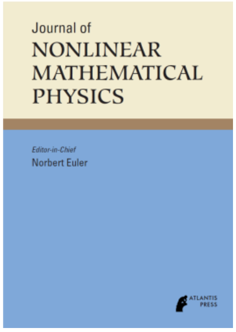

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Hyperkähler Structure of the Taub-NUT Metric

G. Gaeta, M. A. Rodríguez

To cite this article: G. Gaeta, M. A. Rodríguez (2012) Hyperkähler Structure of the Taub-NUT Metric, Journal of Nonlinear Mathematical Physics 19:2, 226–235, DOI: https://doi.org/10.1142/S1402925112500143

To link to this article: https://doi.org/10.1142/S1402925112500143

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 19, No. 2 (2012) 1250014 (10 pages)

DOI: 10.1142/S1402925112500143

HYPERKÄHLER STRUCTURE OF THE TAUB-NUT METRIC

G. GAETA

Dipartimento di Matematica, Università degli Studi di Milano
v. Saldini 50, Milano I-20133, Italy
giuseppe.gaeta@unimi.it

M. A. RODRÍGUEZ

Departamento de Física Teórica II
Universidad Complutense, Madrid 28040, Spain
rodrigue@fis.ucm.es

Received 10 November 2011
Accepted 9 January 2012
Published 4 July 2012

The Taub-NUT four-dimensional space-time can be obtained from Euclidean eight-dimensional one through a momentum map construction; the HKLR theorem [9] guarantees the hyperkähler structure of $\mathbf{R}^8$ descends to a hyperkähler structure in the Taub-NUT space. Here we present a detailed and fully explicit construction of the hyperkähler structure of a space-time with a Taub-NUT metric.

Keywords: Hyperkähler manifolds; Taub-NUT metric.

Mathematics Subject Classification 2010: 37J05, 53D05, 53C56

1. Introduction

Hyperkähler manifolds have received in recent years increasing attention both from mathematicians and physicists [1, 6, 16]; their physical relevance is in particular related to supersymmetry and instanton solutions of nonlinear field theories [2–5, 7, 8, 10, 11, 18].

Simple examples of hyperkähler manifolds are provided by Euclidean spaces $\mathbf{R}^{4n}$, which naturally carry a quaternionic structure. It was shown by Hitchin, Karlhede, Lindström and Roček (HKLR) [9] that one can build new hyperkähler manifolds from old ones through a momentum map construction; the reduction of the hyperkähler structure in the source manifold will provide a hyperkähler structure on the reduced manifold.

A specially interesting example of nontrivial hyperkähler manifold is provided by Taub-NUT (Newman, Unti, and Tamburino) space-time [12–14, 17]; this is physically relevant, and of the minimal dimension (four) for hyperkähler manifolds. It provides an explicit example of nontrivial hyperkähler manifold, which can also be used as a test case in the study of hyperhamiltonian dynamics [6] outside of the standard Euclidean cases.

The HKLR theorem mentioned above guarantees the hyperkähler structure can be obtained from the one in $\mathbf{R}^8$ through the HKLR momentum map reduction procedure. It appears this general theorem is not accompanied, in the literature, by many examples for which the hyperkähler structure in the reduced manifold is explicitly provided. In particular, we have not been able to locate an explicit expression for the hyperkähler structure of the Taub-NUT space-times (see [15] for a different approach to this problem).

In this note we will provide such an explicit expression through a direct computation.

2. The Taub-NUT Metric

We will shortly discuss in this section the construction of the Tab-NUT metric (see for instance the details in [19]. We borrow most of our notation from this reference). Let E be a complex line bundle, with fiber $\mathbf{C}$, and base the interval $[0, \ell]$. The coordinates in the bundle are (z, x), where $z \in \mathbf{C}$ and $x \in [0, \ell]$.

We define a connection:

$$\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}t_0, \tag{2.1}$$

where t_0 is a Hermitian endomorphism of $\mathbf{C}$, depending on $x \in [0, \ell]$, (in fact, a real number for each $x \in [0, \ell]$) and three more Hermitian endomorphisms of $\mathbf{C}$, $t_1(x), t_2(x), t_3(x)$. Finally, let us consider two linear maps from the fiber at $x = 0$ to the fiber at $x = \ell$ (b_0) and viceversa (b_ℓ) (all this information can be encoded in a bow diagram [3]).

We consider the action of a local U(1) as a gauge group, in the following way (we skip the details since they are very well known): if $g(x) \in \mathrm{U}(1)$

$$\begin{aligned} t_0(x) &\to g^{-1} t_0 g + \mathrm{i} g^{-1} \frac{\mathrm{d}g(x)}{\mathrm{d}x}, \\ t_i(x) &\to g^{-1} t_i g \quad (i = 1, 2, 3), \\ b_0 &\to g^{-1}(0) b_0 g(\ell), \\ b_\ell &\to g^{-1}(\ell) b_\ell g(0), \end{aligned} \tag{2.2}$$

where (we consider a nontrivial action at the end points and a linear interpolating function for the interior of the interval):

$$g(x) = \mathrm{e}^{\mathrm{i}f(x)} \in \mathrm{U}(1), \quad f(x) = \frac{1}{\ell}((\ell - x)\phi_0 + x\phi_\ell), \quad g(0) = \mathrm{e}^{i\phi_0}, \quad g(\ell) = \mathrm{e}^{\mathrm{i}\phi_\ell}. \tag{2.3}$$

Since the linear maps b_0 and b_ℓ are complex, we will write:

$$b_0 = q_0 + \mathrm{i}q_1, \quad b_\ell = q_2 + \mathrm{i}q_3 \tag{2.4}$$

and if $\theta = \phi_\ell - \phi_0$, we get the following action (all the coordinates are real):

$$\begin{aligned} t_0 &\to t_0 - \frac{\theta}{\ell}, \\ t_i &\to t_i, \quad i = 1, 2, 3, \\ q_0 &\to q_0 \cos\theta - q_1 \sin\theta, \end{aligned}$$

$$\begin{aligned}
q_1 &\longrightarrow q_0 \sin\theta + q_1 \cos\theta, \\
q_2 &\longrightarrow q_2 \cos\theta + q_3 \sin\theta, \\
q_3 &\longrightarrow -q_2 \sin\theta + q_3 \cos\theta.
\end{aligned} \tag{2.5}$$

Using the momentum map associated to this action we can consider the coordinates t_i, $i = 0, 1, 2, 3$ as constants (that is, not depending on x). The Euclidean metric is:

$$\begin{aligned}
\mathrm{d}s^2 &= \int_0^\ell (\mathrm{d}t_0^2 + \mathrm{d}t_1^2 + \mathrm{d}t_2^2 + \mathrm{d}t_3^2)\mathrm{d}x + \mathrm{d}q_0^2 + \mathrm{d}q_1^2 + \mathrm{d}q_2^2 + \mathrm{d}q_3^2 \\
&= \ell(\mathrm{d}t_0^2 + \mathrm{d}t_1^2 + \mathrm{d}t_2^2 + \mathrm{d}t_3^2) + \mathrm{d}q_0^2 + \mathrm{d}q_1^2 + \mathrm{d}q_2^2 + \mathrm{d}q_3^2.
\end{aligned} \tag{2.6}$$

We can consider this space as the sum of two copies of $\mathbf{R}^4$. In the second copy, with coordinates q_i, we introduce quaternionic coordinates, and change the variables, first to a polar form (with angle $\psi/2$) and second to the coordinates r_i, $i = 1, 2, 3$ and ψ given by

$$\begin{aligned}
q_0 &= -\sqrt{\frac{r + r_1}{2}} \sin\frac{\psi}{2}, \\
q_1 &= \sqrt{\frac{r + r_1}{2}} \cos\frac{\psi}{2}, \\
q_2 &= \frac{1}{\sqrt{2(r + r_1)}} \left(r_2 \cos\frac{\psi}{2} + r_3 \sin\frac{\psi}{2} \right), \\
q_3 &= \frac{1}{\sqrt{2(r + r_1)}} \left(-r_2 \sin\frac{\psi}{2} + r_3 \cos\frac{\psi}{2} \right),
\end{aligned} \tag{2.7}$$

where

$$r = \sqrt{r_1^2 + r_2^2 + r_3^2}. \tag{2.8}$$

It is a simple task to write the metric in these coordinates

$$\mathrm{d}s^2 = \ell(\mathrm{d}t_0^2 + \mathrm{d}\vec{t}^{\,2}) + \frac{1}{4}\left[\frac{1}{r}\mathrm{d}\vec{r}^{\,2} + r(\mathrm{d}\psi + \vec{\sigma} \cdot \mathrm{d}\vec{r})^2\right], \tag{2.9}$$

where

$$\vec{t} = (t_1, t_2, t_3), \quad \vec{r} = (r_1, r_2, r_3), \quad \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = \left(0, \frac{r_3}{r(r + r_1)}, -\frac{r_2}{r(r + r_1)}\right). \tag{2.10}$$

Since we will use them in the forthcoming sections, we will write explicitly the matrix of the metric (in the second copy of $\mathbf{R}^4$) in the coordinates (r_1, r_2, r_3, ψ):

$$G^{(1)} = \frac{r}{4} \begin{pmatrix} \frac{1}{r^2} & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} + \sigma_2^2 & \sigma_2\sigma_3 & \sigma_2 \\ 0 & \sigma_2\sigma_3 & \frac{1}{r^2} + \sigma_3^2 & \sigma_3 \\ 0 & \sigma_2 & \sigma_3 & 1 \end{pmatrix} \tag{2.11}$$

and the Jacobian matrix $(\partial q/\partial(\vec{r},\psi))$ of the change of coordinates:

$$\Lambda = \frac{1}{2}\sqrt{\frac{r+r_1}{2}}\left(\Lambda_1 \cos\frac{\psi}{2} + \Lambda_2 \sin\frac{\psi}{2}\right), \tag{2.12}$$

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ \frac{1}{r} & -\sigma_3 & \sigma_2 & 0 \\ \sigma_3 & \frac{1}{r} + r\sigma_2^2 & r\sigma_2\sigma_3 & r\sigma_2 \\ -\sigma_2 & r\sigma_2\sigma_3 & \frac{1}{r} + r\sigma_3^2 & r\sigma_3 \end{pmatrix}, \tag{2.13}$$

$$\Lambda_2 = \begin{pmatrix} -\frac{1}{r} & \sigma_3 & -\sigma_2 & 0 \\ 0 & 0 & 0 & -1 \\ -\sigma_2 & r\sigma_2\sigma_3 & \frac{1}{r} + r\sigma_3^2 & r\sigma_3 \\ -\sigma_3 & -\frac{1}{r} - r\sigma_2^2 & -r\sigma_2\sigma_3 & -r\sigma_2 \end{pmatrix}. \tag{2.14}$$

The relation between the matrices $G^{(1)}$ and Λ is the usual one (since the matrix of the metric in the cartesian coordinates for the Euclidean space is the identity):

$$G^{(1)} = \Lambda^T \Lambda. \tag{2.15}$$

We pass to a quotient space where the Taub-NUT metric is the reduction of the Euclidean metric described in the above paragraphs, using the momentum map (associated to the action of the group U(1)). The inverse image of zero under this map is a submanifold of $\mathbf{R}^8$ given by

$$\vec{t} = -\frac{1}{2}\vec{r} \tag{2.16}$$

and the metric, with coordinates $(t_0, r_1, r_2, r_3, \psi)$ is

$$\mathrm{d}s^2 = \ell \mathrm{d}t_0^2 + \frac{1}{4}\left[\left(\frac{1}{r} + \ell\right)\mathrm{d}\vec{r}^2 + r(\mathrm{d}\psi + \vec{\sigma}\cdot \mathrm{d}\vec{r})^2\right]. \tag{2.17}$$

The action of the gauge group (2.5) on this manifold is:

$$\begin{aligned} t_0 &\to t_0 - \frac{\theta}{\ell}, \\ r_i &\to r_i, \quad i = 1,2,3, \\ \psi &\to \psi + 2\theta, \end{aligned} \tag{2.18}$$

with an invariant given by

$$\tau = 2\ell t_0 + \psi, \quad \mathrm{d}\psi = \mathrm{d}\tau - 2\ell \mathrm{d}t_0, \tag{2.19}$$

which yields the following expression for the metric (in the coordinates $(t_0, r_1, r_2, r_3, \tau)$):

$$\mathrm{d}s^2 = \ell \mathrm{d}t_0^2 + \frac{1}{4}\left[\left(\frac{1}{r} + \ell\right)\mathrm{d}\vec{r}^2 + r(\mathrm{d}\tau - 2\ell \mathrm{d}t_0 + \vec{\sigma}\cdot \mathrm{d}\vec{r})^2\right]. \tag{2.20}$$

Finally, to remove t_0 (which is not invariant under the group action) we take:

$$\mathrm{d}t_0 = \frac{r}{2(1+\ell r)}(\mathrm{d}\tau + \vec{\sigma}\cdot \mathrm{d}\vec{r}) \tag{2.21}$$

and we get the Taub-NUT metric

$$\mathrm{d}s^2 = \frac{1}{4}\left[\left(\frac{1}{r}+\ell\right)\mathrm{d}\vec{r}^2 + \frac{1}{\frac{1}{r}+\ell}(\mathrm{d}\tau + \vec{\sigma}\cdot \mathrm{d}\vec{r})^2\right]. \tag{2.22}$$

We will write the matrix associated to the Taub-NUT metric (in the coordinates (r_1, r_2, r_3, τ)) which will be used in the following sections:

$$G^{\mathrm{TNUT}} = \frac{1}{4(\ell+\frac{1}{r})}\begin{pmatrix} (\ell+\frac{1}{r})^2 & 0 & 0 & 0 \\ 0 & (\ell+\frac{1}{r})^2+\sigma_2^2 & \sigma_2\sigma_3 & \sigma_2 \\ 0 & \sigma_2\sigma_3 & (\ell+\frac{1}{r})^2+\sigma_3^2 & \sigma_3 \\ 0 & \sigma_2 & \sigma_3 & 1 \end{pmatrix}. \tag{2.23}$$

3. Quotient Hyperkähler Structures

We will discuss in the following sections how to construct a hyperkhäler structure in a four-dimensional manifold with a Taub-NUT metric. This is an example of the construction of hyperkähler spaces as quotients.

Let M be a manifold with a metric g and assume we have three complex structures J_α satisfying the quaternionic relations (sum over repeated indices):

$$J_\alpha J_\beta = \epsilon_{\alpha\beta\gamma} J_\gamma - \delta_{\alpha\beta} I. \tag{3.1}$$

Using J_α and the metric, we can define three symplectic forms in the usual way

$$\omega_\alpha = g(J_\alpha\cdot, \cdot), \quad \alpha = 1, 2, 3. \tag{3.2}$$

Our goal is to construct explicitly the complex structures and the symplectic forms when the metric is the Taub-NUT metric written in the coordinates we used in Sec. 2. As in that approach, our starting point will be a standard hyperkhäler structure in $\mathbf{R}^8$. In the following we will refer to standard hyperkähler structures in $\mathbf{R}^{4n}$ which are obtained from standard structures in $\mathbf{R}^4$ (endowed with an Euclidean metric).

There are two such standard structures, differing for their orientation. The positively-oriented standard hyperkähler structure is given by

$$Y_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

with corresponding symplectic structures (3.2)

$$\begin{aligned} \omega_1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4, \\ \omega_2 &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3, \\ \omega_3 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^4 \wedge \mathrm{d}x^2. \end{aligned} \tag{3.4}$$

In matrix form, these symplectic forms read as:

$$K_\alpha^{(0)} = Y_\alpha, \quad \alpha = 1,2,3 \tag{3.5}$$

since the matrix of the Euclidean metric is the identity (in the cartesian coordinates we are using). Note that, in the general case, if G is the matrix of the metric, J_α the matrices of the hyperkähler structures and K_α the matrices of the symplectic forms, the following relations hold:

$$J_\alpha = G^{-1} K_\alpha, \quad \alpha = 1,2,3. \tag{3.6}$$

The negatively-oriented standard hyperkähler structure is given by

$$\hat{Y}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{Y}_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{Y}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.7}$$

with corresponding symplectic structures,

$$\begin{aligned} \hat{\omega}_1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^2 \wedge \mathrm{d}x^4, \\ \hat{\omega}_2 &= \mathrm{d}x^4 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3, \\ \hat{\omega}_3 &= \mathrm{d}x^2 \wedge \mathrm{d}x^1 + \mathrm{d}x^3 \wedge \mathrm{d}x^4. \end{aligned} \tag{3.8}$$

When we change the coordinate system, as we did in Sec. 2 passing from the Cartesian coordinates (q_0, q_1, q_2, q_3) to the coordinates (r_1, r_2, r_3, ψ), the corresponding matrices of the metric, symplectic forms and quaternionic structure change, and we get

$$G^{(1)} = \Lambda^T I_4 \Lambda = \Lambda^T \Lambda, \quad K_\alpha^{(1)} = \Lambda^T K_\alpha^{(0)} \Lambda = \Lambda^T Y_\alpha \Lambda, \quad J_\alpha^{(1)} = \Lambda^{-1} Y_\alpha \Lambda \tag{3.9}$$

and the general relation still holds

$$J_\alpha^{(1)} = (G^{(1)})^{-1} K_\alpha^{(1)}, \quad \alpha = 1,2,3. \tag{3.10}$$

The symplectic form matrices $K_\alpha^{(1)}$ are

$$\begin{gathered} K_1^{(1)} = \frac{1}{4} \begin{pmatrix} 0 & \sigma_2 & \sigma_3 & 1 \\ -\sigma_2 & 0 & \frac{1}{r} & 0 \\ -\sigma_3 & -\frac{1}{r} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K_2^{(1)} = \frac{1}{4} \begin{pmatrix} 0 & \frac{1}{r} & 0 & 0 \\ -\frac{1}{r} & 0 & -\sigma_2 & 0 \\ 0 & \sigma_2 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\ K_3^{(1)} = \frac{1}{4} \begin{pmatrix} 0 & 0 & -\frac{1}{r} & 0 \\ 0 & 0 & \sigma_3 & 1 \\ \frac{1}{r} & -\sigma_3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{gathered} \tag{3.11}$$

and that of the quaternionic matrices $J_\alpha^{(1)}$,

$$J_1^{(1)} = \begin{pmatrix} 0 & r\sigma_2 & r\sigma_3 & r \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{r} & \sigma_3 & -\sigma_2 & 0 \end{pmatrix}, \quad J_2^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & r\sigma_2 & r\sigma_3 & r \\ \sigma_2 & -r\sigma_2\sigma_3 & -\frac{1}{r} - r\sigma_3^2 & -r\sigma_3 \end{pmatrix},$$
$$J_3^{(1)} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & r\sigma_2 & r\sigma_3 & r \\ 1 & 0 & 0 & 0 \\ -\sigma_3 & -\frac{1}{r} - r\sigma_2^2 & -r\sigma_2\sigma_3 & -r\sigma_2 \end{pmatrix} \tag{3.12}$$

and they satisfy the quaternionic relations (3.1).

Since we have two copies of $\mathbf{R}^4$ in our original space $\mathbf{R}^8$ (2.6) we could consider several combinations of positive or negative oriented hyperkähler structures. However, the procedure will be essentially the same and we will restrict to the case of two positive oriented hyperkähler structures. Hence, in the original $\mathbf{R}^8$ and cartesian coordinates, these matrices are (I_4 is the identity matrix in four dimensions):

$$\mathfrak{G}^{(0)} = \begin{pmatrix} \ell I_4 & 0 \\ 0 & I_4 \end{pmatrix}, \quad \mathfrak{K}_\alpha^{(0)} = \begin{pmatrix} \ell Y_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix},$$
$$\mathfrak{J}_\alpha^{(0)} = (\mathfrak{G}^{(0)})^{-1}\mathfrak{K}_\alpha^{(0)} = \begin{pmatrix} Y_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix}, \quad \alpha = 1, 2, 3. \tag{3.13}$$

After changing the coordinates in the second copy of $\mathbf{R}^4$ we get the following set of matrices:

$$\mathfrak{G}^{(1)} = \begin{pmatrix} \ell I_4 & 0 \\ 0 & G^{(1)} \end{pmatrix}, \quad \mathfrak{K}_\alpha^{(1)} = \begin{pmatrix} \ell Y_\alpha & 0 \\ 0 & K_\alpha^{(1)} \end{pmatrix},$$
$$\mathfrak{J}_\alpha^{(1)} = (\mathfrak{G}^{(1)})^{-1}\mathfrak{K}_\alpha^{(1)} = \begin{pmatrix} Y_\alpha & 0 \\ 0 & J_\alpha^{(1)} \end{pmatrix}, \quad \alpha = 1, 2, 3. \tag{3.14}$$

4. Hyperkähler Structure and the Taub-NUT Metric

In the construction of the Taub-NUT metric we reduce an eight-dimensional manifold to a four-dimensional one. Our aim is to study the reduction of the quaternionic structure. A direct approach to this problem is to write the metric and the symplectic forms in the new coordinates. We have solved the problem with the metric, but not with the symplectic forms and we do not have an explicit form for the quotient under the action of the gauge group. But we know explicitly the relation between the forms which provides the quotient space (see Eqs. (2.16), (2.19) and (2.21))

$$\mathrm{d}\vec{t} = -\frac{1}{2}\mathrm{d}\vec{r}, \quad \mathrm{d}\psi = \mathrm{d}\tau - 2\ell \mathrm{d}t_0, \quad \mathrm{d}t_0 = \frac{r}{2(1+\ell r)}(\mathrm{d}\tau + \vec{\sigma}\cdot \mathrm{d}\vec{r}) \tag{4.1}$$

and that is the only fact we need to construct the symplectic forms. In an explicit form

$$\begin{aligned} dt_0 &= \frac{r}{2(1+\ell r)}d\tau + \frac{r}{2(1+\ell r)}\vec{\sigma}\cdot d\vec{r}, \\ dt_\alpha &= -\frac{1}{2}dr_\alpha, \quad \alpha = 1,2,3, \\ d\psi &= \frac{1}{1+\ell r}d\tau - \frac{\ell r}{1+\ell r}\vec{\sigma}\cdot d\vec{r}. \end{aligned} \tag{4.2}$$

Before the reduction, the symplectic forms are (corresponding to the matrices $\mathfrak{K}_\alpha^{(1)}$, (3.14)):

$$\begin{aligned} \omega_1 &= \ell dt_0 \wedge dt_1 + \ell dt_2 \wedge dt_3 + \frac{1}{4}\sigma_2 dr_1 \wedge dr_2 + \frac{1}{4}\sigma_3 dr_1 \wedge dr_3 \\ &\quad + \frac{1}{4r}dr_2 \wedge dr_3 + \frac{1}{4}dr_1 \wedge d\psi, \\ \omega_2 &= \ell dt_0 \wedge dt_3 + \ell dt_1 \wedge dt_2 + \frac{1}{4r}dr_1 \wedge dr_2 - \frac{1}{4}\sigma_2 dr_2 \wedge dr_3 + \frac{1}{4}dr_3 \wedge d\psi, \\ \omega_3 &= \ell dt_0 \wedge dt_2 + \ell dt_3 \wedge dt_1 - \frac{1}{4r}dr_1 \wedge dr_3 + \frac{1}{4}\sigma_3 dr_2 \wedge dr_3 + \frac{1}{4}dr_2 \wedge d\psi \end{aligned} \tag{4.3}$$

and in the quotient space, after substituting (4.2)

$$\begin{aligned} \omega_1 &= \frac{1}{4}dr_1 \wedge d\tau + \frac{1}{4}\sigma_2 dr_1 \wedge dr_2 + \frac{1}{4}\sigma_3 dr_1 \wedge dr_3 + \frac{1}{4}\left(\ell + \frac{1}{r}\right) dr_2 \wedge dr_3, \\ \omega_2 &= \frac{1}{4}dr_3 \wedge d\tau - \frac{1}{4}\sigma_2 dr_2 \wedge dr_3 + \frac{1}{4}\left(\ell + \frac{1}{r}\right) dr_1 \wedge dr_2, \\ \omega_3 &= \frac{1}{4}dr_2 \wedge d\tau + \frac{1}{4}\sigma_3 dr_2 \wedge dr_3 - \frac{1}{4}\left(\ell + \frac{1}{r}\right) dr_1 \wedge dr_3, \end{aligned} \tag{4.4}$$

with matrices

$$\begin{aligned} K_1^{\mathrm{TNUT}} &= \frac{1}{4}\begin{pmatrix} 0 & \sigma_2 & \sigma_3 & 1 \\ -\sigma_2 & 0 & \ell + \frac{1}{r} & 0 \\ -\sigma_3 & -\ell - \frac{1}{r} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K_2^{\mathrm{TNUT}} = \frac{1}{4}\begin{pmatrix} 0 & \ell + \frac{1}{r} & 0 & 0 \\ -1 - \frac{1}{r} & 0 & -\sigma_2 & 0 \\ 0 & \sigma_2 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\ K_3^{\mathrm{TNUT}} &= \frac{1}{4}\begin{pmatrix} 0 & 0 & -\ell - \frac{1}{r} & 0 \\ 0 & 0 & \sigma_3 & 1 \\ \ell + \frac{1}{r} & -\sigma_3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.5}$$

Since we know the matrix of the metric (2.23), we can compute the matrices of the quaternionic structure ($Y_\alpha^{\mathrm{TNUT}} = (G^{\mathrm{TNUT}})^{-1} K_\alpha^{\mathrm{TNUT}}$):

$$J_1^{\mathrm{TNUT}} = \begin{pmatrix} 0 & \sigma_2(\ell+\frac{1}{r})^{-1} & \sigma_3(\ell+\frac{1}{r})^{-1} & (\ell+\frac{1}{r})^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{r}-\ell & \sigma_3 & -\sigma_2 & 0 \end{pmatrix},$$

$$J_2^{\mathrm{TNUT}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & \sigma_2(\ell+\frac{1}{r})^{-1} & \sigma_3(\ell+\frac{1}{r})^{-1} & (\ell+\frac{1}{r})^{-1} \\ \sigma_2 & -\sigma_2\sigma_3(\ell+\frac{1}{r})^{-1} & -\ell-\frac{1}{r}-\sigma_3^2(\ell+\frac{1}{r})^{-1} & -\sigma_3(\ell+\frac{1}{r})^{-1} \end{pmatrix}, \tag{4.6}$$

$$J_3^{\mathrm{TNUT}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & \sigma_2(\ell+\frac{1}{r})^{-1} & \sigma_3(\ell+\frac{1}{r})^{-1} & (\ell+\frac{1}{r})^{-1} \\ 1 & 0 & 0 & 0 \\ -\sigma_3 & -\ell-\frac{1}{r}-\sigma_2^2(\ell+\frac{1}{r})^{-1} & -\sigma_2\sigma_3(\ell+\frac{1}{r})^{-1} & -\sigma_2(\ell+\frac{1}{r})^{-1} \end{pmatrix},$$

which satisfy the quaternionic relations (3.1). It can also be checked that they are covariantly constant under the Levi-Civita connection associated to the Taub-NUT metric.

We have thus explicitly computed the hyperkähler structures, thus implementing the abstract HKLR theorem [9] in the concrete case of Taub-NUT. Note that all these structures reproduce the flat case when $r \to 0$.

Had we chosen as starting point a standard hyperkähler structure in $\mathbf{R}^8 = \mathbf{R}^4 \oplus \mathbf{R}^4$ with different orientations, we would have obtained similar results. In fact, working with a negative orientation we can easily choose a different set of coordinates (2.7) and obtain the same expressions for the hyperkähler and quaternionic structures.

Acknowledgments

This research was supported by the Spanish Ministry of Science and Innovation under grant No. FIS2011-22566 and the Universidad Complutense and Banco Santander under grant No. GR58/08-910556. M. A. Rodríguez thanks the Università di Milano and G. Gaeta the Universidad Complutense de Madrid for their hospitality and support.

References

[1] D. V. Alekseevsky and S. Marchiafava, Hermitian and Kähler submanifolds of a quaternionic Kähler manifold, *Osaka J. Math.* **38** (2001) 869–904.

[2] P.-Y. Casteill, E. Ivanov and G. Valent, Quaternionic extension of the double Taub-NUT metric, *Phys. Lett. B* **508** (2001) 354–364.

[3] S. A. Cherkis, Moduli spaces of instantons on the Taub-NUT space, *Comm. Math. Phys.* **290** (2009) 719–736.

[4] S. A. Cherkis and A. Kapustin, Hyper-Kähler metrics from periodic monopoles, *Phys. Rev. D* **65** (2002) 084015.

[5] D. Cherney, E. Latini and A. Waldron, Quaternionic Kähler detour complexes and $\mathcal{N} = 2$ supersymmetric black holes, *Comm. Math. Phys.* **302** (2011) 843–873.

[6] G. Gaeta and P. Morando, Hyper-Hamiltonian dynamics, *J. Phys. A* **35** (2002) 3925–3943.

[7] G. W. Gibbons, P. Rychenkova and R. Goto, HyperKähler quotient construction of BPS monopole moduli spaces, *Comm. Math. Phys.* **186** (1997) 581–599.

[8] N. Hitchin, Instantons, Poisson structures and generalized Kähler geometry, *Comm. Math. Phys.* **265** (2006) 131–164.

[9] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Hyperkähler metrics and supersymmetry, *Comm. Math. Phys.* **108** (1987) 535–589.

[10] P. B. Kronheimer, The construction of ALE spaces as hyper-Kähler quotients, *J. Differential Geom.* **29** (1989) 665–683.

[11] U. Lindström and M. Roček, Properties of hyperkähler manifolds and their twistor spaces, *Comm. Math. Phys.* **293** (2010) 257–278.

[12] J. G. Miller, M. D. Kruskal and B. B. Godfrey, Taub-NUT (Newman, Unti, Tamburino) metric and incompatible extensions, *Phys. Rev. D* **4** (1971) 2945–2948.

[13] C. Misner, The Flatter Regions of Newman, Unti, and Tamburino's generalized Schwarzschild space, *J. Math. Phys.* **4** (1963) 924–938.

[14] E. Newman, L. Tamburino and T. Unti, Empty-space generalization of the Schwarzschild metric, *J. Math. Phys.* **4** (1963) 915–924.

[15] T. Noda, A special Lagrangian fibration in the Taub-NUT space, *J. Math. Soc. Japan* **60** (2008) 653–663.

[16] A. Swann, HyperKähler and quaternionic Kähler geometry, *Math. Ann.* **289** (1991) 421–450.

[17] A. H. Taub, Empty space-times admitting a three parameter group of motions, *Ann. Math.* **53** (1951) 472–490.

[18] E. Witten, Branes, instantons, and Taub-NUT spaces, preprint (2009), arXiv:0902.0948v2 [hep-th].

[19] http://www.maths.tcd.ie/~islands/index.php?title=Taub-NUT_space_as_a_hyperkahler_quotient.