tau) u_{tt} - (\mathcal{D}_t \xi) u_{tx} - (\mathcal{D}_t \eta) u_{ty} \\ &= \phi_{tt} + (2\phi_{tu} - \tau_{tt}) u_t - \xi_{tt} u_x - \eta_{tt} u_y + (\phi_{uu} - 2\tau_{tu}) u_t^2 - 2\xi_{tu} u_t u_x - 2\eta_{tu} u_t u_y \\ &\quad - \tau_{uu} u_t^3 - \xi_{uu} u_t^2 u_x - \eta_{uu} u_t^2 u_y + (\phi_u - 2\tau_t - 3\tau_u u_t - \xi_u u_x - \eta_u u_y) u_{tt} \\ &\quad - 2(\xi_t + \xi_u u_t) u_{tx} - 2(\eta_t + \eta_u u_t) u_{ty}, \\ \phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t - \eta_{xx} u_y + (\phi_{uu} - 2\xi_{xu}) u_x^2 - 2\tau_{xu} u_t u_x - 2\eta_{xu} u_x u_y \\ &\quad - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t - \eta_{uu} u_x^2 u_y + (\phi_u - 2\xi_x - 3\xi_u u_x - \tau_u u_t - \eta_u u_y) u_{xx} \\ &\quad - 2(\tau_x + \tau_u u_x) u_{tx} - 2(\eta_x + \eta_u u_x) u_{xy}, \\ \phi^{yy} &= \phi_{yy} + (2\phi_{yu} - \eta_{yy}) u_y - \tau_{yy} u_t - \xi_{yy} u_x + (\phi_{uu} - 2\eta_{yu}) u_y^2 - 2\tau_{yu} u_t u_y - 2\xi_{yu} u_x u_y \\ &\quad - \eta_{uu} u_y^3 - \tau_{uu} u_y^2 u_t - \xi_{uu} u_y^2 u_x + (\phi_u - 2\eta_y - 3\eta_u u_y - \tau_u u_t - \xi_u u_x) u_{yy} \\ &\quad - 2(\tau_y + \tau_u u_y) u_{ty} - 2(\xi_y + \xi_u u_y) u_{xy}. \end{aligned}$$

Applying $\mathbf{pr}^{(2)}X$ to eq.(4.1) we find the infinitesimal criterion of invariance (**Theorem 2.31**, pg 104) must be satisfied whenever $u_{tt} - e^{\lambda u} (u_{xx} + u_{yy}) - \lambda u_t^2 = 0$, that is,

$$\mathbf{pr}^{(2)}X(u_{tt} - e^{\lambda u} (u_{xx} + u_{yy}) - \lambda u_t^2) = 0, \quad \text{whenever} \quad u_{tt} - e^{\lambda u} (u_{xx} + u_{yy}) - \lambda u_t^2 = 0.$$

Thus we must have

$$\begin{aligned} 0 &= \left(\phi \frac{\partial}{\partial u} + \phi^t \frac{\partial}{\partial u_t} + \phi^{tt} \frac{\partial}{\partial u_{tt}} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} \right) (u_{tt} - e^{\lambda u} (u_{xx} + u_{yy}) - \lambda u_t^2) \\ &= -\phi \lambda e^{\lambda u} (u_{xx} + u_{yy}) - \phi^t 2\lambda u_t + \phi^{tt} - (\phi^{xx} + \phi^{yy}) e^{\lambda u}. \end{aligned} \quad (5.13)$$

Now we substitute the eq.(4.1) into ϕ^{tt} and the rewritten as

$$\phi^{tt} := \tilde{\phi}^{tt} + (\phi_u - 2\tau_t - 3\tau_u u_t - \xi_u u_x - \eta_u u_y) (e^{\lambda u} (u_{xx} + u_{yy}) + \lambda u_t^2) \quad (5.14)$$

where $\tilde{\phi}^{tt} = 0$.

The coefficients of the quadratic monomials $u_t u_{xx}$, $u_x u_{xx}$ and $u_y u_{yy}$ in (5.13, 5.14) are given by

$$\begin{aligned} u_t u_{xx}: & \quad -3\tau_u \lambda e^{\lambda u} + \tau_u \lambda e^{\lambda u} = 0, \\ u_x u_{xx}: & \quad -\xi_u \lambda e^{\lambda u} + 3\xi_u \lambda e^{\lambda u} = 0, \\ u_y u_{yy}: & \quad -\eta_u \lambda e^{\lambda u} + 3\eta_u \lambda e^{\lambda u} = 0. \end{aligned}$$

From $\lambda \neq 0$ follows that $\tau = \tau(t, x, y)$, $\xi = \xi(t, x, y)$ and $\eta = \eta(t, x, y)$. Then we have

$$\begin{aligned}\phi^t &= \phi_t + (\phi_u - \tau_t)u_t - \xi_t u_x - \eta_t u_y, \\ \phi^{tt} &= \phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t - \xi_{tt} u_x - \eta_{tt} u_y + \phi_{uu} u_t^2 - 2\xi_t u_{tx} - 2\eta_t u_{ty} \\ &\quad + (\phi_u - 2\tau_t)(e^{\lambda u}(u_{xx} + u_{yy}) + \lambda u_t^2) \\ &= \phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t - \xi_{tt} u_x - \eta_{tt} u_y + (\phi_{uu} + \lambda(\phi_u - 2\tau_t))u_t^2 - 2\xi_t u_{tx} - 2\eta_t u_{ty} \\ &\quad + (\phi_u - 2\tau_t)e^{\lambda u}(u_{xx} + u_{yy}), \\ \phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx} u_t - \eta_{xx} u_y + \phi_{uu} u_x^2 + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{tx} \\ &\quad - 2\eta_x u_{xy}, \\ \phi^{yy} &= \phi_{yy} + (2\phi_{yu} - \eta_{yy})u_y - \tau_{yy} u_t - \xi_{yy} u_x + \phi_{uu} u_y^2 + (\phi_u - 2\eta_y)u_{yy} - 2\tau_y u_{ty} \\ &\quad - 2\xi_y u_{xy}, \\ \phi^{xx} + \phi^{yy} &= \Delta\phi - \Delta\tau u_t + (2\phi_{xu} - \Delta\xi)u_x + (2\phi_{yu} - \Delta\eta)u_y + \phi_{uu}(u_x^2 + u_y^2) \\ &\quad + (\phi_u - 2\xi_x)u_{xx} + (\phi_u - 2\eta_y)u_{yy} - 2\tau_x u_{tx} - 2\tau_y u_{ty} - 2(\eta_x + \xi_y)u_{xy}.\end{aligned}$$

Substituting this formulae in (5.13) we obtain

$$\begin{aligned}0 &= -\phi\lambda e^{\lambda u}(u_{xx} + u_{yy}) - (\phi_t + (\phi_u - \tau_t)u_t - \xi_t u_x - \eta_t u_y)2\lambda u_t + \phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t \\ &\quad - \xi_{tt} u_x - \eta_{tt} u_y + (\phi_{uu} + \lambda(\phi_u - 2\tau_t))u_t^2 - 2\xi_t u_{tx} - 2\eta_t u_{ty} \\ &\quad + (\phi_u - 2\tau_t)e^{\lambda u}(u_{xx} + u_{yy}) - [\Delta\phi - \Delta\tau u_t + (2\phi_{xu} - \Delta\xi)u_x + (2\phi_{yu} - \Delta\eta)u_y \\ &\quad + \phi_{uu}(u_x^2 + u_y^2) + (\phi_u - 2\xi_x)u_{xx} + (\phi_u - 2\eta_y)u_{yy} - 2\tau_x u_{tx} - 2\tau_y u_{ty} \\ &\quad - 2(\eta_x + \xi_y)u_{xy}]e^{\lambda u}.\end{aligned}\tag{5.15}$$

From eq. (5.15) we find the determining equations for the symmetry to be the following:

$$\begin{aligned}u_t^2: & \quad -2\lambda(\phi_u - \tau_t) + \phi_{uu} + \lambda(\phi_u - 2\tau_t) = 0, \\ u_x^2 + u_y^2: & \quad \phi_{uu} = 0, \\ u_{xx}: & \quad -\phi\lambda e^{\lambda u} + (\phi_u - 2\tau_t)e^{\lambda u} - (\phi_u - 2\xi_x)e^{\lambda u} = 0, \\ u_{yy}: & \quad -\phi\lambda e^{\lambda u} + (\phi_u - 2\tau_t)e^{\lambda u} - (\phi_u - 2\eta_y)e^{\lambda u} = 0, \\ u_{xy}: & \quad 2(\eta_x + \xi_y)e^{\lambda u} = 0, \\ u_{tx}: & \quad -2\xi_t + 2\tau_x e^{\lambda u} = 0, \\ u_{ty}: & \quad -2\eta_t + 2\tau_y e^{\lambda u} = 0, \\ u_t u_x: & \quad 2\lambda\xi_t = 0, \\ u_t u_y: & \quad 2\lambda\eta_t = 0, \\ u_t: & \quad -2\lambda\phi_t + (2\phi_{tu} - \tau_{tt}) + \Delta\tau e^{\lambda u} = 0, \\ u_x: & \quad -\xi_{tt} - (2\phi_{xu} - \Delta\xi)e^{\lambda u} = 0, \\ u_y: & \quad -\eta_{tt} - (2\phi_{yu} - \Delta\eta)e^{\lambda u} = 0, \\ 1: & \quad \phi_{tt} - \Delta\phi e^{\lambda u} = 0.\end{aligned}$$

The solution of the determining equations is elementary. First substitute the second determining equation into first, so we have $\phi = \phi(t, x, y)$ since $\lambda \neq 0$.

The eighth and ninth equations shows that $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. Substituting this ξ and η in the sixth and seventh equations we find $\tau = \tau(t)$.

Now we solve the third equation for ϕ and we obtain

$$-(\phi \lambda + 2 \tau_t - 2 \xi_x) e^{\lambda u} = 0, \quad \text{that is,} \quad \phi = \frac{2}{\lambda} (\xi_x - \tau_t).$$

Subtracting to third the fourth equation, we have $\xi_x - \eta_y = 0$. This and from fifth equation follow that the functions $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ satisfy the Cauchy-Riemann equations.

Substituting τ and ϕ in tenth equation, we obtain $\tau_{tt} = 4 \tau_{tt}$, that is, $\tau = c_1 + c_4 t$, where c_1 and c_4 are arbitrary constants.

Thus, $\tau = c_1 + c_4 t$, the functions $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ satisfy the Cauchy-Riemann equations, and finally $\phi = \frac{2}{\lambda} (\xi_x - c_4)$. \square

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