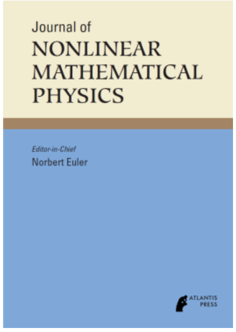

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LEFT-INVARIANT PSEUDO-EINSTEIN METRICS ON LIE GROUPS

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In this article, we focus on left-invariant pseudo-Einstein metrics on Lie groups. To begin with, we give some examples of pseudo-Einstein metrics on Lie groups. Also we calculate the Levi-civita connection, and then Ricci tensor associated with left-invariant pseudo-Riemannian metrics on the unimodular Lie groups of dimension three. Furthermore, we show that the left-invariant pseudo-Einstein metric on SL(2) is unique up to a constant. At last, we study the left-invariant pseudo-Riemannian metrics on compact Lie groups and classify the pseudo-Einstein metrics on the low-dimensional compact Lie groups.

Keywords: Pseudo-Riemannian metric; pseudo-Einstein metric; Levi-civita connection; Ricci curvature.

Mathematics Subject Classification 2010: 53C50, 53C30

1. Introduction

We focus on left-invariant pseudo-Einstein metrics on Lie groups. Let G be a Lie group with the Lie algebra $\mathfrak{g}$, g a left-invariant pseudo-Riemannian metric. Then the unique torsion-free affine connection, i.e. Levi-Civita connection, is determined by

$$g(\nabla_x y, z) = \frac{1}{2}(g([x,y],z) - g([y,z],x) + g([z,x],y)). \tag{1.1}$$

The curvature tensor is defined by

$$R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z. \tag{1.2}$$

The metric g is said to be flat if $R \equiv 0$. For any pair $x, y \in \mathfrak{g}$, the Ricci tensor is defined by

$$\mathrm{Ric}(x,y) = \mathrm{Tr}\{z \mapsto R(z,x)y\}. \tag{1.3}$$

The metric g is said to be pseudo-Einstein if $\mathrm{Ric} \equiv \lambda g$ for some constant λ. Furthermore, the metric g is said to be Ricci flat if $\lambda = 0$.

There are many important results on Einstein manifolds [4, 10–13]. A detailed exposition on Einstein manifolds can be found in the book of Besse [3] and a lot of more recent results on homogeneous Einstein manifolds can be found in the survey of Wang [9]. On solvmanifolds, Lauret proved an important result that Einstein solvmanifolds are standard [7]; Hervik also did some interesting work about homogeneous Einstein solvmanifolds with negative curvature and described some applications in physics [5, 6]. However, the pseudo-Einstein case is totally different from the Einstein case. In this paper, we discuss some lower dimensional examples and give the algebraic formula of Ricci curvature on compact Lie groups. May be it will be a basis of further discussion on this field.

The article is organized as follows. In Sec. 2, we give some examples of pseudo-Einstein metrics on Lie groups. In Sec. 3, we calculate the Levi-civita connection, and then Ricci tensor associated with left-invariant pseudo-Riemannian metrics on the unimodular Lie groups of dimension three. Furthermore, we show that the left-invariant pseudo-Einstein metric on SL(2) is unique up to a constant. In Sec. 4, we focus on the left-invariant pseudo-Riemannian metrics on compact Lie groups. Firstly we get a formula of Ricci curvatures. And then we denote the formula by the structure constants of the Lie algebra of the given Lie group. At last, we apply the formula to calculate the pseudo-Einstein metrics on the low-dimensional compact Lie groups.

2. Examples of Pseudo-Einstein Metrics

Example 2.1. Let G be the nonabelian Lie group of dimension two with the Lie algebra $\mathfrak{g}$, g a left-invariant pseudo-Einstein metric on G. Then there is a basis $\{x, y\}$ of $\mathfrak{g}$ such that

(1) $g(x,x) = a, g(y,y) = b, [x,y] = y$, where $a, b \neq 0$, or
(2) $g(x,y) = 1, [x,y] = y$.

For the first case, $\nabla_x x = \nabla_x y = 0, \nabla_y x = -y, \nabla_y y = \frac{b}{a}x$. Then

$$\mathrm{Ric}(x,x) = -1, \quad \mathrm{Ric}(x,y) = 0, \quad \mathrm{Ric}(y,y) = -\frac{b}{a}.$$

That is, $\mathrm{Ric} = -\frac{1}{a}g$. For the second case, $\nabla_x x = -x, \nabla_x y = y, \nabla_y x = \nabla_y y = 0$. It is easy to see that g is flat, and then Ricci flat.

Remark 2.2. Let g be a left-invariant pseudo-Einstein metric of a Lie group G with the Einstein constant λ. Then for any nonzero constant a, ag is also a left-invariant pseudo-Einstein metric with the constant $\frac{\lambda}{a}$.

Example 2.3. Let G_1 be a compact Lie group with a left-invariant Einstein metric g_1, the corresponding Einstein constant $\lambda_1 (> 0)$. Let G_2 be a solvable Lie group with a left-invariant Einstein metric g_2, the corresponding Einstein constant $\lambda_2 (< 0)$. Let $G = G_1 \times G_2$, for any constant $a \neq 0$, define a left-invariant pseudo-Riemannian metric g on G by

$$g\mid_{\mathfrak{g}_1\times\mathfrak{g}_1} = \frac{\lambda_1}{a}g_1, \quad g\mid_{\mathfrak{g}_1\times\mathfrak{g}_2} = 0, \quad g\mid_{\mathfrak{g}_2\times\mathfrak{g}_2} = \frac{\lambda_2}{a}g_2.$$

Then g is a pseudo-Einstein metric with the constant a.

Conjecture 2.4 ([3]). *Let $M = G/K$ be a noncompact homogeneous Einstein manifold. Then K is a maximal compact subgroup of G.*

Remark 2.5. Conjecture 2.4 is Alekseevskii conjecture on Riemannian metrics. By Example 2.3, Alekseevskii conjecture on pseudo-Riemannian metrics does not hold.

Example 2.6. Let G_1 be a compact Lie group with a left-invariant Einstein metric g_1, the corresponding Einstein constant $\lambda_1(>0)$. Let G_2 be the solvable Lie group of dimension two given in Example 2.1. Assume that g_2 is a left-invariant pseudo-Riemannian metric such that there is a basis x, y of $\mathfrak{g}_2$ satisfying $g_2(x,x) = \lambda_2, g_2(y,y) = \lambda_3, [x,y] = y$, where $\lambda_2\lambda_3 < 0$. Let $G = G_1 \times G_2$, for any constant $a > 0$, define a left-invariant pseudo-Riemannian metric g on G by

$$g\,|_{\mathfrak{g}_1\times\mathfrak{g}_1} = \frac{\lambda_1}{a}g_1, \quad g\,|_{\mathfrak{g}_1\times\mathfrak{g}_2} = 0, \quad g\,|_{\mathfrak{g}_2\times\mathfrak{g}_2} = -\frac{1}{a\lambda_2}g_2.$$

Then g is a Lorentzian Einstein metric with the constant a.

3. Pseudo-Riemannian Metrics on Three-Dimensional Unimodular Lie Groups

Let G be a connected three-dimensional unimodular Lie group with the Lie algebra $\mathfrak{g}$, g a left-invariant pseudo-Riemannian metric with signature $(2,1)$. Then there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$ such that

$$g(e_1,e_1) = g(e_2,e_2) = -g(e_3,e_3) = 1, \tag{3.1}$$

$$[e_2,e_3] = \lambda_1 e_1 + ae_3, \quad [e_3,e_1] = \lambda_2 e_2 + be_3, \quad [e_1,e_2] = ae_1 + be_2 + \lambda_3 e_3. \tag{3.2}$$

Proposition 3.1. *Let notations be as above. Then the Levi-Civita connection is given by*

$$\nabla_{e_1}e_1 = -ae_2, \quad \nabla_{e_1}e_2 = ae_1 + \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3)e_3,$$

$$\nabla_{e_1}e_3 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3)e_2, \quad \nabla_{e_2}e_1 = -be_2 + \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)e_3,$$

$$\nabla_{e_2}e_2 = be_1, \quad \nabla_{e_2}e_3 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)e_1,$$

$$\nabla_{e_3}e_1 = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)e_2 + be_3,$$

$$\nabla_{e_3}e_2 = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)e_1 - ae_3, \quad \nabla_{e_3}e_3 = be_1 - ae_2.$$

Proof. By Eqs. (1.1), (3.1) and (3.2), we have

$$g(\nabla_{e_1}e_1, e_1) = 0, \quad g(\nabla_{e_1}e_1, e_2) = -a, \quad g(\nabla_{e_1}e_1, e_3) = 0.$$

It follows that $\nabla_{e_1}e_1 = -ae_2$. The others are similar. □

Proposition 3.2. *Let notations be as above. The nonzero Ricci curvatures are given by*

$$\mathrm{Ric}(e_1,e_3) = a(\lambda_1 - \lambda_2 - \lambda_3), \tag{3.3}$$

$$\mathrm{Ric}(e_2,e_3) = b(-\lambda_1 + \lambda_2 - \lambda_3), \tag{3.4}$$

$$\mathrm{Ric}(e_1,e_1) = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3), \tag{3.5}$$

$$\mathrm{Ric}(e_2,e_2) = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3), \tag{3.6}$$

$$\mathrm{Ric}(e_3,e_3) = -2a^2-2b^2+\frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3). \tag{3.7}$$

Proof. By the Eqs. (1.3), (3.1), (3.2) and Proposition 3.1,

$$\mathrm{Ric}(e_1,e_1) = g(R(e_1,e_1)e_1,e_1)+g(R(e_2,e_1)e_1,e_2)-g(R(e_3,e_1)e_1,e_3)=0.$$

By similar calculation, we have the proposition. □

Theorem 3.3. *Let G be a connected three-dimensional unimodular Lie group with the Lie algebra $\mathfrak{g}$, g a left-invariant pseudo-Riemannian metric with signature $(2,1)$. If g is pseudo-Einstein, then there exists a basis $\{e_1,e_2,e_3\}$ of $\mathfrak{g}$ such that*

$$g(e_1,e_1)=g(e_2,e_2)=-g(e_3,e_3)=1,$$
$$[e_2,e_3]=\lambda_1e_1,\quad [e_3,e_1]=\lambda_2e_2,\quad [e_1,e_2]=\lambda_3e_3,$$

where λ_i must be one of the following cases:

(1) $\lambda_1=\lambda_2=\lambda_3=0$;
(2) $\lambda_1=0$, $\lambda_2=-\lambda_3\neq 0$;
(3) $\lambda_2=0$, $\lambda_1=-\lambda_3\neq 0$;
(4) $\lambda_3=0$, $\lambda_1=\lambda_2\neq 0$;
(5) $\lambda_1=\lambda_2=-\lambda_3\neq 0$.

Here g is Ricci flat for the first four cases.

Proof. Let $\{e_1,e_2,e_3\}$ be the basis of $\mathfrak{g}$ satisfying the Eqs. (3.1) and (3.2). It is enough to show that $a=b=0$. Since g is pseudo-Einstein, by Proposition 3.2, we have

$$a(\lambda_1-\lambda_2-\lambda_3)=b(-\lambda_1+\lambda_2-\lambda_3)=0.$$

Assume that $a\neq 0$. Then $\lambda_1-\lambda_2-\lambda_3=0$. Thus $\mathrm{Ric}(e_1,e_1)=0$. It follows that

$$\mathrm{Ric}(e_3,e_3)=-2a^2-2b^2=0.$$

Namely $a=b=0$. It is a contradiction. So $a=0$. Similarly, $b=0$.

Thus by Proposition 3.2,

$$\mathrm{Ric}(e_1,e_1) = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3),$$

$$\mathrm{Ric}(e_2,e_2) = \frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3),$$

$$\mathrm{Ric}(e_3,e_3) = \frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3).$$

Assume that $\lambda_1+\lambda_2+\lambda_3=0$. Then $\mathrm{Ric}(e_1,e_1)=\mathrm{Ric}(e_2,e_2)=0$. Therefore

$$\mathrm{Ric}(e_1,e_1)=\frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3)(\lambda_1-\lambda_2+\lambda_3)=2\lambda_1\lambda_2=0.$$

Thus $\lambda_1=0$ or $\lambda_2=0$. Then we have the first three cases.

Assume that $\lambda_1+\lambda_2+\lambda_3\neq 0$. Then by $\mathrm{Ric}(e_1,e_1)=\mathrm{Ric}(e_2,e_2)$, we have $\lambda_1=\lambda_2$. Furthermore by $\mathrm{Ric}(e_2,e_2)=-\mathrm{Ric}(e_3,e_3)$, we have

$$(2\lambda_1+\lambda_3)\lambda_3=-\lambda_3^2.$$

If $\lambda_3=0$, then we have the fourth case. If $\lambda_3\neq 0$, then $\lambda_1=-\lambda_3$, i.e., $\lambda_1=\lambda_2=-\lambda_3\neq 0$. □

Theorem 3.4. *The left-invariant pseudo-Einstein metric on* SL(2) *is unique up to a constant.*

Proof. By the results of [8], we know there is no left-invariant Einstein metric on SL(2). Thus the signature of left-invariant pseudo-Einstein metric on SL(2) must be $(2,1)$ or $(1,2)$. By Remark 2.2, we can assume that the signature is $(2,1)$. By Theorem 3.3, there exists a basis $\{e_1,e_2,e_3\}$ of SL(2) such that

$$g(e_1,e_1)=g(e_2,e_2)=-g(e_3,e_3)=1,\quad [e_2,e_3]=\lambda e_1,\quad [e_3,e_1]=\lambda e_2,\quad [e_1,e_2]=-\lambda e_3.$$

Here $\lambda\neq 0$. Let K be the killing form of sl(2). Then we can check that

$$K(e_i,e_j)=2\lambda^2 g(e_i,e_j).$$

That is, the left-invariant pseudo-Einstein metric on SL(2) is unique up to a constant. □

4. Pseudo-Riemannian Metrics on Compact Lie Groups

Let G be a compact Lie group with the Lie algebra $\mathfrak{g}$. Let $(\ ,\ )$ be a bi-invariant metric on G. Assume that g is a left-invariant pseudo-Riemannian metric on G. There exists a unique endmorphism D of $\mathfrak{g}$ such that

$$g(x,y)=(x,Dy)=(Dx,y).$$

Denote by ∇ the Levi-Civita connection associated with g, i.e.,

$$\nabla_x=\frac{1}{2}(\mathrm{ad}x-D^{-1}\mathrm{ad}Dx+D^{-1}\mathrm{ad}xD).$$

Choose an orthonormal basis $e_1,\ldots,e_n$ of $\mathfrak{g}$ with respect to $(\ ,\ )$ such that $De_i=\lambda_i e_i$. Hence $g(e_i,e_j)=\lambda_i\delta_{ij}$.

Theorem 4.1. *For any $x,y\in\mathfrak{g}$,* $\mathrm{Ric}(x,y)=-\mathrm{Tr}(\nabla_x-\mathrm{ad}x)(\nabla_y-\mathrm{ad}y)$.

Proof. For any basis $e_1,\ldots,e_n$ of $\mathfrak{g}$, e_i^* the dual basis associated to g, we have

$$\begin{aligned}\mathrm{Ric}(x,y)&=\sum_{i=1}^n g(R(e_i,x)y,e_i^*)\\&=\sum_{i=1}^n g(\nabla_{e_i}\nabla_x y-\nabla_x\nabla_{e_i}y-\nabla_{[e_i,x]}y,e_i^*)\end{aligned}$$

$$= \sum_{i=1}^{n} g((\nabla_{\nabla_x y} e_i + [e_i, \nabla_x y]) - \nabla_x(\nabla_y e_i + [e_i, y]) - (\nabla_y [e_i, x] + [[e_i, x], y]), e_i^*)$$
$$= \sum_{i=1}^{n} g((\nabla_x \nabla_y - \nabla_x \mathrm{ad} y - \nabla_{\nabla_x y} + \mathrm{ad}\nabla_x y - \nabla_y \mathrm{ad} x + \mathrm{ad} y \mathrm{ad} x)(e_i), e_i^*)$$
$$= \mathrm{Tr}(\nabla_x \nabla_y - \nabla_x \mathrm{ad} y - \nabla_{\nabla_x y} + \mathrm{ad}\nabla_x y - \nabla_y \mathrm{ad} x + \mathrm{ad} y \mathrm{ad} x)$$
$$= \mathrm{Tr}(\nabla_{\nabla_x y}) - \mathrm{Tr}(\mathrm{ad}\nabla_x y) - \mathrm{Tr}(\nabla_x(\nabla_y - \mathrm{ad} y)) + \mathrm{Tr}((\nabla_y - \mathrm{ad} y)\mathrm{ad} x)$$
$$= \mathrm{Tr}(\nabla_{\nabla_x y}) - \mathrm{Tr}(\mathrm{ad}\nabla_x y) - \mathrm{Tr}(\nabla_x(\nabla_y - \mathrm{ad} y)) + \mathrm{Tr}(\mathrm{ad} x(\nabla_y - \mathrm{ad} y))$$
$$= \mathrm{Tr}(\nabla_{\nabla_x y}) - \mathrm{Tr}(\mathrm{ad}\nabla_x y) - \mathrm{Tr}(\nabla_x - \mathrm{ad} x)(\nabla_y - \mathrm{ad} y).$$

By the Eq. (1.1), we have

$$g(\nabla_x y, z) + g(y, \nabla_x z) = 0.$$

It follows that $\mathrm{Tr}\nabla_x = 0$ for any $x \in \mathfrak{g}$. Furthermore, for any $x, y, z \in \mathfrak{g}$, we have

$$(\mathrm{ad} x(y), z) + (y, \mathrm{ad} x(z)) = 0,$$

so Tr $\mathrm{ad} x = 0$, for any $x \in \mathfrak{g}$. This completes the proof of the assertion. □

Assume that $[e_i, e_j] = \sum_{l=1}^{n} C_{ij}^l e_l$. By the invariancy of (,), one has $C_{ij}^l = C_{jl}^i = C_{li}^j$. Then we have

$$\nabla_{e_i} e_j = \frac{1}{2}([e_i, e_j] - D^{-1}[De_i, e_j] + D^{-1}[e_i, De_j])$$
$$= \frac{1}{2}(\mathrm{id} - \lambda_i D^{-1} + \lambda_j D^{-1})([e_i, e_j])$$
$$= \frac{1}{2} \sum_{l=1}^{n} \frac{\lambda_l - \lambda_i + \lambda_j}{\lambda_l} C_{ij}^l e_l.$$

By the above theorem, one has the following formula.

Theorem 4.2.

$$\mathrm{Ric}(e_j, e_k) = \frac{1}{2} \sum_{i<l} ((\lambda_l - \lambda_i)^2 - \lambda_k \lambda_j) \frac{C_{ki}^l}{\lambda_l} \frac{C_{ji}^l}{\lambda_i}.$$

Proof. By Theorem 4.1,

$$\mathrm{Ric}(e_j, e_k) = -\mathrm{Tr}(\nabla_{e_j} - \mathrm{ad} e_j)(\nabla_{e_k} - \mathrm{ad} e_k)$$
$$= -\sum_{i=1}^{n} g((\nabla_{e_j} - \mathrm{ad} e_j)(\nabla_{e_k} - \mathrm{ad} e_k) e_i, e_i^*)$$
$$= -\sum_{i=1}^{n} g\left((\nabla_{e_j} - \mathrm{ad} e_j)\left(\frac{1}{2}\sum_{l=1}^{n}\left(\frac{\lambda_l - \lambda_k + \lambda_i}{\lambda_l} - 2\right) C_{ki}^l e_l\right), e_i^*\right)$$

$$= -\frac{1}{4}\sum_{i=1}^{n}\sum_{l=1}^{n}\frac{-\lambda_l-\lambda_k+\lambda_i}{\lambda_l}\frac{-\lambda_i-\lambda_j+\lambda_l}{\lambda_i}C_{ki}^{l}C_{jl}^{i}$$

$$= \frac{1}{2}\sum_{i<l}((\lambda_l-\lambda_i)^2-\lambda_k\lambda_j)\frac{C_{ki}^{l}}{\lambda_l}\frac{C_{ji}^{l}}{\lambda_i}. \qquad \square$$

Example 4.3. Let $G = \mathrm{SU}(2)$ and $\{e_1, e_2, e_3\}$ an orthonormal basis of $\mathfrak{su}(2)$ with respect to the given inner product $(\ ,\ )$ and $g(e_i, e_i) = \lambda_i$, $i = 1, 2, 3$. So

$$[e_1, e_2] = ae_3, \quad [e_2, e_3] = ae_1, \quad [e_3, e_2] = ae_2.$$

By Theorem 4.2, Ricci curvatures are given by

$$\begin{aligned}
&\mathrm{Ric}(e_j, e_k) = 0, \quad j \neq k;\\
&\mathrm{Ric}(e_1, e_1) = \frac{a^2}{2\lambda_2\lambda_3}(\lambda_1+\lambda_2-\lambda_3)(-\lambda_1+\lambda_2-\lambda_3),\\
&\mathrm{Ric}(e_2, e_2) = \frac{a^2}{2\lambda_1\lambda_3}(\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2-\lambda_3),\\
&\mathrm{Ric}(e_3, e_3) = \frac{a^2}{2\lambda_1\lambda_2}(\lambda_1-\lambda_2+\lambda_3)(\lambda_1-\lambda_2-\lambda_3).
\end{aligned}$$

If g is pseudo-Einstein, then we have $\lambda_1 = \lambda_2 = \lambda_3$. That is, g is positive definite or negative definite.

Example 4.4. Let $G = \mathrm{SU}(2) \times S^1$ and $\{e_1, e_2, e_3, e_4\}$ an orthonormal basis with respect to the given inner product $(\ ,\ )$ and $g(e_i, e_i) = \lambda_i$, $i = 1, 2, 3, 4$. Then

$$\begin{aligned}
&[e_1, e_2] = ae_3 + be_4, \quad [e_1, e_3] = -ae_2 + ce_4, \quad [e_1, e_4] = -be_2 - ce_3,\\
&[e_2, e_3] = ae_1 + de_4, \quad [e_2, e_4] = be_1 - de_3, \quad [e_3, e_4] = ce_1 + de_2.
\end{aligned}$$

By Theorem 4.2, the Ricci curvatures are given as following.

$$\begin{aligned}
&\mathrm{Ric}(e_1, e_2) = \frac{cd((\lambda_4-\lambda_3)^2-\lambda_1\lambda_2)}{2\lambda_3\lambda_4},\\
&\mathrm{Ric}(e_1, e_3) = \frac{-bd((\lambda_4-\lambda_2)^2-\lambda_1\lambda_3)}{2\lambda_2\lambda_4},\\
&\mathrm{Ric}(e_1, e_4) = \frac{ad((\lambda_3-\lambda_2)^2-\lambda_1\lambda_4)}{2\lambda_2\lambda_3},\\
&\mathrm{Ric}(e_2, e_3) = \frac{bc((\lambda_4-\lambda_1)^2-\lambda_2\lambda_3)}{2\lambda_1\lambda_4},\\
&\mathrm{Ric}(e_2, e_4) = \frac{-ac((\lambda_3-\lambda_1)^2-\lambda_2\lambda_4)}{2\lambda_1\lambda_3},
\end{aligned}$$

$$\mathrm{Ric}(e_3, e_4) = \frac{ab((\lambda_2 - \lambda_1)^2 - \lambda_3\lambda_4)}{2\lambda_1\lambda_2},$$

$$\mathrm{Ric}(e_1, e_1) = \frac{a^2((\lambda_3 - \lambda_2)^2 - \lambda_1^2)}{2\lambda_2\lambda_3} + \frac{b^2((\lambda_4 - \lambda_2)^2 - \lambda_1^2)}{2\lambda_2\lambda_4} + \frac{c^2((\lambda_4 - \lambda_3)^2 - \lambda_1^2)}{2\lambda_3\lambda_4},$$

$$\mathrm{Ric}(e_2, e_2) = \frac{a^2((\lambda_3 - \lambda_1)^2 - \lambda_2^2)}{2\lambda_1\lambda_3} + \frac{b^2((\lambda_4 - \lambda_1)^2 - \lambda_2^2)}{2\lambda_1\lambda_4} + \frac{d^2((\lambda_4 - \lambda_3)^2 - \lambda_2^2)}{2\lambda_3\lambda_4},$$

$$\mathrm{Ric}(e_3, e_3) = \frac{a^2((\lambda_2 - \lambda_1)^2 - \lambda_3^2)}{2\lambda_1\lambda_2} + \frac{c^2((\lambda_4 - \lambda_1)^2 - \lambda_3^2)}{2\lambda_1\lambda_4} + \frac{d^2((\lambda_4 - \lambda_2)^2 - \lambda_3^2)}{2\lambda_2\lambda_4},$$

$$\mathrm{Ric}(e_4, e_4) = \frac{b^2((\lambda_2 - \lambda_1)^2 - \lambda_4^2)}{2\lambda_1\lambda_2} + \frac{c^2((\lambda_3 - \lambda_1)^2 - \lambda_4^2)}{2\lambda_1\lambda_3} + \frac{d^2((\lambda_3 - \lambda_2)^2 - \lambda_4^2)}{2\lambda_2\lambda_3}.$$

Without loss of generality, we can assume that $d \neq 0$.

(1) $a^2 + b^2 + c^2 = 0$.
Then $\mathrm{Ric}(e_1, e_1) = 0$. Therefore $\mathrm{Ric}(e_i, e_i) = 0$. Namely, we must have

$$(\lambda_4 - \lambda_3)^2 - \lambda_2^2 = (\lambda_4 - \lambda_2)^2 - \lambda_3^2 = (\lambda_3 - \lambda_2)^2 - \lambda_4^2 = 0. \tag{4.1}$$

For any case, we must have $\lambda_1\lambda_2\lambda_3\lambda_4 = 0$. It is a contradiction.

(2) $abc \neq 0$.
Then we have

$$(\lambda_4 - \lambda_3)^2 - \lambda_1\lambda_2 = (\lambda_4 - \lambda_2)^2 - \lambda_1\lambda_3 = 0,$$
$$(\lambda_3 - \lambda_2)^2 - \lambda_1\lambda_4 = (\lambda_4 - \lambda_1)^2 - \lambda_2\lambda_3 = 0,$$
$$(\lambda_3 - \lambda_1)^2 - \lambda_2\lambda_4 = (\lambda_2 - \lambda_1)^2 - \lambda_3\lambda_4 = 0.$$

Then $\lambda_i \neq \lambda_j$ for $i \neq j$ since $\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0$. By $(\lambda_4 - \lambda_3)^2 - \lambda_1\lambda_2 = (\lambda_4 - \lambda_2)^2 - \lambda_1\lambda_3$, we have

$$2\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3.$$

Similarly, by $(\lambda_3 - \lambda_1)^2 - \lambda_2\lambda_4 = (\lambda_2 - \lambda_1)^2 - \lambda_3\lambda_4$, we have

$$2\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4.$$

It follows that $\lambda_1 = \lambda_4$, which is a contradiction.

(3) Two of a, b, c are zero. Assume that $a^2 + b^2 = 0$ and $c \neq 0$.
Then we must have

$$\mathrm{Ric}(e_1, e_2) = (\lambda_4 - \lambda_3)^2 - \lambda_1\lambda_2 = 0. \tag{4.2}$$

By $\mathrm{Ric}(x, y) = mg(x, y)$, we have

$$\mathrm{Ric}(e_1, e_1) = \frac{c^2((\lambda_4 - \lambda_3)^2 - \lambda_1^2)}{2\lambda_3\lambda_4} = m\lambda_1,$$

$$\mathrm{Ric}(e_2, e_2) = \frac{d^2((\lambda_4 - \lambda_3)^2 - \lambda_2^2)}{2\lambda_3\lambda_4} = m\lambda_2.$$

It follows that $(c^2+d^2)(\lambda_1-\lambda_2)=0$. That is,

$$\lambda_1=\lambda_2, \quad m=0.$$

Furthermore, we have

$$(\lambda_4-\lambda_3)^2-\lambda_2^2=(\lambda_4-\lambda_2)^2-\lambda_3^2=(\lambda_3-\lambda_2)^2-\lambda_4^2=0.$$

For any case, we must have $\lambda_1\lambda_2\lambda_3\lambda_4=0$. It is a contradiction.

(4) Only one of a,b,c is zero. Assume that $a=0, bc\neq 0$.
Then we have

$$(\lambda_4-\lambda_1)^2-\lambda_2\lambda_3=(\lambda_4-\lambda_2)^2-\lambda_1\lambda_3=(\lambda_4-\lambda_3)^2-\lambda_1\lambda_2=0. \tag{4.3}$$

It follows that

$$(\lambda_2-\lambda_1)(2\lambda_4-\lambda_1-\lambda_2-\lambda_3)=0,$$
$$(\lambda_3-\lambda_2)(2\lambda_4-\lambda_1-\lambda_2-\lambda_3)=0.$$

Assume that $2\lambda_4-\lambda_1-\lambda_2-\lambda_3\neq 0$. Then $\lambda_1=\lambda_2=\lambda_3$. By Eq. (4.3), $\lambda_4=2\lambda_1$. It follows that

$$\mathrm{Ric}(e_1,e_1)=\mathrm{Ric}(e_2,e_2)=\mathrm{Ric}(e_3,e_3)=0.$$

Also

$$\mathrm{Ric}(e_4,e_4)=\frac{(b^2+c^2+d^2)(-\lambda_4^2)}{2\lambda_1^2}=0.$$

Thus $\lambda_4=0$. It is a contradiction. Then we must have

$$2\lambda_4-\lambda_1-\lambda_2-\lambda_3=0.$$

Let $k_i=\frac{\lambda_i}{\lambda_1}, i=2,3,4$. By $(\lambda_4-\lambda_2)^2-\lambda_1\lambda_3=0$ and the above equation, we have

$$2k_4-1-k_2-k_3=0 \quad \text{and} \quad (k_4-k_3)^2-k_2=0.$$

Then we have $k_3=k_2\pm 2\sqrt{k_2}+1$. Let $k=k_2>0$. Then

$$\lambda_2=k\lambda_1, \quad \lambda_3=(k\pm 2\sqrt{k}+1)\lambda_1, \quad \lambda_4=(k\pm\sqrt{k}+1)\lambda_1.$$

Case 1. $\lambda_2=k\lambda_1, \lambda_3=(k+2\sqrt{k}+1)\lambda_1, \lambda_4=(k+\sqrt{k}+1)\lambda_1$. Then we have

$$\mathrm{Ric}(e_1,e_1)=\frac{b^2((\sqrt{k}+1)^2-1)}{2k(k+\sqrt{k}+1)}+\frac{c^2(k-1)}{2(k+2\sqrt{k}+1)(k+\sqrt{k}+1)},$$
$$\mathrm{Ric}(e_2,e_2)=\frac{b^2((k+\sqrt{k})^2-k^2)}{2(k+\sqrt{k}+1)}+\frac{d^2(k-k^2)}{2(k+2\sqrt{k}+1)(k+\sqrt{k}+1)},$$

$$\mathrm{Ric}(e_3,e_3)=\frac{c^2((k+\sqrt{k})^2-(k+2\sqrt{k}+1)^2)}{2(k+\sqrt{k}+1)}+\frac{d^2((\sqrt{k}+1)^2-(k+2\sqrt{k}+1)^2)}{2k(k+\sqrt{k}+1)},$$

$$\mathrm{Ric}(e_4,e_4)=\frac{b^2((k-1)^2-(k+\sqrt{k}+1)^2)}{2k}+\frac{c^2((k+2\sqrt{k})^2-(k+\sqrt{k}+1)^2)}{2(k+2\sqrt{k}+1)}+\frac{d^2((2\sqrt{k}+1)^2-(k+\sqrt{k}+1)^2)}{2k(k+2\sqrt{k}+1)}.$$

It is easy to see that $\mathrm{Ric}(e_3,e_3)<0$. Then we must have $\mathrm{Ric}(e_1,e_1)<0$ and $\mathrm{Ric}(e_2,e_2)<0$. It follows that

$$k-1<0,\quad k-k^2<0.$$

It is a contradiction since $k>0$.

Case 2. $\lambda_2=k\lambda_1, \lambda_3=(k-2\sqrt{k}+1)\lambda_1, \lambda_4=(k-\sqrt{k}+1)\lambda_1$. Then we have

$$\mathrm{Ric}(e_1,e_1)=\frac{b^2((-\sqrt{k}+1)^2-1)}{2k(k-\sqrt{k}+1)}+\frac{c^2(k-1)}{2(k-2\sqrt{k}+1)(k-\sqrt{k}+1)},$$

$$\mathrm{Ric}(e_2,e_2)=\frac{b^2((k-\sqrt{k})^2-k^2)}{2(k-\sqrt{k}+1)}+\frac{d^2(k-k^2)}{2(k-2\sqrt{k}+1)(k-\sqrt{k}+1)},$$

$$\mathrm{Ric}(e_3,e_3)=\frac{c^2((k-\sqrt{k})^2-(k-2\sqrt{k}+1)^2)}{2(k-\sqrt{k}+1)}+\frac{d^2((-\sqrt{k}+1)^2-(k-2\sqrt{k}+1)^2)}{2k(k-\sqrt{k}+1)},$$

$$\mathrm{Ric}(e_4,e_4)=\frac{b^2((k-1)^2-(k-\sqrt{k}+1)^2)}{2k}+\frac{c^2((k-2\sqrt{k})^2-(k-\sqrt{k}+1)^2)}{2(k-2\sqrt{k}+1)}+\frac{d^2((-2\sqrt{k}+1)^2-(k-\sqrt{k}+1)^2)}{2k(k-2\sqrt{k}+1)}.$$

If $k\geq 4$, then $\mathrm{Ric}(e_1,e_1)>0$ and $\mathrm{Ric}(e_2,e_2)<0$. If $1\leq k<4$, then $\mathrm{Ric}(e_3,e_3)>0$ and $\mathrm{Ric}(e_2,e_2)<0$. If $\frac{1}{4}<k<1$, then $\mathrm{Ric}(e_1,e_1)<0$ and $\mathrm{Ric}(e_3,e_3)>0$. If $0<k\leq\frac{1}{4}$, then $\mathrm{Ric}(e_1,e_1)<0$ and $\mathrm{Ric}(e_2,e_2)>0$. It is a contradiction.

So we prove the following assertion.

Proposition 4.5. *There is no Einstein metric on the nonabelian compact Lie group of dimension four.*

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