Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

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To cite this article: Nianhua Li, Q.P. Liu (2013) Bi-Hamiltonian Structure of a Three-Component Camassa-Holm Type Equation, Journal of Nonlinear Mathematical Physics 20:1, 126–134, DOI: <https://doi.org/10.1080/14029251.2013.792477>

To link to this article: <https://doi.org/10.1080/14029251.2013.792477>

Published online: 04 January 2021

Bi-Hamiltonian Structure of a Three-Component Camassa-Holm Type Equation

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Received 9 September 2012

Accepted 22 January 2012

A recently proposed three-component Camassa-Holm equation is considered. It is shown that this system is a bi-Hamiltonian system.

Keywords: bi-Hamiltonian structures; three-component CH equation

2000 Mathematics Subject Classification: 37K10, 35Q51, 35Q58

1. Introduction

In 1981, the following integrable equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.1)$$

was proposed as an abstract equation admitting a bi-Hamiltonian structure [20]. This system, now known as Camassa-Holm (CH) equation, was rediscovered by Camassa and Holm [3] as an approximation to the incompressible Euler equations and its relevance for the shallow water waves modeling was discussed [14, 24–26]. The CH equation has been studied extensively and a number of properties are established for it. Indeed, it is shown that this equation is solvable by inverse scattering transformation [1, 2, 5, 6, 15, 17, 27, 30], possesses a large number of solutions including multi-soliton solutions, algebraic-geometric solutions and multi-peakon solutions [22, 33, 34, 37], can be regarded as the geodesic flow on the diffeomorphism group [29, 35] (see also [12, 13, 28]), to mention just a few (also see [21, 32] and the references there).

Also, it is worthwhile to remark that while the CH equation is reciprocally associated to a particular flow of the celebrated Korteweg-de Vries (KdV) equation [19, 32], there are fundamental differences between the CH equation and the KdV equation. Unlike the KdV equation, the CH equation has breaking wave solutions which develop singularities in finite time [8]. Indeed, the construction of such an equation, which can model both breaking waves and peaked traveling waves, was a long-standing open problem in hydrodynamics [39]. Furthermore, the search of models which have peaked wave solutions is also motivated by the quest for waves of great height (see [7, 9, 10, 38]). It is interesting to note that the peaked solitons or peakons of the CH equation are orbitally stable [16, 31].

The CH equation has several generalizations. A two-component and a three-component extensions were proposed in [4, 36] and in [18] respectively and both extensions are shown to be bi-Hamiltonian systems. Very recently, a new three-component generalization of the CH equation was constructed by Geng and Xue [21]. It reads as

$$\begin{aligned} u_t &= -vp_x + u_xq + \frac{3}{2}uq_x - \frac{3}{2}u(p_xr_x - pr), \\ v_t &= 2vq_x + v_xq, \\ w_t &= vr_x + w_xq + \frac{3}{2}wq_x + \frac{3}{2}w(p_xr_x - pr), \end{aligned} \quad (1.2)$$

with

$$\begin{aligned} u &= p - p_{xx}, \\ v &= \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_xr - 3pr_x), \\ w &= r_{xx} - r. \end{aligned} \quad (1.3)$$

Above system was derived from a 3×3 matrix spectral problem. It was further shown that (1.2) is a Hamiltonian system with infinite number of conservation laws. The purpose of this paper is to construct two Hamiltonian structures for (1.2) and to prove that it constitutes a bi-Hamiltonian system.

2. Bi-Hamiltonian structure of the three-component Camassa-Holm equation

As demonstrated in [21], the system (1.2) arises as a zero curvature equation

$$M_t - N_x + [M, N] = 0, \quad (2.1)$$

which is the compatibility condition of the linear system

$$\varphi_x = M\varphi, \quad \varphi_t = N\varphi. \quad (2.2)$$

where

$$\begin{aligned} M &= \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix}, \\ N &= \begin{pmatrix} \frac{1}{2}(-q_x - p_xr_x + pr) & \frac{1}{\lambda} + q & \frac{p_x}{\lambda} \\ \frac{1}{\lambda} - pr_x + p_xr - q + \lambda qv & \frac{1}{2}(q_x - p_xr_x + pr) & \frac{p}{\lambda} + qu \\ -r + \lambda qw & r_x & p_xr_x - pr \end{pmatrix}. \end{aligned}$$

Now, we construct Hamiltonian structures for the equations associated with this 3×3 spectral problem. To this end, we take M as above and N as follows

$$N = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \quad (2.3)$$

with A_k, B_k, C_k ($k = 1, 2, 3$) are functions depending on the spectral parameter λ and the field variables.

Through tedious calculations, we find that the zero-curvature representation (2.1) yields

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = (\lambda^{-1} \mathcal{E} + \mathcal{F}) \begin{pmatrix} C_2 \\ A_2 \\ \lambda A_3 \end{pmatrix}, \quad (2.4)$$

where

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & \partial^2 - 1 \\ \frac{1}{2}u_x + \frac{3}{2}u\partial & 2\partial - \frac{1}{2}\partial^3 & \frac{1}{2}w_x + \frac{3}{2}w\partial \\ 1 - \partial^2 & 0 & 0 \end{pmatrix}, \quad (2.5)$$

$$\mathcal{F} = \begin{pmatrix} \frac{3}{2}u\partial^{-1}u & u_x + \frac{3}{2}u\partial & -\frac{3}{2}u\partial^{-1}w - v \\ 0 & v\partial + \partial v & 0 \\ -\frac{3}{2}w\partial^{-1}u + v & w_x + \frac{3}{2}w\partial & \frac{3}{2}w\partial^{-1}w \end{pmatrix}. \quad (2.6)$$

While neither $\mathcal{E}$ nor $\mathcal{F}$ could serve as Hamiltonian operators since both of them are even not skew-symmetric, we may formulate a operator

$$\mathcal{R} = \mathcal{F} \mathcal{E}^{-1}, \quad (2.7)$$

which is the candidate for a recursion operator. To justify this, we notice that a Hamiltonian operator was obtained in [21], which reads as

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & \partial^2 - 1 \\ 0 & -\partial v - v\partial & 0 \\ 1 - \partial^2 & 0 & 0 \end{pmatrix}. \quad (2.8)$$

Then we define a new operator by

$$\mathcal{K} = \mathcal{F} \mathcal{E}^{-1} \mathcal{J}, \quad (2.9)$$

whose explicit form is given by

$$\mathcal{K} = \mathcal{T} - 2\mathcal{S}(\partial^3 - 4\partial)^{-1}(\mathcal{S}^\dagger), \quad (2.10)$$

with

$$\mathcal{S} = \begin{pmatrix} \frac{3}{2}u\partial + u_x \\ v\partial + \partial v \\ \frac{3}{2}w\partial + w_x \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} \frac{3}{2}u\partial^{-1}u & 0 & -v - \frac{3}{2}u\partial^{-1}w \\ 0 & 0 & 0 \\ v - \frac{3}{2}w\partial^{-1}u & 0 & \frac{3}{2}w\partial^{-1}w \end{pmatrix},$$

and $\mathcal{S}^\dagger$ denotes the formal adjoint of the operator $\mathcal{S}$.

Our claim is that $\mathcal{J}$ and $\mathcal{K}$ constitutes a compatible Hamiltonian pair. In fact, we have

Theorem 2.1. *The system (1.2) is a bi-Hamiltonian system, namely there exist two compatible Hamiltonian operators $\mathcal{J}$ and $\mathcal{K}$ such that*

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t = \mathcal{J} \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta v} \\ \frac{\delta H_1}{\delta w} \end{pmatrix} = \mathcal{K} \begin{pmatrix} \frac{\delta H_0}{\delta u} \\ \frac{\delta H_0}{\delta v} \\ \frac{\delta H_0}{\delta w} \end{pmatrix}, \quad (2.11)$$

where $\mathcal{J}$ and $\mathcal{K}$ are the operators given by (2.8) and (2.9), and

$$H_0 = \frac{1}{4} \int (4q^2 - qq_{xx} - p_x^2 r_x^2 + 6pp_x rr_x + 3p^2 r^2) dx, \\ H_1 = \frac{1}{2} \int (2v + ur_x + wp_x) dx.$$

Proof. The first operator $\mathcal{J}$ is a local and linear operator, which already appeared in [21]. Next, we turn to the more complicated operator $\mathcal{K}$. Also, it is easy to show that this operator is skew-symmetric, therefore we prove that the related Jacobi identity holds. Since

$$\begin{aligned} & \langle f, \mathcal{K}'[\mathcal{K}g]h \rangle + \text{c.p.} \\ &= \langle f, \mathcal{T}'[\mathcal{K}g]h - 2\mathcal{S}'[\mathcal{K}g](\partial^3 - 4\partial)^{-1}(\mathcal{S}^\dagger)h - 2\mathcal{S}(\partial^3 - 4\partial)^{-1}(\mathcal{S}^\dagger)'[\mathcal{K}g]h \rangle + \text{c.p.} \end{aligned}$$

Let A, B, C be the rows of $\mathcal{K}$, or $\mathcal{K} = (A, B, C)^T$, we find

$$\begin{aligned} & \langle f, \mathcal{T}'[\mathcal{K}g]h \rangle + \text{c.p.} \\ &= \left\langle \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \begin{pmatrix} \frac{3}{2}Ag\partial^{-1}(uh_1 - wh_3) + \frac{3}{2}u\partial^{-1}Agh_1 - Bgh_3 - \frac{3}{2}u\partial^{-1}Cgh_3 \\ 0 \\ Bgh_1 - \frac{3}{2}Cg\partial^{-1}(uh_1 - wh_3) - \frac{3}{2}w\partial^{-1}Agh_1 + \frac{3}{2}w\partial^{-1}Cgh_3 \end{pmatrix} \right\rangle + \text{c.p.} \\ &= \langle \frac{3}{2}(f_1Ag\partial^{-1}(uh_1 - wh_3) + u f_1\partial^{-1}Agh_1 - u f_1\partial^{-1}Cgh_3) - f_1Bgh_3 \rangle \\ & \quad + \langle \frac{3}{2}(-w f_3\partial^{-1}h_1Ag - f_3Cg\partial^{-1}(uh_1 - wh_3) + w f_3\partial^{-1}Cgh_3) + f_3Bgh_1 \rangle + \text{c.p.} \\ &= \langle \frac{3}{2}(f_1Ag - g_1Af + Cf g_3 - f_3Cg)\partial^{-1}(uh_1 - wh_3) + f_3Bgh_1 - f_1Bgh_3 \rangle + \text{c.p.} \\ &= \frac{9}{4}(\langle u g_1\partial^{-1}w f_3 + w g_3\partial^{-1}w f_3, \partial^{-1}uh_1 \rangle + \langle u g_1\partial^{-1}u f_1 + w g_3\partial^{-1}u f_1, \partial^{-1}h_3w \rangle) \\ & \quad + 3 \langle f_1(\frac{3}{2}u\partial + u_x)g_0 - f_3(\frac{3}{2}w\partial + w_x)g_0, \partial^{-1}(uh_1 - wh_3) \rangle \\ & \quad + 2 \langle (v_x + 2v\partial)h_0, f_1g_3 \rangle - (f \leftrightarrow g) + \text{c.p.} \\ &= 3 \langle (\frac{3}{2}f_1u\partial + f_1u_x)g_0 - (\frac{3}{2}f_3w\partial + f_3w_x)g_0, \partial^{-1}(uh_1 - h_3w) \rangle \\ & \quad + 2 \langle (v_x + 2v\partial)h_0, f_1g_3 \rangle - (f \leftrightarrow g) + \text{c.p.} \\ &= \frac{3}{2} \langle g_0, (f_3w_x + 3f_3xw - f_1u_x - 3f_{1x}u)\partial^{-1}(uh_1 - wh_3) \rangle \\ & \quad + 2 \langle g_0, v_x f_1 h_3 + 2v(f_1 h_3)_x \rangle - (f \leftrightarrow h) + \text{c.p.}, \end{aligned} \quad (2.12)$$

where $(f \leftrightarrow g)$ denotes the resulted term from the previous ones by permutating f and g , and

$$g_0 \equiv (\partial^3 - 4\partial)^{-1} \left(\frac{3}{2}u g_{1x} + \frac{1}{2}u_x g_1 + v_x g_2 + 2v g_{2x} + \frac{3}{2}w g_{3x} + \frac{1}{2}w_x g_3 \right), \quad (2.13)$$

and similarly for f_0, h_0 . Also, we have

$$\begin{aligned}
 & \langle f, -2\mathcal{S}'[\mathcal{K}g](\partial^3 - 4\partial)^{-1}(\mathcal{S}^\dagger)h - 2\mathcal{S}(\partial^3 - 4\partial)^{-1}(\mathcal{S}^\dagger)'[\mathcal{K}g]h \rangle + \text{c.p.} \\
 &= 2 \langle h, \mathcal{S}'[\mathcal{K}f]g_0 \rangle - 2 \langle h, \mathcal{S}'[\mathcal{K}g]f_0 \rangle + \text{c.p.} \\
 &= 2 \langle h^T \mathcal{S}'[\mathcal{K}f]g_0 - f^T \mathcal{S}'[\mathcal{K}h]g_0 \rangle + \text{c.p.} \\
 &= 2 \langle (\frac{3}{2}h_1Af\partial + h_1(Af)_x + h_2((Bf)_x + 2Bf\partial) + h_3(\frac{3}{2}Cf\partial + (Cf)_x))g_0 \rangle \\
 &\quad - (f \leftrightarrow h) + \text{c.p.} \\
 &= \langle g_0, f_1(Ah)_x + 3f_{1x}Ah + 2f_2(Bh)_x + 4f_{2x}Bh + f_3(Ch)_x + 3f_{3x}Ch \rangle \\
 &\quad - (f \leftrightarrow h) + \text{c.p.} \\
 &\equiv \Omega_1 + \Omega_2,
 \end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
 \Omega_1 &= \langle g_0, f_1(\frac{3}{2}u_x\partial^{-1}(uh_1 - wh_3) + \frac{3}{2}u^2h_1 - v_xh_3 - vh_{3x} - \frac{3}{2}uwh_3) \rangle \\
 &\quad + \langle g_0, 3f_{1x}(\frac{3}{2}u\partial^{-1}(uh_1 - wh_3) - vh_3) + 3f_{3x}(vh_1 - \frac{3}{2}w\partial^{-1}(uh_1 - wh_3)) \rangle \\
 &\quad + \langle g_0, f_3(v_xh_1 + vh_{1x} - \frac{3}{2}w_x\partial^{-1}(uh_1 - wh_3) - \frac{3}{2}wuh_1 + \frac{3}{2}w^2h_3) \rangle \\
 &\quad - (f \leftrightarrow h) + \text{c.p.} \\
 &= \frac{3}{2} \langle g_0, (f_1u_x + 3f_{1x}u - f_3w_x - 3f_{3x}w)\partial^{-1}(uh_1 - wh_3) \rangle \\
 &\quad + \langle g_0, 3h_{1x}vf_3 - h_1(\frac{3}{2}u^2f_1 - v_xf_3 - vf_{3x} - \frac{3}{2}uwf_3) \rangle \\
 &\quad - \langle g_0, 3h_{3x}vf_1 + h_3(v_xf_1 + vf_{1x} - \frac{3}{2}uwf_1 + \frac{3}{2}w^2f_3) \rangle - (f \leftrightarrow h) + \text{c.p.} \\
 &= \frac{3}{2} \langle g_0, (f_1u_x + 3f_{1x}u - f_3w_x - 3f_{3x}w)\partial^{-1}(uh_1 - wh_3) \rangle \\
 &\quad + 2 \langle g_0, v_x(f_3h_1) + 2v(f_{3x}h_1 + f_3h_{1x}) \rangle - (f \leftrightarrow h) + \text{c.p.},
 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
 \Omega_2 &= 2 \langle g_0, (f_1\partial + 3f_{1x})(\frac{3}{2}u\partial + u_x)h_0 + (2f_2\partial + 4f_{2x})(v_x + 2v\partial)h_0 \rangle \\
 &\quad + 2 \langle g_0, (f_3\partial + 3f_{3x})(\frac{3}{2}w\partial + w_x)h_0 \rangle - (f \leftrightarrow h) + \text{c.p.}
 \end{aligned} \tag{2.16}$$

Taking account of (2.12)-(2.14) with (2.15)-(2.16), we obtain

$$\begin{aligned}
 & \langle f, \mathcal{K}'[\mathcal{K}g]h \rangle + \text{c.p.} = \Omega_2 \\
 &= 2 \langle g_0, (f_1(\frac{3}{2}u\partial^2 + \frac{5}{2}u_x\partial + u_{xx}) + 3f_{1x}(\frac{3}{2}u\partial + u_x))h_0 \rangle \\
 &\quad + 2 \langle g_0, (f_3(\frac{3}{2}w\partial^2 + \frac{5}{2}w_x\partial + w_{xx}) + 3f_{3x}(\frac{3}{2}w\partial + w_x))h_0 \rangle \\
 &\quad + 4 \langle g_0, f_2(v_{xx} + 3v_x\partial + 2v\partial^2) + 2f_{2x}(v_x + 2v\partial))h_0 \rangle - (f \leftrightarrow h) + \text{c.p.}
 \end{aligned}$$

$$\begin{aligned}
&= 2 < h_0, (\frac{3}{2}\partial^2 f_1 u - \frac{5}{2}\partial f_1 u_x + u_{xx}f_1 - \frac{9}{2}\partial u f_{1x} + 3f_{1x}u_x)g_0 > \\
&\quad + 2 < h_0, (\frac{3}{2}\partial^2 w f_3 - \frac{5}{2}\partial f_3 w_x + w_{xx}f_3 - \frac{9}{2}\partial w f_{3x} + 3f_{3x}w_x)g_0 > \\
&\quad + 2 < h_0, (2f_2 v_{xx} - 6\partial v_x f_2 + 4\partial^2 f_2 v + 4f_{2x}v_x - 8\partial f_{2x}v)g_0 > \\
&\quad - 2 < g_0, (h_1(\frac{3}{2}u\partial^2 + \frac{5}{2}u_x\partial + u_{xx}) + 3h_{1x}(\frac{3}{2}u\partial + u_x))f_0 > \\
&\quad - 2 < g_0, (h_3(\frac{3}{2}w\partial^2 + \frac{5}{2}w_x\partial + w_{xx}) + 3h_{3x}(\frac{3}{2}w\partial + w_x))f_0 > \\
&\quad - 2 < g_0, (2h_2(v_{xx} + 3v_x\partial + 2v\partial^2) + 4h_{2x}(v_x + 2v\partial))f_0 > + \text{c.p.} \\
&= -2 < g_0, \partial(h_1 u_x + 3h_{1x}u + 2h_2 v_x + 4h_{2x}v + w_x h_3 + 3w h_{3x})f_0 > \\
&\quad - 2 < g_0, (h_1 u_x + 3h_{1x}u + 2h_2 v_x + 4h_{2x}v + w_x h_3 + 3w h_{3x})\partial f_0 > + \text{c.p.} \\
&= -4 < g_0, \partial(h_{0xx} - 4h_{0x})f_0 + (h_{0xxx} - 4h_{0x})\partial f_0 > + \text{c.p.} \\
&= -4 < (h_{0xxx} - 4h_{0x})(f_{0x}g_0 - f_0g_{0x}) > + \text{c.p.} \\
&= -4 < h_{0xx}(f_{0xx}g_0 - f_0g_{0xx}) > + \text{c.p.} \\
&= 0.
\end{aligned}$$

Therefore, $\mathcal{K}$ is a Hamiltonian operator. Finally, we prove that $\mathcal{J}$ and $\mathcal{K}$ form a compatible Hamiltonian pair, that is, we have to show that these operators satisfy the following condition

$$< f, \mathcal{J}'[\mathcal{K}g]h > + < f, \mathcal{K}'[\mathcal{J}g]h > + \text{c.p.} = 0. \quad (2.17)$$

Indeed, by direct calculation, we find

$$\begin{aligned}
< f, \mathcal{J}'[\mathcal{K}g]h > + \text{c.p.} &= -2 < f_2, ((v_x g_0 + 2v g_{0x})_x + 2(v_x g_0 + 2v g_{0x})\partial)h_2 > + \text{c.p.} \\
&= 2 < v_x g_0 + 2v g_{0x}, h_2 f_{2x} - h_{2x} f_2 > + \text{c.p.}, \quad (2.18)
\end{aligned}$$

and

$$\begin{aligned}
&< f, \mathcal{K}'[\mathcal{J}g]h > + \text{c.p.} \\
&= < f, \mathcal{J}'[\mathcal{J}g]h - 2\mathcal{J}'[\mathcal{J}g](\partial^3 - 4\partial)^{-1}\mathcal{J}^\dagger h - 2\mathcal{J}(\partial^3 - 4\partial)^{-1}(\mathcal{J}^\dagger)'[\mathcal{J}g]h > + \text{c.p.} \\
&= < f_1, \frac{3}{2}(g_{3xx} - g_3)\partial^{-1}(uh_1 - wh_3) + \frac{3}{2}u\partial^{-1}((g_{3xx} - g_3)h_1 - (g_1 - g_{1xx})h_3) > \\
&\quad + < f_1, 2v g_{2x}h_3 + v_x g_{2x}h_3 > - < f_3, 2v g_{2x}h_1 + v_x g_{2x}h_1 > \\
&\quad - < f_3, \frac{3}{2}(g_1 - g_{1xx})\partial^{-1}(uh_1 - wh_3) + \frac{3}{2}w\partial^{-1}((g_{3xx} - g_3)h_1 - (g_1 - g_{1xx})h_3) > \\
&\quad - 2 < f, \mathcal{J}'[\mathcal{J}g](\partial^3 - 4\partial)^{-1}\mathcal{J}^\dagger h + \mathcal{J}(\partial^3 - 4\partial)^{-1}(\mathcal{J}^\dagger)'[\mathcal{J}g]h > \\
&= < \frac{3}{2}(f_1(g_{3xx} - g_3) - f_3(g_1 - g_{1xx}))\partial^{-1}(uh_1 - wh_3) + 2v f_1 h_3 g_{2x} + v_x f_1 g_{2x} h_3 > \\
&\quad - < \frac{3}{2}(h_1(g_{3xx} - g_3) - h_3(g_1 - g_{1xx}))\partial^{-1}(uf_1 - wf_3) + 2v f_3 g_{2x} h_1 + f_3 v_x g_{2x} h_1 > \\
&\quad - 2 < f, \mathcal{J}'[\mathcal{J}g](\partial^3 - 4\partial)^{-1}\mathcal{J}^\dagger h + \mathcal{J}(\partial^3 - 4\partial)^{-1}(\mathcal{J}^\dagger)'[\mathcal{J}g]h >
\end{aligned}$$

$$\begin{aligned}
 &= \langle \frac{3}{2}(f_1(g_{3xx} - g_3) - f_3(g_1 - g_{1xx}))\partial^{-1}(uh_1 - wh_3) + 2vf_1h_3g_{2x} + v_xf_1g_2h_3 \rangle \\
 &\quad + 2 \langle g_0, f^T \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} (\mathcal{K}h)_x + f_x^T \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \mathcal{K}h \rangle - (f \leftrightarrow h) + \text{c.p.}
 \end{aligned} \tag{2.19}$$

Thus making use of (2.18) with (2.19), we obtain

$$\begin{aligned}
 &\langle f, \mathcal{K}'[\mathcal{J}g]h \rangle + \langle f, \mathcal{J}'[\mathcal{K}g]h \rangle + \text{c.p.} \\
 &= \langle \frac{3}{2}(f_1(g_{3xx} - g_3) - f_3(g_1 - g_{1xx}))\partial^{-1}(uh_1 - wh_3) + 2vf_1h_3g_{2x} + v_xf_1g_2h_3 \rangle \\
 &\quad + 2 \langle v_xg_0 + 2vg_{0x}, h_2f_{2x} \rangle + 2 \langle g_0, f^T \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} (\mathcal{K}h)_x + f_x^T \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \mathcal{K}h \rangle \\
 &\quad - (f \leftrightarrow h) + \text{c.p.} \\
 &= \langle \frac{3}{2}(f_1(g_{3xx} - g_3) - f_3(g_1 - g_{1xx}))\partial^{-1}(uh_1 - wh_3) + 2vf_1h_3g_{2x} + v_xf_1g_2h_3 \rangle \\
 &\quad + \langle g_0, f_1(h_{3xx} - h_{3x}) + f_3(h_{1x} - h_{1xx}) + 3f_{1x}(h_{3xx} - h_3) + 3f_{3x}(h_1 - h_{1xx}) \rangle \\
 &\quad - (f \leftrightarrow h) + \text{c.p.} \\
 &= \langle 2vg_{2x}f_1h_3 + f_1v_xg_2h_3 - 2f_3g_{2x}h_1v - f_3v_xg_2h_1 \rangle + \langle g_0, (\partial^3 - 4\partial)(f_1h_3 - f_3h_1) \rangle \\
 &\quad + \frac{3}{2} \langle uh_1 - wh_3, f_{3x}g_1 - f_1g_{3x} + f_{1x}g_3 - f_3g_{1x} \rangle + \text{c.p.} \\
 &= \langle 2vg_{2x}f_1h_3 + f_1v_xg_2h_3 - 2vg_{2x}h_1f_3 - f_3v_xg_2h_1 \rangle \\
 &\quad + \frac{3}{2} \langle uh_1 - wh_3, f_{3x}g_1 - f_1g_{3x} + f_{1x}g_3 - f_3g_{1x} \rangle \\
 &\quad + \frac{1}{2} \langle f_3h_1 - f_1h_3, 3ug_{1x} + u_xg_1 + 2v_xg_2 + 4vg_{2x} + 3wg_{3x} + g_3w_x \rangle \\
 &= \frac{3}{2} \langle uh_1 - wh_3, f_{3x}g_1 - f_1g_{3x} + f_{1x}g_3 - f_3g_{1x} \rangle \\
 &\quad + \frac{1}{2} \langle f_3h_1 - f_1h_3, 3ug_{1x} + u_xg_1 + 3wg_{3x} + g_3w_x \rangle + \text{c.p.} \\
 &= \frac{1}{2} \langle u, 3h_1(f_{3x}g_1 - f_1g_{3x} + f_{1x}g_3 - f_3g_{1x}) + 3(f_3h_1 - f_1h_3)g_{1x} - ((f_3h_1 - f_1h_3)g_1)_x \rangle \\
 &\quad + \frac{1}{2} \langle w, 3h_3(f_1g_{3x} + f_3g_{1x} - f_{3x}g_1 - f_{1x}g_3) + 3g_{3x}(f_3h_1 - f_1h_3) \\
 &\quad - \frac{1}{2} \langle w, ((f_3h_1 - f_1h_3)g_3)_x \rangle + \text{c.p.} \\
 &= 0.
 \end{aligned}$$

Therefore, the Hamiltonian operators $\mathcal{J}$ and $\mathcal{K}$ are compatible and the proof of the theorem is completed.

According to the bi-Hamiltonian theory, the operator $\mathcal{R}$ is a recursion operator with hereditary property, which may be used to generate a hierarchy of commuting flows. $\square$

Acknowledgments

This work is supported by the National Natural Science Foundation of China (grant numbers: 10971222 and 11271366) and the Fundamental Research Funds for Central Universities.

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