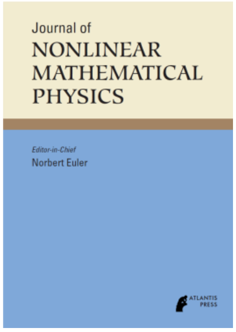

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CONSTRUCTION OF MODULATED AMPLITUDE WAVES VIA AVERAGING IN COLLISIONALLY INHOMOGENEOUS BOSE–EINSTEIN CONDENSATES

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We apply the averaging method to analyze spatio-temportal structures in nonlinear Schrödinger equations and thereby study the dynamics of quasi-one-dimensional collisionally inhomogeneous Bose–Einstein condensates with the scattering length varying periodically in space and crossing zero. Infinitely many modulated amplitude waves with nontrivial phases are shown.

Keywords: Modulated amplitude waves; Gross–Pitaevskii equations; collisionally inhomogeneous Bose–Einstein condensates; periodic potentials; averaging method.

Mathematics Subject Classification 2010: 34C35, 35B10, 35B27, 35B36, 35B40

1. Introduction

Since the experimental realization of Bose–Einstein condensates (BECs) in the mid-1990s [1, 13], the study of matter-wave patterns including existence and stability in BECs has drawn a great deal of interest from experimentalists [35, 36] and theorists [20, 30, 17, 39].

In atomic physics, the Feshbach resonance of the scattering length of interatomic interactions is used for control of Bose–Einstein condensates [12, 16]. We consider the main model of this paper given by the perturbed Gross–Pitaevskii (GP) equation of the dimensionless form [32]

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \tilde{g}(x)|\psi|^2\psi + \tilde{V}(x)\psi, \tag{1.1}$$

where the nonlinearity coefficient $\tilde{g}(x)$ varies in space. In Eq. (1.1), ψ is the mean-field condensate wave function (with density $|\psi|^2$ measured in units of the peak 1D density n_0), x and t are normalized, respectively, to the healing length $\xi = \hbar/\sqrt{n_0|g_1|m}$ and ξ/c (where $c = \hbar\sqrt{n_0|g_1|/m}$ is the Bogoliubov speed of sound), and energy is measured in units of the chemical potential $\delta = g_1 n_0$. In the above expressions, $g_1 = 2\hbar\omega_\perp a_0$, where $\omega_\perp$ denotes the confining frequency in the transverse direction, and a_0 is a characteristic (constant) value of the scattering length relatively close to the Feshbach resonance. Finally, $\tilde{V}(x)$ is the rescaled external trapping potential, and the x-dependent nonlinearity is given by $\tilde{g}(x) = a(x)/a_0$, where $a(x)$ is the spatially varying scattering length.

In the past few years, GP equation (1.1) has been widely studied, such as the stability and dynamics of bright, dark solitary waves [34, 37, 2, 3, 25, 24, 18, 14, 5], modulated amplitude waves (MAWs) [32] and exact solutions [38, 4, 23]. Also we can refer a number of books [26, 27, 19].

In order to study the dynamics of BECs with scattering length subjected to a spatially periodic variation, Porter *et al.* [32] transform Eq. (1.1) into a new GP equation with a constant coefficient and an additional effective potential

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + |\psi|^2\psi + \tilde{V}(x)\psi + \tilde{V}_{\mathit{eff}}(x)\psi,$$

$$\tilde{V}_{\mathit{eff}}(x) = \frac{1}{2}\frac{f''}{f} - \frac{(f')^2}{f^2} + \frac{f'}{f}\frac{\partial}{\partial x}$$

with $f(x) = \sqrt{\tilde{g}(x)}$, then the transformed equation was investigated. For weak underlying inhomogeneity, the effective potential takes a form resembling a superlattice, and the amplitude dynamics of the solutions of the constant-coefficient GP equation obey a nonlinear generalization of the Ince equation. In the small-amplitude limit, they use averaging to construct analytical solutions for modulated amplitude waves (MAWs), whose stability was subsequently examined using both numerical simulations of the original GP equation and fixed-point computations with the MAWs as numerically exact solutions. However, mentioned in their paper, the transformation $\tilde{\psi} = \sqrt{\tilde{g}(x)}\psi$ applies only in the case that $\tilde{g}(x)$ does not cross zero. A natural question is that for general periodic function $\tilde{g}(x)$, whether the similar results upon the dynamics can be obtained. On the other hand, the phases of MAWs considered in [32] are trivial, which are correspond to standing waves. Thus, another question is whether there exist the MAWs with nontrivial phases in collisionally inhomogeneous BECs.

With these questions discussed above, in this paper we investigate the existence and stability of MAWs with nontrivial phases in collisionally inhomogeneous BECs modeled by GP equation (1.1) for general small periodic function $\tilde{g}(x)$. The method is based on averaging, and we use the averaging principle to replace a GP equation by the corresponding averaged system. Along this paper, we assume that $g(x)$ and $V(x)$ are analytic and periodic functions with the least positive period $T = \pi/\sqrt{\delta}$.

The rest of the paper is organized as follows. In Sec. 2, we introduce modulated amplitude waves involving periodic and quasi-periodic, and an averaging theorem is obtained in Sec. 3. In Sec. 4 we investigate the existence and stability of equilibrium points for the averaged system and thereby study the periodic orbits and a numerical simulation is presented as prescribed parameters in Sec. 5. Finally, we summarize our results in Sec. 6.

2. Coherent Structure

We consider uniformly propagating coherent structures with the ansatz

$$\psi(t,x) = R(x)\exp(i[\Theta(x) - \mu t]), \tag{2.1}$$

where $R(x) \in \mathbb{R}$ gives the amplitude dynamics of the condensate wave function, $\theta(x)$ determines the phase dynamics, and the "chemical potential" μ, defined as the energy it takes to add one more particle to the system, is proportional to the number of atoms trapped in the condensate. When the (temporally periodic) coherent structure (2.1) is also spatially periodic, it is called a *modulated amplitude wave* (MAW) [9, 8]. Similarly, a solution of Eq. (1.1) with the (temporally periodic) coherent structure (2.1) is called a *quasi-periodic modulated amplitude wave* (QMAW) if it is also spatially quasi-periodic.

Inserting (2.1) into (1.1), we obtain the following two couple nonlinear ordinary differential equations

$$R'' + \delta R - \frac{c^2}{R^3} + \varepsilon g(x)R^3 + \varepsilon V(x)R = 0, \tag{2.2}$$

$$\Theta'' + 2\Theta' R'/R = 0 \;\Rightarrow\; \Theta'(x) = \frac{c}{R^2}, \tag{2.3}$$

where

$$\varepsilon g(x) := -2\tilde{g}(x), \quad \varepsilon V(x) := -2\tilde{V}(x), \quad \delta = 2\mu$$

and the integration constant c, determined by the velocity and number density, plays the role of "angular momentum" [7].

In case of $c = 0$, the phase of the condensate wave function (standing wave) is trivial and constant. In the general case, $c \neq 0$, the system (2.2) becomes more complicated and the phase is no longer constant [11], which implies nonzero current of the matter — it is proportional to $R^2(x)\theta(x) = c$, for amplitude $R(x)$ of MAW and nonzero constant c — along x-axis, and hence seems to have no direct relation to present experimental setting for BECs with a parabolic trap [21, 10]. For each given c, we will obtain a solution for the original GP equation. In fact, if c varies from some open interval, all solutions forms a continuum. Even the amplitude $R(x)$, a solution of (2.2), is T-periodic, the corresponding condensate wave function $\psi(x,t)$ may be not periodic, but quasi-periodic, with respect to the spatial variable x [33].

3. Averaging Theorem

Let us rewrite Eq. (2.2) in the planar equivalent form

$$\begin{cases} R' = S, \\ S' = -\delta R + \dfrac{c^2}{R^3} - \varepsilon g(x)R^3 - \varepsilon V(x)R. \end{cases} \tag{3.1}$$

In order to proceed we need to transform (3.1) to a standard form for the method of averaging.

Lemma 3.1. *Under the transformation*

$$\Psi : \mathbb{T} \times (-\infty, -\sqrt[4]{c^2/\delta}) \cup (\sqrt[4]{c^2/\delta}, +\infty) \to (-\infty, 0) \cup (0, +\infty) \times \mathbb{R}$$

defined by

$$\begin{cases} R = \rho\sqrt{\cos^2(\sqrt{\delta}x + \theta) + \dfrac{c^2}{\delta\rho^4}\sin^2(\sqrt{\delta}x + \theta)}, \\ S = \rho\sqrt{\delta}\left(\dfrac{c^2}{\delta\rho^4} - 1\right)\dfrac{\cos(\sqrt{\delta}x + \theta)\sin(\sqrt{\delta}x + \theta)}{\sqrt{\cos^2(\sqrt{\delta}x + \theta) + \frac{c^2}{\delta\rho^4}\sin^2(\sqrt{\delta}x + \theta)}}, \end{cases}$$

system (3.1) *changes into a new system*

$$\begin{cases} \rho' = \varepsilon\left\{\dfrac{g(x)\rho^3}{\sqrt{\delta}}\left[\dfrac{1}{4}\left(1 + \dfrac{c^2}{\delta\rho^4}\right)\sin 2(\sqrt{\delta}x + \theta) + \dfrac{1}{8}\left(1 - \dfrac{c^2}{\delta\rho^4}\right)\sin 4(\sqrt{\delta}x + \theta)\right]\right. \\ \qquad \left. + \dfrac{\rho}{2\sqrt{\delta}}V(x)\sin 2(\sqrt{\delta}x + \theta)\right\}, \\ \theta' = \varepsilon\left\{\dfrac{g(x)(\delta\rho^4 + c^2)}{8\delta^2\rho^2}(3 + \cos 4(\sqrt{\delta}x + \theta)) + \dfrac{g(x)(\delta^2\rho^8 + c^4)}{2\delta^2\rho^2(\delta\rho^4 - c^2)}\right. \\ \qquad \left. \times \cos 2(\sqrt{\delta}x + \theta) + \dfrac{1}{2\delta}V(x)\left(1 + \dfrac{\delta\rho^4 + c^2}{\delta\rho^4 - c^2}\cos 2(\sqrt{\delta}x + \theta)\right)\right\} \end{cases} \tag{3.2}$$

with the new coordinates (θ, ρ) *in the plane* $\mathbb{T} \times (-\infty, -\sqrt[4]{c^2/\delta}) \cup (\sqrt[4]{c^2/\delta}, +\infty)$.

The proof of Lemma 3.1 follows from the basic computation (may be lengthy), and it can be found in [33] where only the half-plane is considered. The transformation Ψ arises from the variation of constant by using the solutions of the unperturbed system ($\varepsilon = 0$), and ρ plays the role of "energy". When taking the integration constant $c = 0$, the transformation Ψ is the usual change of polar coordinates. Because the transformation in Lemma 3.1 is already periodic in x, if (ρ, θ) are equilibrium points at the averaged system (3.2) with $\varepsilon = 0$, then the corresponding solution of (3.1) are already periodic solutions.

Now write the T-periodic functions $g(x), V(x)$ as the Fourier series

$$g(x) = g_0 + \sum_{k=1}^{\infty}(\alpha_k \sin 2k\sqrt{\delta}x + \beta_k \cos 2k\sqrt{\delta}x), \tag{3.3}$$

$$V(x) = v_0 + \sum_{k=1}^{\infty}(a_k \sin 2k\sqrt{\delta}x + b_k \cos 2k\sqrt{\delta}x). \tag{3.4}$$

After inserting (3.3) and (3.4) into (3.2) and then multiplying the right side of (3.2) by $1/T$ and integrating from 0 to T, we obtain the averaged system

$$\begin{cases} \bar{\rho}' = \varepsilon\left\{\dfrac{\delta\bar{\rho}^4 + c^2}{8\delta\sqrt{\delta}\bar{\rho}}A\sin(2\bar{\theta}+\phi_1) + \dfrac{\bar{\rho}}{4\sqrt{\delta}}B\sin(2\bar{\theta}+\phi_2)\right\} := \varepsilon F_1(\bar{\theta},\bar{\rho}), \\ \bar{\theta}' = \varepsilon\left\{\dfrac{3g_0(\delta\bar{\rho}^4 + c^2)}{8\delta^2\bar{\rho}^2} + \dfrac{(\delta^2\bar{\rho}^8 + c^4)}{4\delta^2\bar{\rho}^2(\delta\bar{\rho}^4 - c^2)}A\cos(2\bar{\theta}+\phi_1)\right. \\ \qquad \left. + \dfrac{v_0}{2\delta} + \dfrac{1}{4\delta}\dfrac{\delta\bar{\rho}^4 + c^2}{\delta\bar{\rho}^4 - c^2}B\cos(2\bar{\theta}+\phi_2)\right\} := \varepsilon F_2(\bar{\theta},\bar{\rho}), \end{cases} \tag{3.5}$$

where

$$A = \sqrt{\alpha_1^2 + \beta_1^2}, \quad B = \sqrt{a_1^2 + b_1^2},$$

$$\phi_1 = \arctan\frac{\alpha_1}{\beta_1}, \quad \left(\phi_1 = \frac{\pi}{2}\cdot\mathrm{sign}(\phi_1),\ \text{if } \beta_1 = 0\right),$$

$$\phi_1 = \arctan\frac{a_1}{b_1}, \quad \left(\phi_1 = \frac{\pi}{2}\cdot\mathrm{sign}(a_1),\ \text{if } b_1 = 0\right).$$

Theorem 3.2 (Averaging theorem). *There exists a $c^r, r \geq 2$, change of variables*

$$\rho = \bar{\rho} + \varepsilon w_1(\bar{\theta},\bar{\rho},x,\varepsilon), \quad \theta = \bar{\theta} + \varepsilon w_2(\bar{\theta},\bar{\rho},x,\varepsilon)$$

with w_1, w_2 T-periodic functions of x, transforming (3.2) *into*

$$\begin{cases} \bar{\rho}' = \varepsilon F_1(\bar{\theta},\bar{\rho}) + \varepsilon^2 g_1(\bar{\theta},\bar{\rho},x,\varepsilon) \\ \bar{\theta}' = \varepsilon F_2(\bar{\theta},\bar{\rho}) + \varepsilon^2 g_2(\bar{\theta},\bar{\rho},x,\varepsilon), \end{cases} \tag{3.6}$$

with g_1, g_2 T-periodic functions of x. Moreover,

(i) *If $(\theta_\varepsilon(x),\rho_\varepsilon(x))$ and $(\bar{\theta}(x),\bar{\rho}(x))$ are solutions of the original system* (3.2) *and averaged system* (3.5) *respectively, with the initial value such that*

$$|(\rho_\varepsilon(0),\theta_\varepsilon(0)) - (\bar{\rho}(0),\bar{\theta}(0))| = \mathcal{O}(\varepsilon),$$

then

$$|(\rho_\varepsilon(x),\theta_\varepsilon(x)) - (\bar{\rho}(x),\bar{\theta}(x))| = \mathcal{O}(\varepsilon),$$

for the spatial variable x of order $1/\varepsilon$.

(ii) *If P_0 is an equilibrium point of system* (3.5) *and there exists a neighborhood $U(r;P_0)$ of P_0 (with radius r and center P_0) such that there is not another equilibrium point in the closure of U and*

$$\deg(F,U,P_0) \neq 0$$

with $F = (F_1,F_2)$. Then for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\varphi_\varepsilon(x)$ of system (3.2) *such that*

$$\varphi_\varepsilon(\cdot,\varepsilon) \to P_0 \quad \text{as } \varepsilon \to 0.$$

(iii) *If* P_0 *is an equilibrium point of* (3.5) *such that the corresponding Jacobian matrix has no eigenvalue equal to zero, then* (3.2) *admits a* T*-periodic solution* $(\theta_\varepsilon(x), \rho_\varepsilon(x))$ *such that* $|(\rho_\varepsilon(x), \theta_\varepsilon(x)) - P_0| = \mathcal{O}(\varepsilon)$, *for sufficiently small* ε; *if* P_0 *is a hyperbolic equilibrium point of* (3.5), *then there exists* $\epsilon_0 > 0$ *such that, for all* $0 < \varepsilon < \varepsilon_0$, *system* (3.2) *possesses a hyperbolic periodic orbit* $\gamma_\varepsilon(x) = P_0 + \mathcal{O}(\varepsilon)$ *of the same stability type as* P_0.

(iv) *If* $(\theta_\varepsilon(x), \rho_\varepsilon(x)) \in W^s(\gamma_\varepsilon(x))$ *is a solution of system* (3.2) *lying in the stable manifold of the hyperbolic periodic orbit* $\gamma_\varepsilon(x) = P_0 + \mathcal{O}(\varepsilon)$, $(\bar\theta(x), \bar\rho(x)) \in W^s(P_0)$ *is a solution of system* (3.5) *lying in the stable manifold of the hyperbolic equilibrium point* P_0 *and*

$$|(\theta_\varepsilon(0)), \rho_\varepsilon(0) - (\bar\theta(0), \bar\rho(0))| = \mathcal{O}(\varepsilon),$$

then

$$|(\theta_\varepsilon(x), \rho_\varepsilon(x)) - (\bar\theta(x), \bar\rho(x))| = \mathcal{O}(\varepsilon),$$

for $x \in [0, +\infty)$. *Similar results apply to solutions lying in the instable manifold on the interval* $x \in (-\infty, 0]$.

Proof. The proof of (i), (iii) and (iv) follows directly from [6] (also see [15]). The proof of (ii) is based on a framework of coincidence degree theory. Without loss of generality, we assume $P_0 = 0$. We define homotopy operator $\mathcal{H} : C([0,T], \mathbb{R}^2) \times [0,1] \to L_1([0,T], \mathbb{R}^2)$ by

$$\mathcal{H}(\bar\theta, \bar\rho, \lambda) := \varepsilon F(\bar\theta, \bar\rho) + (1-\lambda)\varepsilon^2 G(\bar\theta, \bar\rho, x, \varepsilon),$$

where $G := (g_1, g_2)$. According to [22], $\mathcal{H}$ is L-compact on $\Omega \times [0,T]$, where Ω is a bounded open set of $C([0,T], \mathbb{R}^2)$ defined by

$$\Omega := \{(\bar\theta, \bar\rho) \in C([0,T], \mathbb{R}^2) : \|(\bar\theta, \bar\rho)\| < r\}.$$

We remark that $(\bar\theta, \bar\rho) \in \overline{\Omega}$ is a T-periodic solution of system (3.6) if and only if $(\bar\theta, \bar\rho)$ is a solution of $\mathcal{L}(\bar\theta, \bar\rho) = \mathcal{H}(\bar\theta, \bar\rho, 0)$ in $\overline{\Omega}$. Since there is not another equilibrium point in the closure of U, we let

$$M_1 = \min_{(\bar\theta, \bar\rho) \in \partial U} |F_1(\bar\theta, \bar\rho) - F_2(\bar\theta, \bar\rho)| > 0,$$

$$M_2(\varepsilon) = \varepsilon \max\{|g_1(\bar\theta, \bar\rho, x, \varepsilon) - g_1(\bar\theta, \bar\rho, x, \varepsilon)| : (\bar\theta, \bar\rho, x, \varepsilon) \in \overline{U} \times [0,T]\}$$

with $M_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. We also assume $M_2(\varepsilon) > 0$, otherwise P_0 is a solution of system (3.6) and the result is proved.

First, we claim that, for each $\varepsilon \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0)$ with $\varepsilon_0 = \max\{|\varepsilon| : M_2(\varepsilon) \le M_1\}$, there exists no solution $(\bar\theta, \bar\rho) \in \partial\Omega$ for the operator equation

$$\mathcal{L}(\bar\theta, \bar\rho) = \mathcal{H}(\bar\theta, \bar\rho, \lambda), \quad \lambda \in (0,1]. \tag{3.7}$$

In fact, if $(\bar\theta, \bar\rho) \in \partial\Omega$ is a solution of (3.7), then there exists $\xi \in [0,T]$ such that

$$\bar\theta'(\xi) + \bar\rho'(\xi) = 0,$$

$$\|(\bar\theta, \bar\rho)\| = \max_{x \in [0,T]} \sqrt{\bar\theta^2(x) + \bar\rho^2(x)} = \sqrt{\bar\theta^2(\xi) + \bar\rho^2(\xi)}$$

and

$$\bar{\theta}'(\xi)\bar{\rho}'(\xi) \le 0.$$

Thus, it follows that

$$\begin{aligned} 0 &= |\bar{\theta}'(\xi) + \bar{\rho}'(\xi)| \\ &\ge |\varepsilon| \cdot |F_1(\bar{\theta}, \bar{\rho}) - F_2(\bar{\theta}, \bar{\rho})| - \varepsilon^2 |g_1(\bar{\theta}, \bar{\rho}, x, \varepsilon) - g_2(\bar{\theta}, \bar{\rho}, x, \varepsilon)| \\ &\ge |\varepsilon| M_1 - \varepsilon M_2(\varepsilon) > 0, \end{aligned}$$

which is a contradiction.

Without loss of generality, we suppose that

$$\mathcal{L}(\bar{\theta}, \bar{\rho}) \ne \mathcal{H}(\bar{\theta}, \bar{\rho}, \lambda), \quad (\bar{\theta}, \bar{\rho}) \in \partial\Omega \tag{3.8}$$

holds for $\lambda \in [0, 1]$. Otherwise, the result is proved for $(\bar{\theta}, \bar{\rho}) \in \partial\Omega$. Thus, we can apply the homotopy property of the coincidence degree and obtain

$$\begin{aligned} |D_{\mathcal{L}}(\mathcal{L} - \mathcal{H}(\cdot, 0), \Omega)| &= |D_{\mathcal{L}}(\mathcal{L} - \mathcal{H}(\cdot, 1), \Omega)| \\ &= |\deg(F, \Omega \cap \mathbb{R}^2, 0)| \ne 0. \end{aligned}$$

Hence, by the existence property of the coincidence degree, there is $(\bar{\theta}, \bar{\rho}) \in \overline{\Omega}$ such that $\mathcal{L}(\bar{\theta}, \bar{\rho}) = \mathcal{H}(\bar{\theta}, \bar{\rho}, 0)$. Then $(\bar{\theta}, \bar{\rho})$ is a T-periodic solution of (3.6). Thus, owing to the change of variables in this theorem, there is a T-periodic solution (θ, ρ) for system (3.2). □

We remark that the proof of part (iv) of Theorem 3.2 does not need the smoothness condition upon F. So, it is convenient to deal with the existence of periodic solutions for nonlinear systems with loss of smoothness by Theorem 3.2.

4. Periodic Orbits and Spatial Stability

To study the dynamics of MAWs or QMAWs for system (1.1), we must investigate the behavior of the periodic orbits for system (3.1). According to the method of averaging, the equilibrium point of the averaged system determines the properties of the periodic orbit of the corresponding perturbed system. For example, the equilibrium point with its eigenvalue of linearization nonzero implies that there exists at least one periodic orbit for the original system; in addition, if the equilibrium point is hyperbolic, then the periodic orbit has the same type of stability as the equilibrium point, for sufficiently small parameter ε.

In order to find periodic orbits of system (3.2), it is sufficient to find equilibrium points of the averaged system (3.5). For simplification of computation, we assume that

$$\phi_2 = \pi + \phi_1, \quad \phi_1 = -\pi/2, \quad g_0 = A/3, \quad v_0 = -B/2.$$

Recalling system (3.5), together with the assumption above, we have the averaged system

$$\begin{cases} \bar{\rho}' = \varepsilon \left\{ \dfrac{A}{8\sqrt{\delta}\bar{\rho}} \left(\bar{\rho}^4 - 2\dfrac{B}{A}\bar{\rho}^2 + \dfrac{c^2}{\delta} \right) \sin(2\bar{\theta} + \phi_1) \right\}, \\ \bar{\theta}' = \varepsilon \left\{ \dfrac{1}{4\delta\bar{\rho}^2 \cdot (\delta\bar{\rho}^4 - \frac{c^2}{\delta})} \left(\dfrac{A}{2} \left(\bar{\rho}^8 - \dfrac{c^4}{\delta^2} \right) - B\bar{\rho}^2 \left(\bar{\rho}^4 - \dfrac{c^2}{\delta} \right) \right. \right. \\ \qquad \left. \left. + \left[A \left(\bar{\rho}^8 + \dfrac{c^4}{\delta^2} \right) - B\bar{\rho}^2 \left(\bar{\rho}^4 + \dfrac{c^2}{\delta} \right) \right] \cos(2\bar{\theta} + \phi_2) \right) \right\}. \end{cases} \tag{4.1}$$

Notice that there exists a constant $c_0 > 0$ such that for each $c \in (0, c_0)$, the equation

$$\bar{\rho}^4 - 2\frac{B}{A}\bar{\rho}^2 + \frac{c^2}{\delta} = 0 \tag{4.2}$$

has at least two real roots

$$\rho_{1,2} = \pm\sqrt{\frac{B}{A} + \sqrt{\frac{B^2}{A^2} - \frac{c^2}{\delta}}} \in (-\infty, -\sqrt[4]{c^2/\delta}) \cup (\sqrt[4]{c^2/\delta}, +\infty).$$

Moreover, Eq. (4.2) implies that

$$\frac{A}{2}\left(\bar{\rho}^8 - \frac{c^4}{\delta^2} \right) - B\bar{\rho}^2 \left(\bar{\rho}^4 - \frac{c^2}{\delta} \right) = 0.$$

Thus, we can find four equilibrium points as follows

$$P_{1,2}: \left(\pm\sqrt{\frac{B}{A} + \sqrt{\frac{B^2}{A^2} - \frac{c^2}{\delta}}},\ k\pi + \frac{\pi}{4} - \frac{\phi_1}{2} \right),$$

$$P_{3,4}: \left(\pm\sqrt{\frac{B}{A} + \sqrt{\frac{B^2}{A^2} - \frac{c^2}{\delta}}},\ k\pi - \frac{\pi}{4} - \frac{\phi_1}{2} \right), \quad k \in \mathbb{Z}.$$

The eigenvalues of the linearization at the equilibrium points $P_{1,2}$ and $P_{3,4}$ are given by

$$\lambda_{1,2}^{(1)} = \frac{\varepsilon A}{2\sqrt{\delta}}\sqrt{\frac{B^2}{A^2} - \frac{c^2}{\delta}} > 0, \quad \lambda_{1,2}^{(2)} = -\frac{\varepsilon A(\rho_{1,2}^4 - \frac{c^2}{\delta})}{4\delta\rho_{1,2}^2} < 0$$

and $\lambda_{3,4}^{(1)} = -\lambda_{1,2}^{(1)}$, $\lambda_{3,4}^{(2)} = -\lambda_{1,2}^{(2)}$, respectively. So, the equilibrium points $P_{1,2}$ and $P_{3,4}$ are hyperbolic, and as a consequence persist as periodic orbits for system (3.2); in addition, these periodic orbits are hyperbolic.

If $\bar{\theta} = k\pi - \phi_1/2$, to find the equilibrium points, one will solve the following algebraic equation

$$\frac{A}{2}\left(\bar{\rho}^8 - \frac{c^4}{\delta^2} \right) - B\bar{\rho}^2 \left(\bar{\rho}^4 - \frac{c^2}{\delta} \right) + A\left(\bar{\rho}^8 + \frac{c^4}{\delta^2} \right) - B\bar{\rho}^2 \left(\bar{\rho}^4 + \frac{c^2}{\delta} \right) = 0. \tag{4.3}$$

Equation (4.3) has at least two roots

$$\rho_{3,4} = \pm\sqrt{\frac{4B}{3A} + o(c)} \in (-\infty, -\sqrt[4]{c^2/\delta}) \cup (\sqrt[4]{c^2/\delta}, +\infty),$$

for sufficiently small integration constant $c > 0$.

If $\bar{\theta} = k\pi + \pi/2 - \phi_1/2$, the algebraic equation

$$f(\rho) := \rho^8 - \frac{B}{A}\rho^2 + \frac{3c^4}{\delta^2} = 0$$

needs to be solved. Note that

$$f\left(\pm\sqrt[4]{\frac{c^2}{\delta}}\right) = 4\frac{c^4}{\delta^2} - \frac{B}{A}\left(\frac{c^4}{\delta^2}\right)^{3/2} < 0 \quad \text{and} \quad f(\pm\infty) = +\infty, \tag{4.4}$$

for sufficiently small positive constant c. By the mean value theorem, Eq. (4.4) has two roots $\rho_{5,6}$ such that

$$\rho_{5,6} = \pm\sqrt[6]{\frac{c^2 B}{\delta A} + o(c^2)} \in (-\infty, -\sqrt[4]{c^2/\delta}) \cup (\sqrt[4]{c^2/\delta}, +\infty),$$

for sufficiently small c. As a consequence, four equilibrium points of the averaged system (4.1) are obtained as follows

$$P_{5,6}: \left(\pm\sqrt{\frac{4B}{3A} + o(c)},\ k\pi - \frac{\phi_1}{2}\right),$$

$$P_{7,8}: \left(\pm\sqrt[6]{\frac{c^2 B}{\delta A} + o(c^2)},\ k\pi + \frac{\pi}{2} - \frac{\phi_1}{2}\right), \quad k \in \mathbb{Z}.$$

The eigenvalues of the linearization at the equilibrium points $P_{5,6}$ and $P_{7,8}$ are given by

$$\lambda_{5,6}^{(1,2)} = \pm i\varepsilon\sqrt{\frac{4B^2(\frac{8B^2}{9A} - \frac{c^2}{\delta})}{3\delta(\rho_{3,4}^4 - \frac{c^2}{\delta})}}$$

and

$$\lambda_{7,8}^{(1,2)} = \pm\varepsilon\sqrt{\frac{2A^2c^4}{4\delta^{7/2}\rho_{5,6}^4(\rho_{5,6}^4 - \frac{c^2}{\delta})}\left(\frac{2B}{A}\rho_{5,6}^2 - \rho_{5,6}^4 - \frac{c^2}{\delta}\right)},$$

respectively. The equilibrium points $P_{7,8}$ imply that two hyperbolic periodic orbit of system (3.2) exist; while the equilibrium points $P_{5,6}$ are nonlinear centers, and also persist as periodic orbits for system (3.2). The phase portrait for system (4.1) is given in Fig. 1.

In summary, for the equilibrium P_i $(i = 1, \ldots, 4, 7, 8)$, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$, system (3.2) possesses hyperbolic periodic orbits $\gamma_\varepsilon(x) = P_i + \mathcal{O}(\varepsilon)$, which are unstable with respect to spatial variable x.

We remark that, if $(\theta_\varepsilon(x), \rho_\varepsilon(x)) \in W_s(\gamma_\varepsilon(x))$ is a solution of system (3.2) lying in the stable manifold of the hyperbolic periodic orbit $\gamma_\varepsilon(x) = P_i + \mathcal{O}(\varepsilon)$, $(\bar{\theta}(x), \bar{\rho}(x)) \in W_s(P_0)$ is a solution of system (4.1) lying in the stable manifold of the hyperbolic equilibrium P_i $(i = 1, \ldots, 4, 7, 8)$ and $|(\theta_\varepsilon(0), \rho_\varepsilon(0)) - (\bar{\theta}(0), \bar{\rho}(0))| = \mathcal{O}(\varepsilon)$, then $|(\theta_\varepsilon(x), \rho_\varepsilon(x)) - (\bar{\theta}(x),$

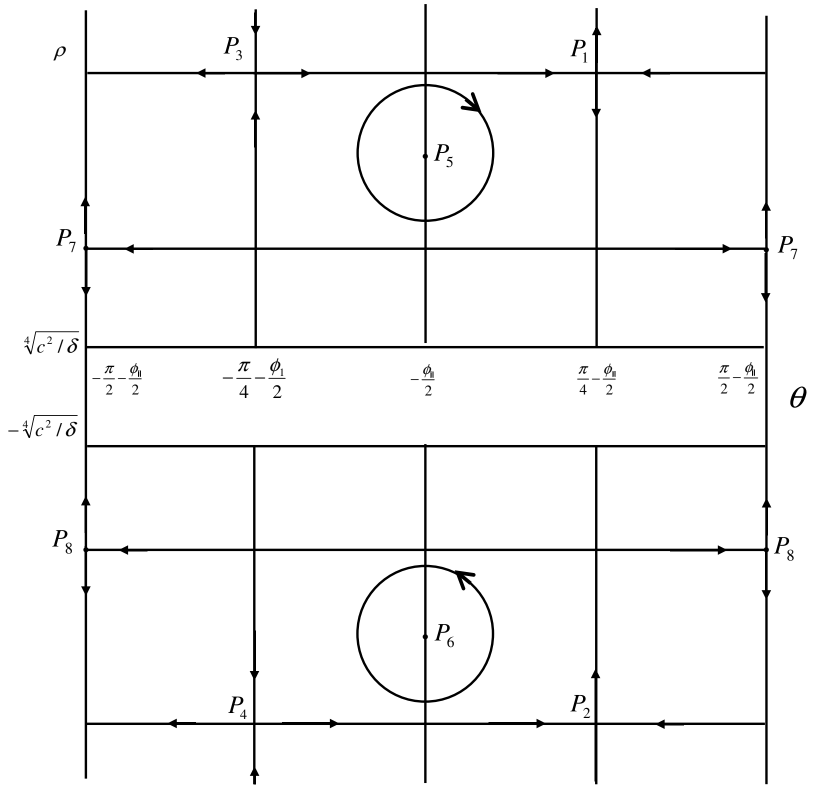

Fig. 1. The phase portrait associated with the averaged system (4.1). All the equilibrium points persist as periodic orbits for system (3.2).

$\bar{\rho}(x))| = \mathcal{O}(\varepsilon)$, for all $x \in [0, +\infty)$. Similar results apply to solutions lying in the unstable manifold on the interval $x \in (-\infty, 0]$.

We also remark that Eq. (1.1) is invariant with respect to transformation $\psi \to \mathrm{e}^{i\theta_0}\psi$, for any constant value θ_0. Since $R(x)$ has the same sign to $\rho(x)$, $-R(x) = R(x)\mathrm{e}^{i\pi}$ and $\Theta(x) = \int_0^x \frac{c^2}{R^2(s)}\mathrm{d}s$, then we have

$$-R(x)\mathrm{e}^{i[\Theta(x)-\mu t]} = -\psi(t,x) = \mathrm{e}^{i\pi}R(x)\mathrm{e}^{i[\Theta(x)-\mu t]} = \mathrm{e}^{i\pi}\psi(t,x).$$

Thus, the points in Fig. 1 with different signs of ρ may be correspond to the same physical solution.

Although we study system (4.1) for sufficiently small $c > 0$, c also can be taken on a open interval $(0, \bar{c})$, for some positive constant $\bar{c}$. By continuous dependence of solutions with respect to the parameters, there is a connected set $\mathcal{C}$ of T-periodic solutions for system (3.2) and then for system (3.1). Since

$$\begin{aligned}\psi(t,x) &= R(x)\exp i[\Theta(x) - \mu t]\\ &= R(x)(\cos[\bar{\Theta}(x) + \nu x - \mu t] + i\sin[\bar{\Theta}(x) + \nu x - \mu t]),\end{aligned}$$

where

$$\nu = \frac{1}{T}\int_{x_0}^{x_0+T} \frac{c}{R^2(\xi)}\mathrm{d}\xi$$

and $\bar{\Theta}(x) = \Theta(x) - \nu$ is a T-periodic function with zero mean value, whether $\psi(t,x)$ is a MAW or QMAW depends on the choosing of the integration constant c. Precisely, if $2\pi/\nu$ and T are rationally related, then $\psi(x,t)$ is a MAW; if $2\pi/\nu$ and T are rationally irrelevant, then $\psi(x,t)$ is not periodic but quasi-periodic, which is corresponding to a QMAW with the frequency $\omega = \langle 2\pi/\nu, T\rangle$.

5. Numerical Simulation

To demonstrate the process of averaging to BECs, a specific example of numerical computation is given in the following. We take

$$g(x) = \frac{1}{2} - \frac{3}{2}\sin 2x,$$
$$V(x) = 2\sin 2x - 1$$

and the parameters $\delta = 1, c = 0.5, \varepsilon = 0.01$. Obviously, $g(x)$ crosses zero. We can find equilibrium points for system (4.1) in the (θ,ρ)-coordinates as follows

$$P_{1,2}: (k\pi, \pm 1.60), \quad P_{3,4}: \left(k\pi + \frac{\pi}{2}, \pm 1.60\right),$$
$$P_{5,6}: \left(k\pi + \frac{3\pi}{4}, \pm 1.02\right), \quad P_{7,8}: \left(k\pi + \frac{\pi}{4}, \pm 1.33\right).$$

$P_{7,8}$ are nonlinear centers with eigenvalues of the linearization $\lambda_7^{(1,2)} = \lambda_8^{(1,2)} = \pm i\varepsilon$.

Using the transformation Ψ, these equilibrium points in the (R,S)-coordinates with $x = 0$ are given by

$$\tilde{P}_{1,2}: (\pm 1.60, 0), \quad \tilde{P}_{3,4}: (\pm 0.31, 0),$$
$$\tilde{P}_{5,6}: (\pm 0.80, \pm 0.48), \quad \tilde{P}_{7,8}: (\pm 0.98, \mp 0.82).$$

We plot the solutions of system (3.1) starting from $\tilde{P}_i, i = 1, 2, \ldots, 8$, according to the averaged theorem, which are good approximations to the periodic orbits, see Fig. 2.

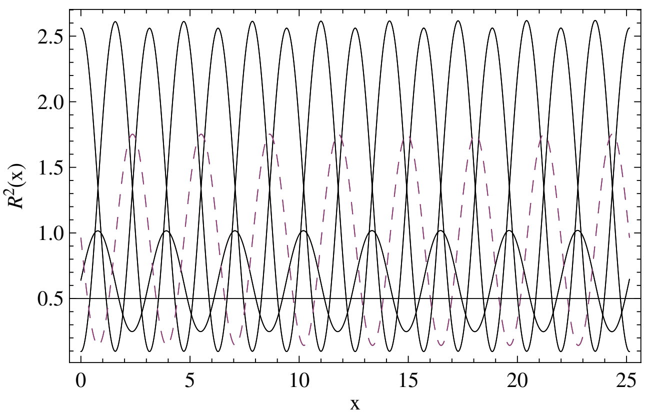

Fig. 2. The solutions portrait of system (3.1) with initial value $(R(0), R'(0)) = \tilde{P}_i, i = 1, 2, \ldots, 8$. Here, we take $g(x) = \frac{1}{2} - \frac{3}{2}\sin 2x, V(x) = 2\sin 2x - 1$ and the parameters $\delta = 1, c = 0.5, \varepsilon = 0.01$. The solutions with solid lines are unstable with respect to spatial variable x. The plot of $R^2(x)$ for $\tilde{P}_i$ and $\tilde{P}_{i+1}$ correspond to the same Curve.

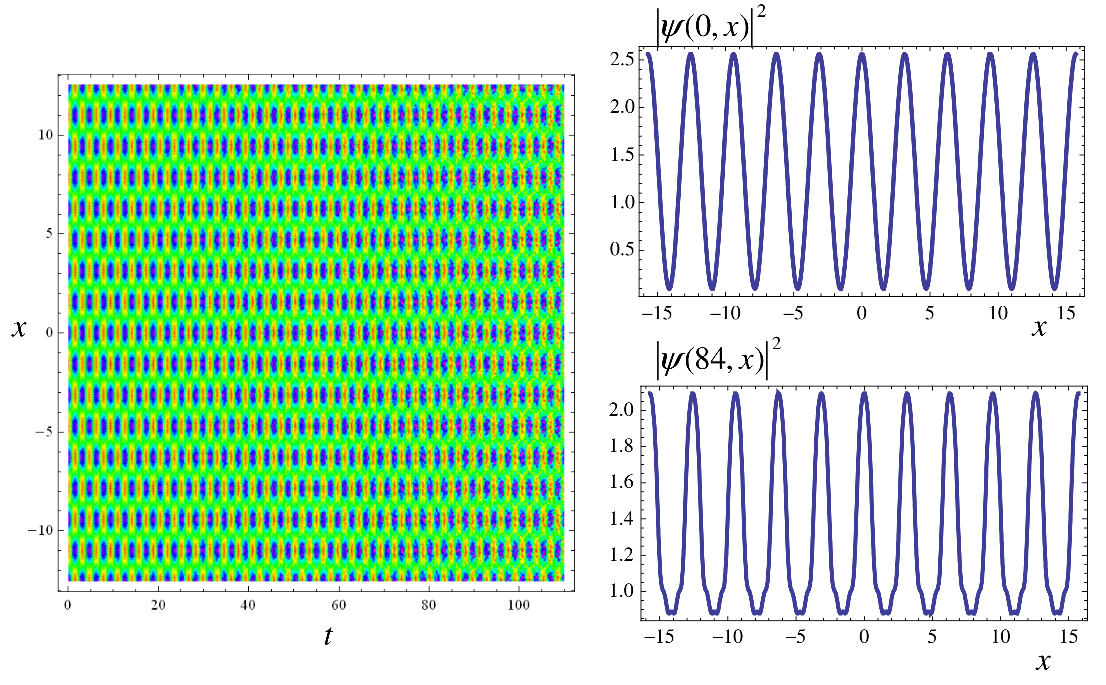

Fig. 3. Dynamics evolution of MAW of GP equation (1.1) for $\varepsilon = 0.01, g(x) = \frac{1}{2} - \frac{3}{2}\sin 2x, V(x) = 2\sin 2x - 1$. The left panel shows the space-time contour plot of the square modulus (density) $|\psi|^2$ of the solution, and the right panels show two snapshots (spatial profiles) of the spatio-temporal evolution.

Returning to the original GP equation, in Fig. 3 we examine its dynamics with direct simulations of Eq. (1.1) with the initial value $\psi(0, x)$ near the solution $R(x)\mathrm{e}^{i\Theta(x)}$ corresponding to the equilibrium $P_i, i = 1, 2$. From the simulation, the points in Fig. 1 with different sings of ρ correspond to the same physical solution.

6. Conclusion

In conclusion, we have presented first-order averaging theorem in the periodic case for dynamics of MAWs (or QMAWs) in collisionally inhomogeneous BECs. The questions as mentioned in the introduction have been answered. When the sufficiently small scattering length $a(x)$ varies periodically in spatial variable x and crosses zero, infinitely many (positive measure set) MAWs and QMAWs can be proved to exist by adjusting the integration constant c on some open interval.

A numerical approximation of periodic orbits is given for some prescribed parameters. We remark that, expanding at each equilibrium point and combining with multiple scale perturbed theory, such as work in [30, 28, 29, 31], there may be a better approximation for continuation of each periodic orbit.

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