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## TIDAL TENSORS IN THE DESCRIPTION OF GRAVITY AND ELECTROMAGNETISM

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In 2008–2009, L. F. O. Costa and C. A. R. Herdeiro proposed a new gravito-electromagnetic analogy, based on tidal tensors. We show that connections on the tangent bundle of the space-time manifold help in finding an advantageous geometrization of their ideas. Moreover, the combination tidal tensors — connections on tangent bundle can underlie a common mathematical description of the main equations of gravity and electromagnetism.

*Keywords:* Tangent bundle; spray; Ehresmann connection; tidal tensor; Einstein–Maxwell equations.

Mathematics Subject Classification 2010: 53Z05, 53B05, 53B40, 53C60, 83C22

### 1. Introduction

In two recent papers, [7, 8], L. F. O. Costa and C. A. R. Herdeiro provided a new gravito-electromagnetic analogy, meant to overcome the limitations of the two classical ones — the linearized approach, which is only valid in the case of a weak gravitational field and the one based on Weyl tensors, which compares tensors of different ranks (an interesting, related approach is also made in [9]). The central role in this analogy is played by worldline deviation equations and the resulting tidal tensors; it is in terms of these tensors that the fundamental equations of the gravitational and electromagnetic fields are expressed and compared. We argue that this idea is a natural one and it can underlie not only an analogy between the two fields, but also a common geometric language for them.

Still, in the cited papers, in order to be able to make such an analogy, it is imposed a restriction: in the case of worldline deviation for charged particles in flat Minkowski space, covariant derivatives of the deviation vector field  $w$  are required to identically vanish along the initial worldline.

We have shown in a previous paper, [18], that, by raising to the tangent bundle  $TM$  of the space-time manifold and using an appropriate 1-parameter family of Ehresmann

connections  $\overset{\alpha}{N}$ , any restriction upon  $w$  becomes unnecessary — the “work” of eliminating the unwanted term in the worldline deviation equation is taken by the adapted frame. Moreover, the obtained tidal tensor expressions for the Einstein and Maxwell equations are valid not only in the case when we have either gravity only, or electromagnetism only (as in [7, 8]), but also in the general case, when both fields are present; thus, connections on the space-time tangent bundle, together with tidal tensors, can provide a common geometric language for the two physical fields.

Ehresmann connections give rise to very convenient frames on  $TM$ : adapted frames. Also, to a given Ehresmann connection (not necessarily a linear one), one can naturally attach a covariant differentiation law for tensors on  $TM$ . This mechanism of building covariant differentiation laws starting from an Ehresmann connection is currently used in a geometric theory of differential equations, [1, 6]. In the present paper, we will use it (in [18], we had used different covariant derivatives), because it offers a series of advantages: (1) lifts of worldlines of charged particles are autoparallel curves; (2) worldline deviation equations become similar to the usual Jacobi equations (with no restrictions needed upon the deviation vector field); (3) the obtained Ricci tensors, to be further used in a variational theory, can be very easily obtained from tidal tensors.

Thus, information regarding gravity will be encoded in the metric, while information regarding the electromagnetic field will be contained in the connections  $\overset{\alpha}{N}$  and in the corresponding covariant differentiation laws  $\overset{\alpha}{D}$ . This idea was first proposed by Miron and collaborators<sup>a</sup> in [15, 13, 14] and we adopted it here as it leads to simpler computations than other approaches on  $TM$  and to elegant equations; still, we use different connections  $\overset{\alpha}{N}$ ,  $\overset{\alpha}{D}$ , meant to offer a more convenient expression for worldline deviation equations.

The paper is organized as follows. In Sec. 2, we present the elements of the gravito-electromagnetic analogy by Costa and Herdeiro ([7, 8]) which are necessary in the subsequent. In Sec. 3, we present known results regarding Ehresmann connections and attached covariant differentiation laws. In Sec. 4, we determine those connections on  $TM$  which lead to the simplest geodesic deviation equations; the results in this section are an application of the equation ([3]) of deviations of affine geodesics on fiber bundles. In Sec. 5, we introduce the connections which “fit” the equations of charged particles subject to gravitational and electromagnetic fields. Section 6 is devoted to the expressions of the basic equations of the two physical fields in terms of tidal tensors. In the last section, we rewrite in terms of adapted frames Costa and Herdeiro’s equations and point out that the restriction imposed in ([7, 8]) upon the deviation vector field is no longer necessary.

## 2. Tidal Tensors and Gravito-Electromagnetic Analogy

Consider a 4-dimensional Lorentzian manifold  $(M, g)$ , with signature  $(+, -, -, -)$ , regarded as space-time manifold, with local coordinates  $x = (x^i)_{i=\overline{0,3}}$  and Levi-Civita connection  $\nabla$ ,

<sup>a</sup>Other attempts of unifying gravity and electromagnetism, based on tangent bundle geometry, try to include also information regarding electromagnetism in the metric tensor — thus getting Finslerian (Randers-type, Beil-type etc.) metrics, ([5, 4]). Also, recently, Wanas, Youssef and Sid-Ahmed produced a description, [19], based on teleparallelism on  $TM$ . Another version, using complex Lagrange geometry, is proposed by Munteanu, [16].

with coefficients  $\gamma^i_{jk}$  and curvature tensor  $r$ . Throughout the paper, we will mean by  $(\partial_i)$  the natural basis of the module of vector fields on  $M$ ; the speed of light  $c$  and the gravitational constant  $k$  will be considered equal to 1.

Worldlines of particles subject to *gravity only* are geodesics  $s \mapsto (x(s))$  of  $(M, g)$ :

$$\frac{\nabla \dot{x}^i}{ds} = 0, \tag{2.1}$$

where  $s$  is the natural parameter (i.e.  $g_{ij}\dot{x}^i\dot{x}^j = 1$ ). Curvature of space-time becomes manifest in the *geodesic deviation equation*:

$$\frac{\nabla^2 w^i}{ds^2} = e^i_k w^k, \tag{2.2}$$

where  $w = w^i \partial_i$  is the deviation vector field and  $e = e^i_j dx^j \otimes \partial_i$ ,  $e^i_k = r^i_{jkl} \dot{x}^j \dot{x}^l$  is the *tidal (electrogravitic) tensor*.<sup>b</sup>

On the other side, in special relativity (where  $g_{ij} = \text{diag}(1, -1, -1, -1)$ ), the electromagnetic field is described by the 4-potential 1-form  $A = A_i(x) dx^i$  and the electromagnetic 2-form  $F = dA$ , i.e.

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j, \quad F_{ij} = \partial_i A_j - \partial_j A_i. \tag{2.3}$$

Worldlines of charged particles subject to an electromagnetic field are solutions of the Lorentz equations:

$$\frac{\nabla \dot{x}^i}{ds} = \frac{q}{m} F^i_j \dot{x}^j; \tag{2.4}$$

here,  $\frac{\nabla \dot{x}^i}{ds} = \frac{d^2 x^i}{ds^2}$ ,  $q$  is the electric charge of the particle and  $m$ , its mass. For families of worldlines of particles with same ratio  $\frac{q}{m}$ , one can determine the *worldline deviation equation*:

$$\frac{\nabla^2 w^i}{ds^2} = \frac{q}{m} \left( E^i_k w^k + F^i_k \frac{\nabla w^k}{ds} \right), \tag{2.5}$$

where

$$E^i_k = \dot{x}^j \nabla_{\partial_k} F^i_j. \tag{2.6}$$

Obviously, Eqs. (2.4) and (2.5) also hold true in the case when both gravity and electromagnetism are present, with the only difference that, in this case, the covariant derivative is no longer trivial.

We notice the appearance of a term containing the derivatives  $\frac{\nabla w^k}{ds}$  in the right-hand side of (2.5), which hinders an analogy between (2.2) and (2.5). This problem is solved in

<sup>b</sup>Here, we have used a different sign convention for the curvature tensor ( $r^i_{jkl} = \partial_l \gamma^i_{jk} - \partial_k \gamma^i_{jl} + \gamma^h_{jk} \gamma^i_{hl} - \gamma^h_{jl} \gamma^i_{hk}$ ) than in [7, 8], resulting in a different sign for  $e^i_j$ .

[7], by imposing the restriction that, along the initial worldline:

$$\frac{\nabla w^i}{ds} = 0; \tag{2.7}$$

under this assumption, the worldline deviation equations reduce to:

$$\frac{\nabla^2 w^i}{ds^2} = \frac{q}{m} E^i_k w^k, \tag{2.8}$$

which makes it possible to compare (2.2) and (2.5). Following the analogy  $E^i_k \sim e^i_k$ , [7, 8], Maxwell's equations are written (after contracting with the 4-velocity  $\dot{x}$ ) as:

$$\begin{aligned} \nabla_{\partial_i} F^{ij} = 4\pi J^j &\Rightarrow E^i_i = 4\pi \rho_c, \\ \nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} = 0 &\Rightarrow E_{[ij]} = \dot{x}^k \nabla_{\partial_k} F_{ij}, \end{aligned} \tag{2.9}$$

(where  $E_{[ij]} = E_{ij} - E_{ji}$ ; and  $\rho_c := J_i \dot{x}^i$ ), while gravitational field equations take the form:

$$\begin{aligned} r_{ij} = 8\pi \left( T_{ij} - \frac{1}{2} g_{ij} T^l_l \right) &\Rightarrow e^i_i = -4\pi (2\rho_m - T^i_i), \\ r_{jk} = r_{kj} &\Rightarrow e_{[ij]} = 0, \end{aligned} \tag{2.10}$$

where  $T_{ij}$  is the stress-energy tensor and  $\rho_m := T_{ij} \dot{x}^i \dot{x}^j$ .

In the cited papers, the authors also use the analogues of the above tidal tensors, built from the Hodge duals of the 2-forms  $r$  and  $F$ , leading to two more pairs of Eqs. (2.9) and (2.10). As we will see in Sec. 6, (2.9) and (2.10) are also sufficient for our purposes, so we have omitted these two extra pairs of equations.

### 3. Connections on Tangent Bundle

In the following three sections, we will try to find a notion of covariant derivative  $D$  such that:

- the Lorentz equations of motion (2.4) are equivalent to geodesic equations for  $D$ ;
- worldline deviation Eqs. (2.5) become formally similar to the classical Jacobi equation (i.e.  $\frac{D^2 w}{ds^2}$  is equal to a linear expression in  $w$ , which does not contain any of the derivatives  $\frac{D\dot{x}}{ds}$  or  $\frac{Dw}{ds}$ ).

**Remark 3.1.** Since the right-hand sides of both (2.4) and (2.5) depend on the parameter  $\frac{q}{m}$ , a single connection is not enough — we actually need a 1-parameter family of connections  $D$ .

To this aim, we will raise to the tangent bundle  $(TM, \pi, M)$  of the space-time manifold; there, we denote the local coordinates by  $(x \circ \pi, y) =: (x^i, y^j)_{i=0,3}$  and by  $\cdot_i$  and  $\cdot_j$ , partial differentiation with respect to  $x^i$  and  $y^j$  respectively.

An Ehresmann connection on  $TM$  is defined as a splitting  $T_u TM = H_u TM \oplus V_u TM$  of the tangent space at each  $u \in TM$ , where  $V_u TM$  is the *vertical subspace*, spanned by  $\frac{\partial}{\partial y^i}|_u$ . A first advantage of the use of Ehresmann connection is the presence of adapted frames and subsequently, of tensor components with simple transformation rules.

Any Ehresmann connection  $N$  on  $TM$ , [13, 17], gives rise to the adapted basis

$$\left( \delta_i = \frac{\partial}{\partial x^i} - N^j_i(x, y) \frac{\partial}{\partial y^j}, \dot{\delta}_i = \frac{\partial}{\partial y^i} \right), \tag{3.1}$$

(where the *horizontal subspace*  $H_u TM$  is spanned by  $\delta_i$ 's calculated at  $u$ ) and to its dual  $(dx^i, \delta y^i = dy^i + N^i_j dx^j)$ . With respect to coordinate transformations on  $TM$ , induced by coordinate transformations  $x^{i'} = x^{i'}(x)$  on  $M$ , the elements of the adapted basis/cobasis transform by the same rules as vector fields/covector fields on  $M$ , i.e.

$$\delta_i = \frac{\partial x^{i'}}{\partial x^i} \delta_{i'}, \quad \dot{\delta}_i = \frac{\partial x^{i'}}{\partial x^i} \dot{\delta}_{i'}, \quad dx^i = \frac{\partial x^i}{\partial x^{i'}} dx^{i'}, \quad \delta y^i = \frac{\partial x^i}{\partial x^{i'}} \delta y^{i'}$$

(to a difference from the elements  $\frac{\partial}{\partial x^i}$  of the natural basis  $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ , which obey a more complicated rule, [13]).

Any vector field  $X$  on  $TM$  can be (uniquely) written as:  $X = X^i \delta_i + \tilde{X}^i \dot{\delta}_i$ ; its *horizontal component*  $hX = X^i \delta_i$  and its *vertical component*  $vX = \tilde{X}^i \dot{\delta}_i$ , taken separately, are vector fields on  $TM$ . Similarly, 1-forms (and accordingly, tensors of any rank) split into components which are tensor fields.<sup>c</sup>

The adapted basis  $\{\delta_i, \dot{\delta}_i\}$  is generally nonholonomic, the Lie brackets of its elements are:

$$[\delta_j, \delta_k] = R^i_{jk} \dot{\delta}_i, \quad [\delta_j, \dot{\delta}_k] = L^i_{kj} \dot{\delta}_i, \quad [\dot{\delta}_j, \dot{\delta}_k] = 0;$$

where, [17],

$$R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k \tag{3.2}$$

provides the *curvature object* of  $N$  and

$$L^i_{kj} = \dot{\delta}_k N^i_j \tag{3.3}$$

is called the *connection object* of  $N$  (for reasons we will see several paragraphs below).

In the presence of an Ehresmann connection, vector fields on  $M$  can be mapped to horizontal vector fields on  $TM : X = X^i(x) \partial_i \mapsto l_h(X) = X^i(x) \delta_i$  by means of the *horizontal lift*  $l_h$ ; similarly, 1-forms  $\omega = \omega_i(x) dx^i$  on  $M$  can be “raised” into horizontal 1-forms  $l_h(\omega) = \omega_i(x) dx^i$  on  $TM$ .

Another advantage of Ehresmann connections is the possibility of defining covariant derivatives of vector (and tensor) fields on  $TM$ . Such a notion of covariant derivative of vector fields  $TM$  is given by the rule:

$$(X, Y) \mapsto D_X Y := v[hX, vY] + h[vX, hY] + J[vX, \theta Y] + \theta[hX, JY], \tag{3.4}$$

where  $J = \dot{\delta}_i \otimes dx^i$  and  $\theta = \delta_i \otimes \delta y^i$ .

<sup>c</sup>Hence, the coordinates  $X^i, \tilde{X}^i$  in the adapted basis (and accordingly, coordinates of 1-forms and of tensors of higher rank) also transform by the same rule as the components of vector fields/1-forms/tensors on the base manifold  $M$ .

The above defined mapping  $D$  is called the *Berwald covariant differentiation law*<sup>d</sup> attached to the Ehresmann connection  $N$ . Moreover, for any two vector fields  $X, Y$  on  $TM$ , there holds, [1]:

$$D_X(hY) = hD_XY, \quad D_X(vY) = vD_XY. \tag{3.5}$$

The action of  $D$  on the vectors of the adapted basis is given by:

$$D_{\delta_k} \delta_j = L^i_{jk} \delta_i, \quad D_{\delta_k} \dot{\delta}_j = L^i_{jk} \dot{\delta}_i, \quad D_{\dot{\delta}_k} \delta_j = D_{\dot{\delta}_k} \dot{\delta}_j = 0, \tag{3.6}$$

where the functions  $L^i_{jk}$  define the connection object (3.3).

The torsion tensor of  $D$  is

$$\mathbb{T} = T^i_{jk} dx^k \otimes dx^j \otimes \delta_i + R^i_{jk} dx^k \otimes dx^j \otimes \dot{\delta}_i,$$

where  $T^i_{jk} = L^i_{jk} - L^i_{kj}$  and  $R^i_{jk}$  is the curvature object (3.2). The curvature tensor is very easily expressed in terms of the curvature and connection objects:

$$\begin{aligned} \mathbb{R} = & R_j^i{}_{kl} \delta_i \otimes dx^l \otimes dx^k \otimes dx^j + R_j^i{}_{kl} \dot{\delta}_i \otimes dx^l \otimes dx^k \otimes \delta y^j \\ & + P_j^i{}_{kl} \delta_i \otimes \delta y^l \otimes dx^k \otimes dx^j + P_j^i{}_{kl} \dot{\delta}_i \otimes \delta y^l \otimes dx^k \otimes \delta y^j, \end{aligned} \tag{3.7}$$

where:

$$R_j^i{}_{kl} = \dot{\delta}_j R^i{}_{kl}, \quad P_j^i{}_{kl} = \dot{\delta}_l L^i{}_{jk}. \tag{3.8}$$

#### 4. Geodesics and their Deviations

Consider a curve  $c : t \mapsto (x^i(t))$  on the base manifold  $M$ . We denote by  $c' : t \mapsto (x^i(t), \dot{x}^i(t))$  its lift to  $TM$  and by  $V = \frac{dc'}{dt}$ , its tangent vector field (identified along  $c'$  with the operator  $\frac{d}{dt}$ ). In the adapted basis,  $V$  is written as:

$$V = y^i \delta_i + \frac{\delta y^i}{dt} \dot{\delta}_i, \quad y^i = \dot{x}^i. \tag{4.1}$$

The curve  $c : t \mapsto (x^i(t))$  is called an autoparallel curve (a *geodesic*), of the Ehresmann connection  $N$ , if

$$vV = 0.$$

In local writing, this is, [13]:

$$\frac{\delta y^i}{dt} \equiv \frac{dy^i}{dt} + N^i_j(x, y) y^j = 0, \quad y^i = \dot{x}^i. \tag{4.2}$$

<sup>d</sup>The Berwald covariant differentiation law (*Berwald connection*) is defined in [2, 1] for a special class of Ehresmann connections; still, the generalization to arbitrary ones is straightforward — and it obeys all the properties of an affine (linear, Cartan-type, [11]) connection on  $TM$ .

On the other side, the lift  $c'$  of  $c$  is a geodesic of  $D$  if  $D_V V = 0$ , that is, if:

$$D_V(hV) = 0, \quad D_V(vV) = 0, \tag{4.3}$$

or, in local coordinates:

$$\begin{cases} \frac{dy^i}{dt} + L^i_{jk}(x, y)y^j y^k = 0, & y^i = \dot{x}^i, \\ \frac{d}{dt} \frac{\delta y^i}{dt} + L^i_{jk}(x, y)y^j \frac{\delta y^k}{dt} = 0. \end{cases} \tag{4.4}$$

The systems (4.2) and (4.4) are, generally, not equivalent: lifts of geodesics of  $N$  are generally not geodesics of  $D$  and vice versa. This gives rise — at least for the moment — to two (generally, inconsistent) notions of geodesic. Moreover, comparing the system (4.4) to (2.4), we notice that the former contains four extra equations, which would thus impose extra (possibly, non-physical) restrictions on the trajectories.

All these problems are solved if we require the connection  $N$  to be *homogeneous*, i.e. the coefficients  $N^i_j$  should be homogeneous functions of degree 1 in the fiber coordinates  $y^i$ . Thus, by Euler's theorem, we have:

$$L^i_{kj} y^k = N^i_j. \tag{4.5}$$

In this case, there holds the following result.

**Proposition 4.1.** *Suppose that the Ehresmann connection  $N$  on  $TM$  is homogeneous. Then, the following statements are equivalent:*

- (1) *the curve  $c: t \mapsto x(t)$  is a geodesic of  $N: vV = 0$ ;*
- (2) *the lift  $c'$  of  $c$  satisfies:  $D_V(hV) = 0$ ;*
- (3) *the lift  $c'$  of  $c$  is a geodesic of  $D$ , i.e.  $D_V V = 0$ .*

**Proof.** (1) is equivalent to (2) by virtue of (4.5).

(2)  $\rightarrow$  (3) Relation  $D_V(hV) = 0$  implies  $vV = 0$ ; thus, we have  $D_V(hV) = 0$  and  $D_V(vV) = 0$ , which, put together, yields  $D_V V = 0$ .

(3)  $\rightarrow$  (2) is obvious. □

In the following, we will assume that  $N$  is homogeneous.

Assume that  $c: t \mapsto x(t)$  is a geodesic of  $N$  and we have a variation  $\alpha: (t, \varepsilon) \mapsto \alpha(t, \varepsilon)$ ,  $\alpha(t, 0) = x(t)$  (with  $\varepsilon$  in a neighborhood of  $0 \in \mathbb{R}$ ) of  $c$ , with deviation vector field  $w(t) = \left. \frac{\partial \alpha}{\partial \varepsilon} \right|_{\varepsilon=0}$ .

Raising to  $TM$ , to the lift  $\alpha'(t, \varepsilon) := (\alpha(t, \varepsilon), \frac{\partial \alpha}{\partial t}(t, \varepsilon))$  of  $\alpha$ , it corresponds the deviation vector field:  $W := \left. \frac{\partial \alpha'}{\partial \varepsilon} \right|_{\varepsilon=0} = w^i \frac{\partial}{\partial x^i} + \frac{dw^i}{dt} \frac{\partial}{\partial y^i}$ . In the adapted basis, this is:

$$W = w^i \delta_i + \left( \frac{dw^i}{dt} + N^i_j(x, y)w^j \right) \hat{\delta}_i,$$

Thus, by (4.5), we have  $vW = D_V(hW)$ .



We will express (though, in a redundant manner) the fact that  $c$  is a geodesic as:

$$D_V V = 0. \tag{4.6}$$

Differentiating this relation with respect to  $W$ , we get:

$$0 = D_W D_V V = D_V D_W V + R(W, V)V \tag{4.7}$$

(where we have taken into account that  $[V, W] \equiv [\frac{\partial}{\partial t}, \frac{\partial}{\partial \varepsilon}] = 0$ ); further,  $D_W V = D_V W + T(W, V)$ . Thus, geodesic deviation equations are, [3]:

$$D_V^2 W = R(V, W)V + D_V T(V, W). \tag{4.8}$$

Taking into account (3.5), relation (4.8) splits into horizontal and vertical components:

$$D_V^2(hW) = R(V, W)(hV) + D_V(hT(V, W)); \tag{4.9}$$

$$D_V^2(vW) = R(V, W)(vV) + D_V(vT(V, W)). \tag{4.10}$$

In (4.9), if the torsion term  $D_V(hT(V, W))$  is nonzero, we obtain in the right-hand side a term in the derivatives  $D_V(hW)$  (more exactly,  $T^i_{jk}y^k \frac{Dw^j}{dt}$ ). Hence, if we want geodesic deviation equations (4.9) to be formally similar to the usual Jacobi equations, we have to impose that  $T^i_{jk}y^k = 0$ , i.e.

$$(L^i_{jk} - L^i_{kj})y^k = 0 \Rightarrow N^i_{k,j}y^k = N^i_j$$

(where we had in view (4.5)), which is:  $N^i_j = \frac{1}{2}(N^i_k y^k)_{.j}$ ,  $j, k = \overline{0, 3}$ . Denoting  $2G^i := N^i_k y^k$ , we obtain that there exist some smooth, 2-homogeneous in  $y$  functions  $G^i = G^i(x, y)$  such that:

$$N^i_j = G^i_{.j}. \tag{4.11}$$

If this holds true, then relations (4.9) and (4.10) are, actually, redundant. Indeed, taking into account in (4.10) that  $vW = D_V(hW)$ , its left-hand side is  $D_V^3(hW)$ . On the other side, since  $vV$  is zero, the right-hand side is  $D_V(vT(V, W))$ . Taking into account (3.8) and the homogeneity of the connection  $N$ , we get that  $vT(V, W) = D_V(R(V, W)hV)$ ; this, together with the remark that  $hT(V, W) = 0$ , leads to the conclusion that (4.10) is a consequence of (4.9).

Thus, (4.9) is sufficient for characterizing geodesic deviations.

Conversely, if the 1-homogeneous functions  $N^i_j$  can be written as in (4.11), then deviation equations can be written as:  $D_V^2(hW) = R(V, W)(hV)$ . We have thus obtained the following proposition.

**Proposition 4.2.** *For a 1-homogeneous Ehresmann connection  $N$ , the following statements are equivalent:*

- (i) *There exist some functions  $G^i = G^i(x, y)$ , homogeneous of degree 2 in  $y$ , such that<sup>e</sup>  $N^i_j = G^i_{.j}$ ;*

<sup>e</sup>Ehresmann connections with the property (i) (called *spray connections*), [2, 6], are used in the theory of dynamical systems.

(ii) Lifts to TM of the geodesics of N are geodesics of the associated Berwald connection D:

$$D_V V = 0; \tag{4.12}$$

and deviations of these lifts obey the Jacobi equation:

$$D_V^2(hW) = R(V, W)(hV). \tag{4.13}$$

**Remark 4.3.** If N satisfies the properties in Proposition 4.2, then

$$D_V(hV) = 0, \tag{4.14}$$

is enough to characterize geodesics.

In the following, we will assume that N is as in Proposition 4.2. Equation (4.14) is locally written as:

$$\frac{D^2 w^i}{dt^2} = R_j^i{}_{kl} y^j y^l w^k, \quad y^i = \dot{x}^i.$$

Thus, with

$$E^i{}_k := R_j^i{}_{kl} y^j y^l = R^i{}_{kl} y^l, \tag{4.15}$$

we obtain a tensor

$$E = E^i{}_k(x, y) dx^k \otimes \delta_i,$$

which encompasses all the information regarding geodesic deviation. We will call this tensor, the *tidal tensor*<sup>f</sup> attached to N. In terms of the tidal tensor, geodesic deviation equation (4.14) is written as:

$$D_V^2(hW) = E(W). \tag{4.16}$$

### 5. A Special Family of Connections

Let us go back to the equations of motion of charged particles (2.4). As is well known, they arise from the variation of the Lagrangian:

$$L = \sqrt{g_{ij}(x) \dot{x}^i \dot{x}^j} + \frac{q}{m} A_i \dot{x}^i \tag{5.1}$$

with respect to the trajectory  $t \mapsto x(t)$ . The value of the integral  $\int L dt$  does not depend on the choice of the parameter  $t$  along the curve, thus, we can freely choose this parameter. With  $t = const \cdot s$ , where  $s$  is the arc length, extremal curves of  $L$  are given by:

$$\frac{d\dot{x}^i}{dt} + \gamma^i{}_{jk} \dot{x}^j \dot{x}^k - \frac{q}{m} \|\dot{x}\| F^i{}_j \dot{x}^j = 0, \tag{5.2}$$

where  $F^i{}_j := g^{ih} F_{hj}$ ,  $\|\dot{x}\| = \sqrt{g_{ij} \dot{x}^i \dot{x}^j}$ . (Usually, the factor  $\|\dot{x}\|$  is “discarded” by choosing  $t = s$  and hence,  $\|\dot{x}\| = 1$ . But, passing to the tangent bundle, where velocity can be treated as a coordinate in its own right, this factor will matter.)

<sup>f</sup>The tidal tensor is related to the *Jacobi endomorphism*  $\Phi$  in [6] by the relation  $\Phi = -J \circ E$ .

Equations (5.2) suggest us to define the following functions on  $TM$ :

$$G^\alpha(x, y) = \frac{1}{2}(\gamma^i_{jk}y^jy^k - \alpha\|y\|F^i_jy^j), \tag{5.3}$$

where  $y$  is timelike:  $g_{ij}y^iy^j > 0$ ,  $\|y\| := \sqrt{g_{ij}y^iy^j}$  and  $\alpha$  is a real parameter.

The functions  $G^\alpha$  are homogeneous of degree 2 in  $y$  and it can be easily checked that the functions

$$N^\alpha_j := G^\alpha_{\cdot j} \tag{5.4}$$

obey, with respect to coordinate changes on  $TM$ , the rule of transformation of the coefficients of an Ehresmann connection, [13].

We thus obtained a 1-parameter family of Ehresmann connections  $(N^\alpha)_{\alpha \in \mathbb{R}}$ , satisfying the properties in Proposition 4.2; also, we notice that the Eqs. (5.2) of charged particles coincide with the geodesic equations

$$\frac{dy^i}{dt} + N^\alpha_j(x, y)y^j = 0, \quad y = \dot{x},$$

for the connection of the family corresponding to  $\alpha = \frac{q}{m}$ . We obtain the following result.

**Proposition 5.1.** *The lift to  $TM$  of the trajectory of any charged particle with mass  $m$  and charge  $q$ , subject to gravitational and electromagnetic field, is a geodesic of the connection  $\overset{\alpha}{D}$ , where  $\alpha = \frac{q}{m}$ :*

$$\overset{\alpha}{D}_V V = 0, \tag{5.5}$$

For particles having the same ratio  $\frac{q}{m}$ , worldline deviation equations are given by

$$\overset{\alpha}{D}_V^2(hW) = E(W), \quad \alpha = \frac{q}{m}. \tag{5.6}$$

The writing (5.5) is redundant. Instead of it, we can also use its horizontal component only:

$$\overset{\alpha}{D}_V(hV) = 0, \quad \alpha = \frac{q}{m}. \tag{5.7}$$

Thus, when dealing with equations of motion of particles, the parameter  $\alpha$  will take specific values. In the following sections, connections  $\overset{\alpha}{D}$  will be also used in order to characterize fields alone — and to this aim, we will leave  $\alpha$  as arbitrary.

**Remark 5.2.** For  $\alpha = 0$ , we get:  $\overset{0}{N}^i_j = \gamma^i_{jk}y^k$ ,  $\overset{0}{L}^i_{jk} = \gamma^i_{jk}$ . For vector fields  $X, Y$  on the base manifold  $M$ , the horizontal lift  $l_h(\nabla_X Y)$  of the Levi-Civita covariant derivative  $\nabla_X Y$  and the  $\overset{0}{D}$ -covariant derivative  $\overset{0}{D}_{l_h(X)}l_h(Y)$  of the lifted vector fields coincide:  $l_h(\nabla_X Y) = \overset{0}{D}_{l_h(X)}l_h(Y)$ . In this sense,  $\overset{0}{D}$  can be considered as a  $TM$ -“equivalent” of

the Levi-Civita connection  $\nabla$ . It is worth noticing some of its properties:

$$\overset{0}{\mathbb{R}} = r, \quad \overset{0}{E} = e; \tag{5.8}$$

$$\overset{0}{D}g = 0, \quad \overset{0}{D}_{\delta_k}y^i = 0, \quad \overset{0}{D}\|y\| = 0. \tag{5.9}$$

All the other connections  $\overset{\alpha}{N}, \overset{\alpha}{D}$  of the family can be regarded as perturbations of  $\overset{0}{N}$  and  $\overset{0}{D}$ :

$$\overset{\alpha}{N}^i_j = \overset{0}{N}^i_j + \overset{\alpha}{B}^i_j, \quad \overset{\alpha}{L}^i_{jk} = \overset{0}{L}^i_{jk} + \overset{\alpha}{B}^i_{jk},$$

with contortion tensors given by:

$$B^i_j = -\frac{\alpha}{2}(F^i l_j + \|y\|F^i_j), \quad \overset{\alpha}{B}^i_{jk} := -\frac{\alpha}{2}(l_{.jk}F^i + l_j F^i_k + l_k F^i_j) \tag{5.10}$$

where  $F^i = F^i_j y^j$ . We notice the property:  $\overset{\alpha}{B}^i_{jk} y^k = \overset{\alpha}{B}^i_j$ .

In the following, if there is no risk of confusion, we will not explicitly indicate in the notation of connections, covariant derivatives, tidal tensors etc., the parameter  $\alpha$  (i.e. we will use the notations  $N, D, N^i_j, \dots$ , instead of  $\overset{\alpha}{N}, \overset{\alpha}{D}, \overset{\alpha}{N}^i_j, \dots$ , etc.).

For  $\alpha \neq 0$ , connections  $\overset{\alpha}{D}$  are generally, non-metrical. Also, they also have non-vanishing torsion:

$$\mathbb{T} = R^i_{jk} \dot{\delta}_i \otimes dx^k \otimes dx^j, \tag{5.11}$$

where  $R^i_{jk}$  are the components of the curvature object for  $N = \overset{\alpha}{N}$ .

The curvature of  $\overset{\alpha}{\mathbb{R}}$  of  $D = \overset{\alpha}{D}$  is as in (3.7), where:

$$R_j^i{}_{kl} = \frac{1}{2}(E^i_k)_{.jl}, \quad P_j^i{}_{kl} = B^i_{j.kl}. \tag{5.12}$$

In particular, the Ricci tensor  $R_{jl} = R^i_{jl}{}_i$  is obtained from the Hessian with respect to  $y$  of the trace of the tidal tensor as  $\text{Ric} = -\frac{1}{2}\text{Hess}_y E$ , i.e.:

$$R_{jl} = -\frac{1}{2}(E^i_i)_{.jl}. \tag{5.13}$$

Conversely, the tidal tensor  $E$  can be written in terms of  $\mathbb{R}$  as:

$$E^i_k = R_j^i{}_{kl} y^j y^l, \quad E^i_i = -R_{jl} y^j y^l. \tag{5.14}$$

## 6. Basic Equations of Gravitational and Electromagnetic Fields

### 6.1. Expression of electromagnetic 2-form

The differential forms  $A = A_i dx^i$  and  $F = dA = \frac{1}{2}F_{ij} dx^i \wedge dx^j$ , (2.3), will be lifted to horizontal forms on  $TM$ , which we will denote in the same manner. Unless elsewhere specified, the parameter  $\alpha$  is arbitrary.

Consider the horizontal 1-form on  $TM$

$$l = l_i dx^i, \quad l_i = \frac{y_i}{\|y\|}.$$

Let us calculate the covariant derivatives of  $l$  with respect to the vectors of the adapted basis.

(1) First, we have:  $D_{\delta_j} l_i = l_{i;j} = \|y\| h_{ij}$ , where

$$h_{ij} = g_{ij} - l_i l_j.$$

are the components of the *angular metric*  $h = h_{ij} dx^i \otimes dx^j$ , [2]. The (symmetric) tensor  $h$  has the property

$$h_{ij} y^j = 0. \tag{6.1}$$

This property means the following: for any vector field  $X$  on  $TM$ ,  $h(X, \cdot)^\sharp$  (where  $\sharp$  means raising indices by means of  $g$ ) is a vector field orthogonal to  $l^\sharp = l^i \delta_i$ ; the angular metric is actually a  $TM$  version of the so-called *projection tensor* appearing, for instance, in the Raychaudhuri equation.

(2) On the other side,  $D_{\delta_j} l_i = \overset{0}{D}_{\delta_j} l_i - B_j^k l_{i;k} - B_{ij}^k l_k$ .

From (5.9), we deduce that  $\overset{0}{D}_{\delta_j} l_i = 0$ . Evaluating the remaining terms with the help of (5.10) and taking into account (6.1), we get that:

$$D_{\delta_j} l_i = \frac{\alpha}{2} F_{ij}. \tag{6.2}$$

These results can be put together, if we notice that the exterior derivative  $dl$  can be expressed as, [12]:  $dl = \frac{1}{2}(D_{\delta_j} l_i - D_{\delta_i} l_j) dx^j \wedge dx^i + l_{i;j} \delta y^j \wedge dx^i$ .

Thus, we get that the exterior derivative  $dl$  is

$$dl = \frac{1}{2} \alpha F_{ij} dx^j \wedge dx^i + \|y\| h_{ij} \delta y^j \wedge dx^i, \tag{6.3}$$

that is, the electromagnetic 2-form can be expressed as:

$$\alpha F = -h(dl).$$

In the following, we will relate Einstein–Maxwell equations to tidal tensors attached to  $D = \overset{\alpha}{D}, \alpha \neq 0$ .

### 6.2. Homogeneous Maxwell equations

We apply the Ricci identity  $D_X D_Y \omega - D_Y D_X \omega = -\omega \circ \mathbb{R}(X, Y) + D_{[X, Y]} \omega$  (where  $\omega$  is a 1-form and  $X, Y$  are vector fields on  $TM$ ) to the 1-form  $l$  and the basis vectors  $\delta_k, \delta_j$ :

$$D_{\delta_k} D_{\delta_j} l_i - D_{\delta_j} D_{\delta_k} l_i = -R_i^h{}_{jk} l_h - R_{jk}^h l_{h \cdot i}.$$

We recognize in the left-hand side the difference  $\frac{\alpha}{2}(D_{\delta_k} F_{ij} - D_{\delta_j} F_{ik})$ . Performing cyclic summation over  $i, j, k$  and taking into account that  $\sum R_i^h{}_{jk} = 0$  (by first Bianchi identity

$\sum\{(D_X\mathbb{T})(Y, Z) - \mathbb{R}(X, Y)Z + \mathbb{T}(\mathbb{T}(X, Y), Z)\} = 0$ ), we get

$$\alpha(D_{\delta_k}F_{ij} + D_{\delta_j}F_{ki} + D_{\delta_i}F_{jk}) = -(R^h_{jk}l_{h\cdot i} + R^h_{ij}l_{h\cdot k} + R^h_{ki}l_{h\cdot j}).$$

Multiplying the latter relations with  $\|y\|y^k$  and taking into account (4.15) and (6.1), we have:

$$\alpha\|y\|(D_{\delta_k}F_{ij} + D_{\delta_j}F_{ki} + D_{\delta_i}F_{jk})y^k = -\tilde{E}_{[ij]}, \quad (6.4)$$

where:

$$\tilde{E}_{ij} = h_{ik}E^k_j. \quad (6.5)$$

Expressing  $D = \overset{0}{D} + B$ , the terms involving the contortion  $B$  cancel out and the above relation is actually:

$$\tilde{E}_{[ij]} = -\alpha\|y\|(\overset{0}{D}_{\delta_k}F_{ij} + \overset{0}{D}_{\delta_j}F_{ki} + \overset{0}{D}_{\delta_i}F_{jk})y^k. \quad (6.6)$$

Since  $F_{ij} = F_{ij}(x)$  is projectable to  $M$ , we can write (6.6) as:

$$\tilde{E}_{[ij]} = -\alpha\|y\|(\nabla_{\delta_k}F_{ij} + \nabla_{\delta_j}F_{ki} + \nabla_{\delta_i}F_{jk})y^k. \quad (6.7)$$

Thus, homogeneous Maxwell equations imply:  $\tilde{E}_{[ij]} = 0$ .

Conversely, if  $\tilde{E}_{[ij]} = 0$  holds true for any timelike vector  $y$ , then by first dividing it to  $\|y\|$  and then differentiating it with respect to  $y^k$  (regarded as a fiber coordinate), we obtain  $\nabla_{\delta_k}F_{ij} + \nabla_{\delta_j}F_{ki} + \nabla_{\delta_i}F_{jk} = 0$ .

We have thus proven the following proposition.

**Proposition 6.1.** *Homogeneous Maxwell equations equivalent to the fact that  $\tilde{E}$  is symmetric:*

$$\tilde{E}_{[ij]} = 0. \quad (6.8)$$

### 6.3. Inhomogeneous Maxwell equations

Maxwell's equations with sources, as well as Einstein field equations, can be expressed in terms of the trace  $\text{tr}E = E^i_i$  of the tidal tensor. In the following, we will explicitly calculate this trace.

The curvature object  $R^i_{jk} = \delta_k N^i_j - \delta_j N^i_k$  can be decomposed in terms of  $r^i_{jk} = \overset{0}{R}^i_{jk}$  and of the contortion tensor  $B$ . In detail,  $R^i_{jk} = r^i_{jk} + \overset{0}{D}_{\delta_k}B^i_j - \overset{0}{D}_{\delta_j}B^i_k + B^l_j B^i_{kl} - B^l_k B^i_{jl}$ . Summing over  $i = j$ , noticing that  $B^i_i = 0$ ,  $B^i_{il} = 0$  and then contracting by  $y^k$ , we obtain:

$$E^i_i = e^i_i - \overset{0}{D}_{\delta_i}(2B^i) + B^l_i B^i_l, \quad (6.9)$$

where  $2B^i := B^i_j y^j = -\alpha\|y\|F^i$ .

The derivative term in (6.9) is:  $-\overset{0}{D}_{\delta_i}(2B^i) = \overset{0}{D}_{\delta_i}(\alpha\|y\|F^i_j y^j) = \alpha\|y\|y^j \overset{0}{D}_{\delta_i}F^i_j$ . Further, we can write  $\alpha\|y\|y^j \overset{0}{D}_{\delta_i}F^i_j = \alpha\|y\|y^j \nabla_{\partial_i}F^i_j$ . According to Maxwell's equations with sources

N. Voicu

$\nabla_{\partial_i} F^i_j = 4\pi J_j$ , we thus have:

$$-D_{\delta_i}^0(2B^i) = 4\pi\alpha\rho_c\|y\|^2, \quad (6.10)$$

where, as in [7],  $J$  and  $\rho_c = J^i l_i$  represent, respectively, the (charge) current 4-vector and the charge density as measured by an observer with 4-velocity  $l^\sharp$ . Conversely, if (6.10) holds true for arbitrary  $y$ , then dividing it by  $\alpha\|y\|$  and subsequently differentiating it with respect to  $y^j$ , we get (taking into account the equalities  $\|y\|_{,j} = l_j$ ),  $\nabla_{\partial_i} F^i_j = 4\pi J_j$ . This gives us the right to state the following proposition.

**Proposition 6.2.** *Inhomogeneous Maxwell equations are expressed in terms of tidal tensors as:*

$$E^i_i = e^i_i + 4\pi\alpha\rho_c\|y\|^2 + B^l_i B^i_l. \quad (6.11)$$

**Remark 6.3.** It can be proven by direct computation that the trace  $\tilde{E}^i_i = g^{ij}\tilde{E}_{ji}$  of  $\tilde{E}$  coincides with the trace of  $E$ ; in other words, we can use in (6.11) either of the versions  $\tilde{E}^i_i$  or  $E^i_i$ .

#### 6.4. Einstein field equations

Contracted with  $y^i y^j$ , the Einstein field equations  $r_{ij} = 8\pi(T_{ij} - \frac{1}{2}T^l_l g_{ij})$  imply:

$$e^i_i = -8\pi \left( T_{ij} y^i y^j - \frac{1}{2} T^l_l \|y\|^2 \right). \quad (6.12)$$

Conversely, if (6.12) hold true for any (timelike)  $y$ , then, by differentiating it twice with respect to  $y$ , we get again the classical form  $r_{ij} = 8\pi(T_{ij} - \frac{1}{2}T^l_l g_{ij})$  of the Einstein field equations. Thus, we can use (6.12) as an equivalent expression of these.

The stress energy tensor  $T_{ij}$  can be decomposed as:  $T_{ij} = T_{ij}^{em} + T_{ij}^m$ , where  $T_{ij}^{em}$  is the stress-energy tensor of the electromagnetic field, [10]:

$$T_{ij}^{em} = \frac{1}{4\pi} \left( -F^h_i F_{hj} + \frac{1}{4} F^{kh} F_{kh} g_{ij} \right)$$

and  $T_{ij}^m$  is the stress-energy tensor of matter (and/or other fields). The electromagnetic stress-energy tensor  $T_{ij}^{em}$  has zero trace  $T^l_l{}^{em} = 0$ , hence, in (6.12),  $T^l_l = T^l_l{}^m$ .

On the other side, taking in (6.2) covariant derivative by  $\delta_k$  and successively contracting by  $g^{jk}$  and  $\|y\|^2 l^i$ , we get:

$$4\pi\alpha^2 T_{ij}^{em} y^i y^j = \|y\|^2 l^i \square l_i - D_{\delta_i}^0 B^i,$$

where  $\square l_i := g^{jk} D_{\delta_k} D_{\delta_j} l_i$ . Consequently, (6.12) is equivalent to:

$$e^i_i = -\frac{2}{\alpha^2} (\|y\|^2 l^i \square l_i - D_{\delta_i}^0 B^i) - 8\pi \left( T_{ij}^m y^i y^j - \frac{1}{2} T^l_l{}^m \|y\|^2 \right), \quad (6.13)$$

Substituting  $e^i_i$  into the expression (6.9) of  $E^i_i$  and denoting  $\rho_m := T_{ij}^m l^i l^j$ , we finally have the following proposition.

**Proposition 6.4.** *Einstein field equations are equivalent to the following relation on TM:*

$$\frac{1}{\|y\|^2} E^i{}_i = \frac{2}{\|y\|^2} \left( \frac{1 - \alpha^2}{\alpha^2} D_{\delta_i}^0 B^i + \frac{1}{2} B^i{}_l B^l{}_i \right) - \frac{2}{\alpha^2} l^i \square l_i - 8\pi \left( \rho_m - \frac{1}{2} T^l{}_l \right). \quad (6.14)$$

## 7. Particular Cases

### A. Gravity only

In this case, we have  $B^i{}_j = 0$ , which means that all the affine connections coincide:  $\overset{\alpha}{D} = D, \forall \alpha \in \mathbb{R}$ . The tidal tensor is given by  $e^i{}_j = r_l{}^i{}_{jk} y^l y^k$  and Eqs. (2.10) become:

$$\begin{aligned} r_{ij} = 8\pi \left( T_{ij} - \frac{1}{2} g_{ij} T^l{}_l \right) &\Leftrightarrow \frac{1}{\|y\|^2} e^i{}_i = -4\pi(2\rho_m - T^i{}_i), \\ r_{jk} = r_{kj} &\Leftrightarrow \tilde{e}_{[ij]} = 0, \end{aligned} \quad (7.1)$$

where, this time,  $\rho_m = T_{ij} l^i l^j$ .

### B. Electromagnetism in flat Minkowski space

In this case, we have  $\gamma^i{}_{jk} = 0, e^i{}_j = 0$  and  $L^i{}_{jk} = B^i{}_{jk}$ . The curvature of  $\overset{\alpha}{D}$  ( $\alpha \neq 0$ ), only depends on  $B$ .

Maxwell equations are written in terms of tidal tensors as:

$$\begin{aligned} \nabla_{\partial_i} F^{ij} = 4\pi J^i &\Leftrightarrow \frac{1}{\|y\|^2} E^i{}_i = 4\pi\alpha\rho_c + \frac{1}{\|y\|^2} B^i{}_h B^h{}_i, \\ \nabla_{\partial_i} F_{jk} + \nabla_{\partial_k} F_{ij} + \nabla_{\partial_j} F_{ki} = 0 &\Leftrightarrow \tilde{E}_{[ij]} = 0. \end{aligned} \quad (7.2)$$

Thus, we found analogous equations to the ones determined by Costa and Herdeiro, [7, 8], without resorting to any restriction upon the derivatives of the deviation vector  $w$ .

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