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## CLOSED FORM SOLUTIONS TO THE INTEGRABLE DISCRETE NONLINEAR SCHRÖDINGER EQUATION

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In this article we derive explicit solutions of the matrix integrable discrete nonlinear Schrödinger equation by using the inverse scattering transform and the Marchenko method. The Marchenko equation is solved by separation of variables, where the Marchenko kernel is represented in separated form, using a matrix triplet  $(A, B, C)$ . Here  $A$  has only eigenvalues of modulus larger than one. The class of solutions obtained contains the  $N$ -soliton and breather solutions as special cases. We also prove that these solutions reduce to known continuous matrix NLS solutions as the discretization step vanishes.

*Keywords:* Ablowitz–Ladik model; exact solutions; Marchenko method; integrable discrete nonlinear Schrödinger equation.

### 1. Introduction

In this article we derive explicit solutions of the system of *integrable discrete nonlinear Schrödinger (IDNLS) equations*

$$i \frac{d}{d\tau} \mathbf{u}_n = \mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1} - \mathbf{u}_{n+1} \mathbf{w}_n \mathbf{u}_n - \mathbf{u}_n \mathbf{w}_n \mathbf{u}_{n-1}, \quad (1.1a)$$

$$-i \frac{d}{d\tau} \mathbf{w}_n = \mathbf{w}_{n+1} - 2\mathbf{w}_n + \mathbf{w}_{n-1} - \mathbf{w}_{n+1} \mathbf{u}_n \mathbf{w}_n - \mathbf{w}_n \mathbf{u}_n \mathbf{w}_{n-1}, \quad (1.1b)$$

where  $n$  is an integer labeling “position” and  $\mathbf{u}_n$  and  $\mathbf{w}_n$  are  $N \times M$  and  $M \times N$  matrix functions depending on “time”  $\tau \in \mathbb{R}$ . The *focusing case* occurs if  $\mathbf{w}_n = -\mathbf{u}_n^\dagger$  for each integer  $n$ , where the dagger denotes conjugate matrix transposition. Equations (1.1) can in principle be viewed as a discretization of the matrix nonlinear Schrödinger (NLS) equations (5.2) below, where the “position” variable  $x \in \mathbb{R}$ . As such, they have applications to electromagnetic wave propagation in nonlinear media [21, 27], surface waves on deep waters [27], and signal propagation in optical fibers [15, 16]. On their own behalf, these equations have applications to Heisenberg spin chains [18], self-trapping on a dimer [19], the dynamics of a discrete curve on an ultraspherical surface [11], the dynamics of triangulations of surfaces [17], and Hamiltonian flows [20, 24].

The IST method associates (1.1) to the *discrete Zakharov–Shabat system*

$$\mathbf{v}_{n+1} = \begin{pmatrix} zI_N & \mathbf{u}_n \\ \mathbf{w}_n & z^{-1}I_M \end{pmatrix} \mathbf{v}_n, \quad (1.2)$$

where  $z$  is the (complex) spectral parameter and  $I_N - \mathbf{u}_n \mathbf{w}_n$  is assumed nonsingular for each  $n \in \mathbb{Z}$  (which is always true in the focusing case) and the potentials  $\{\mathbf{u}_n\}_{n=-\infty}^{\infty}$  and  $\{\mathbf{w}_n\}_{n=-\infty}^{\infty}$  satisfy the  $\ell^1$ -condition

$$\sum_{n=-\infty}^{\infty} \{\|\mathbf{u}_n\| + \|\mathbf{w}_n\|\} < +\infty. \quad (1.3)$$

Here  $\|\cdot\|$  denotes any matrix norm. The direct and inverse scattering of the discrete Zakharov–Shabat system (1.2) has been studied as early as in 1981 [13, 14]. More complete accounts have been given in [26; 4, Chap. 5]. In all of these sources it is assumed that the discrete eigenvalues of (2.2) are algebraically and geometrically simple.

In most of the IDNLS literature it appeared convenient to make the so-called assumption (always satisfied for  $N = M = 1$ ) that  $N = M$  and

$$\mathbf{u}_n \mathbf{w}_n = \mathbf{w}_n \mathbf{u}_n = c_n I_N, \quad n \in \mathbb{Z}, \quad (1.4)$$

where  $\{1 - c_n\}_{n=-\infty}^{\infty}$  is a sequence of nonzero complex numbers. In fact, a major part of [4, Chap. 5] is only valid under condition (1.4). In particular, the way to pass from the solutions of their Marchenko equations (5.2.161) to the potentials  $\mathbf{u}_n$  and  $\mathbf{w}_n$  (i.e., [4, Eqs. (5.2.160)]) can only be applied under condition (1.4). Nevertheless, a thorough analysis of [4, Chap. 5] taught us that their Jost solution and transition coefficient material does not require condition (1.4). Thus in this article we do not use condition (1.4).

The scalar ( $N = M = 1$ ) IDNLS equation was first studied by Ablowitz and Ladik [1–3] by the inverse scattering transform (IST) method. The matrix equations (1.1) were studied in detail using the IST method by Ablowitz, Prinari, and Trubatch [4, Chap. 5] and, assuming condition (1.4), by Tsuchida, Ujino and Wadati [26]. Under condition (1.4), Tsuchida *et al.* have also derived the  $N$ -soliton and breather solutions to (1.1) in terms of solutions to  $N \times N$  linear systems [26, Eq. (3.43)]. Some of these breather solutions were constructed before by using the Hirota method [8].

If we allow the potentials to be time dependent in such a way that (1.1) are satisfied, then the time evolution of the scattering data is such that the Marchenko kernels  $\mathbf{F}(n; \tau)$  and  $\bar{\mathbf{F}}(n; \tau)$  known in the literature satisfy the discrete evolution equations

$$i \frac{d}{d\tau} \mathbf{F}(n; \tau) = \mathbf{F}(n+2; \tau) - 2\mathbf{F}(n; \tau) + \mathbf{F}(n-2; \tau), \quad (1.5a)$$

$$-i \frac{d}{d\tau} \bar{\mathbf{F}}(n; \tau) = \bar{\mathbf{F}}(n+2; \tau) - 2\bar{\mathbf{F}}(n; \tau) + \bar{\mathbf{F}}(n-2; \tau). \quad (1.5b)$$

As can be verified by substitution, explicit solutions to (1.5) can be written as follows:

$$\mathbf{F}(n; \tau) = C e^{-i\tau(A-A^{-1})^2} A^{-(n+1)} B, \quad (1.6a)$$

$$\bar{\mathbf{F}}(n; \tau) = \bar{C} e^{i\tau(\bar{A}-\bar{A}^{-1})^2} \bar{A}^{n-1} \bar{B}, \quad (1.6b)$$

where

- (i)  $A$ ,  $B$  and  $C$  are complex  $p \times p$ ,  $p \times N$  and  $M \times p$  matrices, respectively, and  $A$  is a matrix having only eigenvalues of modulus larger than one;
- (ii)  $\bar{A}$ ,  $\bar{B}$  and  $\bar{C}$  are complex  $\bar{p} \times \bar{p}$ ,  $\bar{p} \times M$  and  $N \times \bar{p}$  matrices, respectively, and  $\bar{A}$  is a nonsingular matrix which has only eigenvalues of modulus less than one.

Equations (1.6) allow us to solve the time evolved Marchenko equations, (3.7) below, explicitly in terms of the two matrix triplets by separation of variables and to derive the explicit matrix IDNLS solutions  $\mathbf{u}_n(\tau)$  and  $\mathbf{w}_n(\tau)$ .

Representations of Marchenko kernels of the type

$$\mathbf{F}(x, \tau) = C e^{xA} e^{-i\tau\varphi(A)} B,$$

where the time factor  $e^{-i\tau\varphi(A)}$  commutes with  $A$ , have been successfully used to find closed form solutions of integrable nonlinear evolution equations in terms of matrix exponentials and solutions of Lyapunov and/or Sylvester equations. We mention results for the KdV [7], NLS [5, 9, 10], and sine-Gordon equations [6]. Similar results were obtained for the sine-Gordon [23] and Toda lattice equations [22] with the help of matrix or operator triplets, but without using Marchenko theory. If the position variable is an integer,  $n$ , then integer matrix powers take the place of matrix exponentials. Such explicit solutions provide a concise way to write closed form solutions, which can equivalently be expressed in terms of trigonometric and polynomial functions of  $x$  (or  $n$ ) and  $t$  by “unpacking” the matrix exponentials, integer matrix powers and matrix inverses appearing in these formulas.

It appears [4, 26] that the spectrum of the discrete matrix Zakharov–Shabat system (1.2) is invariant under the sign inversion  $z \mapsto -z$  and that, as a result, the Marchenko kernels  $\mathbf{F}(n; \tau)$  and  $\bar{\mathbf{F}}(n; \tau)$  vanish if  $n$  is an even integer. Thus further constraints on the triplets  $(A, B, C)$  and  $(\bar{A}, \bar{B}, \bar{C})$  are required to represent the Marchenko kernels in the form (1.6). In fact, the matrix triplets have to be decomposed as in (4.3) below in terms of matrix triplets  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  and  $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{C}})$ , where  $\mathcal{A}$  and  $\bar{\mathcal{A}}$  have half the matrix orders that  $A$  and  $\bar{A}$  have.

In this article we state many results regarding the scattering theory of the discrete Zakharov–Shabat system without further ado. Only if an almost clone of the proof is not available in [4, 26], we give a complete proof, especially if it is needed in deriving an alternative and new pair of Marchenko equations. In contrast to [4, 26], we introduce transmission coefficients, left reflection coefficients, and, more importantly, an alternative pair of Marchenko equations whose solutions yield the potentials  $\mathbf{u}_n$  and  $\mathbf{w}_n$  without having to assume condition (1.4). We have given more details on the sign inversion symmetry reduction of the Marchenko equations to make the derivation of our solution formulas more transparent.

Let us discuss the contents of the various sections. In Sec. 2 we introduce preliminaries on the Jost solutions and scattering coefficients along with their basic properties. We formulate the various analyticity properties by writing the Jost solutions and scattering coefficients as sums of absolutely convergent Fourier series. In Sec. 3 we apply sign inversion symmetry to reduce the Marchenko equations and discuss conjugation symmetry to get a further reduction specific to the focusing case. In Sec. 4 we write the matrix IDNLS solutions  $\mathbf{u}_n(\tau)$  in terms of matrix triplets  $(A, B, C)$  and  $(\bar{A}, \bar{B}, \bar{C})$ , both without symmetries on the potential and in the focusing case. In Sec. 5 we prove that our IDNLS solutions converge

to the matrix NLS solutions derived in [5, 9, 10] as the discretization step tends to zero. In Sec. 6 we provide two interesting examples. Finally, the discrete Gronwall inequality applied in Sec. 2 and the alternative Marchenko equations are derived in Appendices A and B.

We notice that overlined quantities constitute an established notation inherited from [4] which has nothing to do with complex conjugation. The complex conjugate of a complex number  $z$  is written as  $z^*$ , whereas the conjugate transpose of a matrix  $A$  is written as  $A^\dagger$ .

## 2. Jost Solutions and Scattering Coefficients

In this section we define the Jost solutions, the transition coefficients expressing their linear dependence, and the reflection and transmission coefficients. We essentially follow [4, Chap. 5], although, unlike the authors of [4], we emphasize continuity and analyticity properties as the natural consequence of dealing with sums of absolutely convergent Fourier series and define transmission coefficients and scattering matrices explicitly. Occasionally we state (and prove) some results not found in [4].

Let us define the four *Jost solutions*  $\phi_n(z)$ ,  $\bar{\phi}_n(z)$ ,  $\psi_n(z)$  and  $\bar{\psi}_n(z)$  as those  $(N + M) \times N$ ,  $(N + M) \times M$ ,  $(N + M) \times M$  and  $(N + M) \times N$  matrix solutions to (1.2) satisfying the asymptotic conditions

$$\phi_n(z) \sim z^n \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, \quad \bar{\phi}_n(z) \sim z^{-n} \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \quad n \rightarrow -\infty, \quad (2.1a)$$

$$\psi_n(z) \sim z^{-n} \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \quad \bar{\psi}_n(z) \sim z^n \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, \quad n \rightarrow +\infty. \quad (2.1b)$$

Since the discrete matrix Zakharov–Shabat system is a homogeneous first order difference equation, we can reduce any pair of  $(N + M) \times (N + M)$  matrix solutions to each other by postmultiplication by a matrix not depending on  $n$ . We thus define the *transition coefficient matrices*  $\mathbf{T}(z)$  and  $\bar{\mathbf{T}}(z)$  by

$$\begin{pmatrix} \phi_n(z) & \bar{\phi}_n(z) \end{pmatrix} = \begin{pmatrix} \bar{\psi}_n(z) & \psi_n(z) \end{pmatrix} \mathbf{T}(z), \quad (2.2a)$$

$$\begin{pmatrix} \bar{\psi}_n(z) & \psi_n(z) \end{pmatrix} = \begin{pmatrix} \phi_n(z) & \bar{\phi}_n(z) \end{pmatrix} \bar{\mathbf{T}}(z), \quad (2.2b)$$

where  $I_N - \mathbf{u}_n \mathbf{w}_n$  (and hence  $I_M - \mathbf{w}_n \mathbf{u}_n$ ) is assumed nonsingular for each  $n \in \mathbb{Z}$  and  $\mathbf{T}(z)$  and  $\bar{\mathbf{T}}(z)$  are each other's inverses. Writing

$$\mathbf{T}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{a}(z) & \bar{\mathbf{b}}(z) \\ \mathbf{b}(z) & \bar{\mathbf{a}}(z) \end{pmatrix}, \quad \bar{\mathbf{T}}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \bar{\mathbf{c}}(z) & \mathbf{d}(z) \\ \bar{\mathbf{d}}(z) & \mathbf{c}(z) \end{pmatrix},$$

we obtain the *transition coefficients*  $\mathbf{a}(z)$  and  $\bar{\mathbf{c}}(z)$ ,  $\mathbf{b}(z)$  and  $\bar{\mathbf{d}}(z)$ ,  $\bar{\mathbf{a}}(z)$  and  $\mathbf{c}(z)$ , and  $\bar{\mathbf{b}}(z)$  and  $\mathbf{d}(z)$ , of respective sizes  $N \times N$ ,  $M \times N$ ,  $M \times M$  and  $N \times M$ .

**Proposition 2.1.** *Suppose  $I_N - \mathbf{u}_n \mathbf{w}_n$  is nonsingular for each  $n \in \mathbb{Z}$ . Then there exist unique Jost solutions  $\psi_n(z)$ ,  $\bar{\psi}_n(z)$ ,  $\phi_n(z)$  and  $\bar{\phi}_n(z)$  of the discrete matrix Zakharov–Shabat system (1.2) that satisfy (2.1).*

**Proof.** We only give the proof for  $\psi_n(z)$  and  $\bar{\psi}_n(z)$ . Let us define the Faddeev functions as follows:

$$\begin{aligned} \mathbf{M}_n(z) &= z^{-n}\phi_n(z), & \bar{\mathbf{M}}_n(z) &= z^n\bar{\phi}_n(z), \\ \mathbf{N}_n(z) &= z^n\psi_n(z), & \bar{\mathbf{N}}_n(z) &= z^{-n}\bar{\psi}_n(z). \end{aligned}$$

Put  $\mathbf{Z} = \begin{pmatrix} zI_N & 0_{NM} \\ 0_{MN} & z^{-1}I_M \end{pmatrix}$  and  $\mathbf{U}_n = \begin{pmatrix} 0_{NN} & \mathbf{u}_n \\ \mathbf{w}_n & 0_{MM} \end{pmatrix}$ . Then (1.2) implies that

$$(\mathbf{M}_{n+1}(z) \quad \bar{\mathbf{M}}_{n+1}(z)) \mathbf{Z} = (\mathbf{Z} + \mathbf{U}_n) (\mathbf{M}_n(z) \quad \bar{\mathbf{M}}_n(z)).$$

Writing this equation in short-hand notation as

$$\mathbf{m}_{n+1}(z)\mathbf{Z} = (\mathbf{Z} + \mathbf{U}_n)\mathbf{m}_n(z),$$

and iterating it backward we get

$$\mathbf{m}_n(z) = \mathbf{Z}^p \mathbf{m}_{n-p}(z) \mathbf{Z}^{-p} + \sum_{k=n-p}^{n-1} \mathbf{Z}^{n-k-1} \mathbf{U}_k \mathbf{m}_k(z) \mathbf{Z}^{k-n}.$$

Letting  $p \rightarrow +\infty$  and using (2.1a) we get

$$(\mathbf{M}_n(z) \quad \bar{\mathbf{M}}_n(z)) = I_{N+M} + \sum_{k=-\infty}^{n-1} \mathbf{Z}^{n-k-1} \mathbf{U}_k (\mathbf{M}_k(z) \quad \bar{\mathbf{M}}_k(z)) \mathbf{Z}^{k-n}.$$

Breaking up this equation into the usual column blocks, we get

$$\mathbf{M}_n(z) = \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix} + z^{-1} \sum_{k=-\infty}^{n-1} \begin{pmatrix} I_N & 0_{NM} \\ 0_{MN} & z^{-2(n-k-1)} I_M \end{pmatrix} \mathbf{U}_k \mathbf{M}_k(z), \tag{2.3a}$$

$$\bar{\mathbf{M}}_n(z) = \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix} + z \sum_{k=-\infty}^{n-1} \begin{pmatrix} z^{2(n-k-1)} I_N & 0_{NM} \\ 0_{MN} & I_M \end{pmatrix} \mathbf{U}_k \bar{\mathbf{M}}_k(z), \tag{2.3b}$$

which completes the proof. □

**Theorem 2.2.** *Suppose  $I_N - \mathbf{u}_n \mathbf{w}_n$  is nonsingular for each  $n \in \mathbb{Z}$ . Then the Jost solutions can be represented as follows:*

$$\psi_n(z) = \sum_{j=n}^{\infty} z^{-j} \mathbf{K}(n, j), \quad \bar{\psi}_n(z) = \sum_{j=n}^{\infty} z^j \bar{\mathbf{K}}(n, j), \tag{2.4a}$$

$$\phi_n(z) = \sum_{j=-\infty}^n z^j \mathbf{L}(n, j), \quad \bar{\phi}_n(z) = \sum_{j=-\infty}^n z^{-j} \bar{\mathbf{L}}(n, j), \tag{2.4b}$$

where

$$\sum_{j=n}^{\infty} \{ \|\mathbf{K}(n, j)\| + \|\bar{\mathbf{K}}(n, j)\| \} + \sum_{j=-\infty}^n \{ \|\mathbf{L}(n, j)\| + \|\bar{\mathbf{L}}(n, j)\| \} < +\infty.$$

As a result,  $z^n \psi_n(z)$  and  $z^{-n} \phi_n(z)$  are continuous in  $|z| \geq 1$ , are analytic in  $|z| > 1$ , and tend to  $\mathbf{K}(n, n)$  and  $\mathbf{L}(n, n)$  as  $|z| \rightarrow +\infty$ . Similarly,  $z^{-n} \bar{\psi}_n(z)$  and  $z^n \bar{\phi}_n(z)$  are continuous in  $|z| \leq 1$  and analytic in  $|z| < 1$ .

**Proof.** Let us only prove (2.4b). Breaking up (2.3) into up and down blocks we get

$$\mathbf{M}_n^{(\text{up})}(z) = I_N + z^{-1} \sum_{k=-\infty}^{n-1} \mathbf{u}_k \mathbf{M}_k^{(\text{dn})}(z), \quad (2.5a)$$

$$\mathbf{M}_n^{(\text{dn})}(z) = \sum_{k=-\infty}^{n-1} z^{-2(n-k)+1} \mathbf{w}_k \mathbf{M}_k^{(\text{up})}(z), \quad (2.5b)$$

$$\bar{\mathbf{M}}_n^{(\text{up})}(z) = \sum_{k=-\infty}^{n-1} z^{2(n-k)-1} \mathbf{u}_k \bar{\mathbf{M}}_k^{(\text{dn})}(z), \quad (2.5c)$$

$$\bar{\mathbf{M}}_n^{(\text{dn})}(z) = I_M + z \sum_{k=-\infty}^{n-1} \mathbf{w}_k \bar{\mathbf{M}}_k^{(\text{up})}(z). \quad (2.5d)$$

We now write (2.4b) in the form

$$\mathbf{M}_n(z) = \sum_{s=0}^{\infty} z^{-s} \mathbf{L}(n, n-s), \quad \bar{\mathbf{M}}_n(z) = \sum_{s=0}^{\infty} z^s \bar{\mathbf{L}}(n, n-s). \quad (2.6)$$

Using the bullet for up and down components, we define

$$\|\mathbf{K}^\bullet(n, \cdot)\|_1 = \sum_{s=0}^{\infty} \|\mathbf{K}^\bullet(n, n+s)\|, \quad \|\mathbf{L}^\bullet(n, \cdot)\|_1 = \sum_{s=0}^{\infty} \|\mathbf{L}^\bullet(n, n-s)\|,$$

and similarly for the overlined quantities.

Substituting (2.6) into (2.5a) we get

$$\mathbf{L}^{(\text{up})}(n, n) = I_N, \quad \mathbf{L}^{(\text{up})}(n, n-s-1) = \sum_{k=-\infty}^{n-1} \mathbf{u}_k \mathbf{L}^{(\text{dn})}(k, k-s),$$

so that

$$\|\mathbf{L}^{(\text{up})}(n, \cdot)\|_1 \leq 1 + \sum_{k=-\infty}^{n-1} \|\mathbf{u}_k\| \|\mathbf{L}^{(\text{dn})}(k, \cdot)\|_1. \quad (2.7a)$$

Substituting (2.6) into (2.5b) we get

$$\bar{\mathbf{L}}^{(\text{dn})}(n, n) = 0_{MN}, \quad \bar{\mathbf{L}}^{(\text{dn})}(n, n-s-1) = \sum_{k=-\infty}^{n-1} \mathbf{w}_k \bar{\mathbf{L}}^{(\text{up})}(k, k-s+2(n-k-1)),$$

so that

$$\|\bar{\mathbf{L}}^{(\text{dn})}(n, \cdot)\|_1 \leq \sum_{k=-\infty}^{n-1} \|\mathbf{w}_k\| \|\bar{\mathbf{L}}^{(\text{up})}(k, \cdot)\|_1. \quad (2.7b)$$

Substituting (2.6) into (2.5c) we get

$$\bar{\mathbf{L}}^{(\text{up})}(n, n) = 0_{NM}, \quad \bar{\mathbf{L}}^{(\text{up})}(n, n-s-1) = \sum_{k=-\infty}^{n-1} \mathbf{u}_k \bar{\mathbf{L}}^{(\text{dn})}(k, k-s+2(n-k-1)),$$

so that

$$\|\bar{\mathbf{L}}^{(\text{up})}(n, \cdot)\|_1 \leq \sum_{k=-\infty}^{n-1} \|\mathbf{u}_k\| \|\bar{\mathbf{L}}^{(\text{dn})}(k, \cdot)\|_1. \quad (2.7c)$$

Substituting (2.6) into (2.5d) we get

$$\bar{\mathbf{L}}^{(\text{dn})}(n, n) = I_M, \quad \mathbf{L}^{(\text{dn})}(n, n-s-1) = \sum_{k=-\infty}^{n-1} \mathbf{w}_k \bar{\mathbf{L}}^{(\text{up})}(k, k-s),$$

so that

$$\|\bar{\mathbf{L}}^{(\text{dn})}(n, \cdot)\|_1 \leq 1 + \sum_{k=-\infty}^{n-1} \|\mathbf{w}_k\| \|\bar{\mathbf{L}}^{(\text{up})}(k, \cdot)\|_1. \quad (2.7d)$$

Adding (2.7a) and (2.7b) we get the estimate

$$\begin{aligned} & \|\mathbf{L}^{(\text{up})}(n, \cdot)\|_1 + \|\mathbf{L}^{(\text{dn})}(n, \cdot)\|_1 \\ & \leq 1 + \sum_{k=-\infty}^{n-1} \max(\|\mathbf{u}_k\|, \|\mathbf{w}_k\|) (\|\mathbf{L}^{(\text{up})}(k, \cdot)\|_1 + \|\mathbf{L}^{(\text{dn})}(k, \cdot)\|_1), \end{aligned}$$

which, by Proposition A.1, yields

$$\|\mathbf{L}^{(\text{up})}(n, \cdot)\|_1 + \|\mathbf{L}^{(\text{dn})}(n, \cdot)\|_1 \leq \exp\left(\sum_{k=-\infty}^{\infty} \max(\|\mathbf{u}_k\|, \|\mathbf{w}_k\|)\right).$$

A similar estimate for the corresponding overlined matrices follows by adding (2.7c) and (2.7d). We have thus proved (2.4b).  $\square$

With some effort one may verify that

$$\begin{aligned} \mathbf{K}(n, n) &= \begin{pmatrix} 0_{NM} \\ \Delta_n^{-1} \end{pmatrix}, & \bar{\mathbf{K}}(n, n) &= \begin{pmatrix} \Omega_n^{-1} \\ 0_{MN} \end{pmatrix}, \\ \mathbf{L}(n, n) &= \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix}, & \bar{\mathbf{L}}(n, n) &= \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix}, \end{aligned}$$

where

$$\Omega_n = \cdots (I_N - \mathbf{u}_{n+2}\mathbf{w}_{n+2})(I_N - \mathbf{u}_{n+1}\mathbf{w}_{n+1})(I_N - \mathbf{u}_n\mathbf{w}_n), \quad (2.8a)$$

$$\Delta_n = \cdots (I_M - \mathbf{w}_{n+2}\mathbf{u}_{n+2})(I_M - \mathbf{w}_{n+1}\mathbf{u}_{n+1})(I_M - \mathbf{w}_n\mathbf{u}_n). \quad (2.8b)$$

Using that  $\|\mathbf{u}_n\mathbf{w}_n\|$  and  $\|\mathbf{w}_n\mathbf{u}_n\|$  are both dominated by  $\frac{1}{2}(\|\mathbf{u}_n\|^2 + \|\mathbf{w}_n\|^2)$ , the absolute convergence of the infinite products in (2.8) can be derived from the  $\ell^2$ -condition

$$\sum_{n=-\infty}^{\infty} \{\|\mathbf{u}_n\|^2 + \|\mathbf{w}_n\|^2\} < +\infty, \quad (2.9)$$

which follows immediately from (1.3).



Using the analyticity properties of the Jost solutions we can write (2.2) as the Riemann–Hilbert problems

$$\begin{pmatrix} \bar{\psi}_n(z) & \bar{\phi}_n(z) \end{pmatrix} = \begin{pmatrix} \phi_n(z) & \psi_n(z) \end{pmatrix} J \mathbf{S}(z) J, \quad |z| = 1, \quad (2.10a)$$

$$\begin{pmatrix} \phi_n(z) & \psi_n(z) \end{pmatrix} = \begin{pmatrix} \bar{\psi}_n(z) & \bar{\phi}_n(z) \end{pmatrix} J \bar{\mathbf{S}}(z) J, \quad |z| = 1, \quad (2.10b)$$

where  $J = \begin{pmatrix} I_N & 0_{NM} \\ 0_{MN} & -I_M \end{pmatrix}$  and

$$\mathbf{S}(z) = \begin{pmatrix} \mathbf{t}_r(z) & \boldsymbol{\ell}(z) \\ \boldsymbol{\rho}(z) & \mathbf{t}_l(z) \end{pmatrix}, \quad \bar{\mathbf{S}}(z) = \mathbf{S}(z)^{-1} = \begin{pmatrix} \bar{\mathbf{t}}_l(z) & \bar{\boldsymbol{\rho}}(z) \\ \bar{\boldsymbol{\ell}}(z) & \bar{\mathbf{t}}_r(z) \end{pmatrix},$$

are called *scattering matrices*. The quantities  $\mathbf{t}_r(z)$ ,  $\mathbf{t}_l(z)$ ,  $\bar{\mathbf{t}}_l(z)$  and  $\bar{\mathbf{t}}_r(z)$  are referred to as *transmission coefficients*, while  $\boldsymbol{\rho}(z)$ ,  $\boldsymbol{\ell}(z)$ ,  $\bar{\boldsymbol{\rho}}(z)$  and  $\bar{\boldsymbol{\ell}}(z)$  are called *reflection coefficients*.

A unimodular complex number  $z$  is called a *spectral singularity* if at least one of the “diagonal” transition coefficients  $\mathbf{a}(z)$ ,  $\bar{\mathbf{a}}(z)$ ,  $\mathbf{c}(z)$  and  $\bar{\mathbf{c}}(z)$  is singular.

**Theorem 2.3.** *Suppose there are no spectral singularities. Then the following is true:*

- (i) *The reflection coefficients are continuous in  $|z| = 1$  and are in fact sums of absolutely convergent Fourier series.*
- (ii) *The transmission coefficients  $\mathbf{t}_r(z)$  and  $\mathbf{t}_l(z)$  are continuous in  $|z| \geq 1$ , are meromorphic in  $|z| > 1$  with at most finitely many poles, and tend to  $I_N$  and  $\lim_{n \rightarrow +\infty} \boldsymbol{\Delta}_n = \prod_{k=-\infty}^{\infty} (I_M - \mathbf{w}_k \mathbf{u}_k)$ , respectively, as  $|z| \rightarrow +\infty$ . They are sums of absolutely convergent Fourier series.*
- (iii) *The transmission coefficients  $\bar{\mathbf{t}}_r(z)$  and  $\bar{\mathbf{t}}_l(z)$  are continuous in  $|z| \leq 1$ , are meromorphic in  $|z| < 1$  with at most finitely many, nonzero, poles, and tend to  $I_M$  and  $\lim_{n \rightarrow +\infty} \boldsymbol{\Omega}_n = \prod_{k=-\infty}^{\infty} (I_N - \mathbf{u}_k \mathbf{w}_k)$ , respectively, as  $z \rightarrow 0$ . They are sums of absolutely convergent Fourier series.*
- (iv) *The transmission coefficients  $\mathbf{t}_l(z)$  and  $\mathbf{t}_r(z)$  have the same poles and pole orders for  $|z| > 1$ , while  $\bar{\mathbf{t}}_l(z)$  and  $\bar{\mathbf{t}}_r(z)$  have the same poles and pole orders for  $0 < |z| < 1$ .*

**Proof.** Using (2.4) and the asymptotic behavior of the various blocks as  $n \rightarrow \pm\infty$ , we obtain

$$\begin{aligned} \mathbf{a}(z) &= I_N + z^{-1} \sum_{k=-\infty}^{\infty} \mathbf{u}_k \mathbf{M}_k^{(\text{dn})}(z) = I_N + \sum_{k=-\infty}^{\infty} z^{-k-1} \mathbf{u}_k \phi_k^{(\text{dn})}(z) \\ &= I_N + z^{-2} \sum_{k=-\infty}^{\infty} \mathbf{u}_k \mathbf{L}^{(\text{dn})}(k, k-1) + O(z^{-3}), \quad z \rightarrow \infty, \end{aligned} \quad (2.11a)$$

$$\begin{aligned} \bar{\mathbf{a}}(z) &= I_M + z \sum_{k=-\infty}^{\infty} \mathbf{w}_k \bar{\mathbf{M}}_k^{(\text{up})}(z) = I_M + \sum_{k=-\infty}^{\infty} z^{k+1} \mathbf{w}_k \bar{\phi}_k^{(\text{up})}(z) \\ &= I_M + z^2 \sum_{k=-\infty}^{\infty} \mathbf{w}_k \bar{\mathbf{L}}^{(\text{up})}(k, k-1) + O(z^3), \quad z \rightarrow 0, \end{aligned} \quad (2.11b)$$

$$\begin{aligned}\bar{\mathbf{c}}(z) &= I_N - z^{-1} \sum_{k=-\infty}^{\infty} \mathbf{u}_k \bar{N}_k^{(\text{dn})}(z) = I_N - \sum_{k=-\infty}^{\infty} z^{-k-1} \mathbf{u}_k \bar{\psi}_k^{(\text{dn})}(z) \\ &= I_N - \sum_{k=-\infty}^{\infty} \mathbf{u}_k \bar{\mathbf{K}}^{(\text{dn})}(n, n+1) + O(z), \quad z \rightarrow 0,\end{aligned}\tag{2.11c}$$

$$\begin{aligned}\mathbf{c}(z) &= I_M - z \sum_{k=-\infty}^{\infty} \mathbf{w}_k N_k^{(\text{up})}(z) = I_M - \sum_{k=-\infty}^{\infty} z^{k+1} \mathbf{w}_k \psi_k^{(\text{up})}(z) \\ &= I_M - \sum_{k=-\infty}^{\infty} \mathbf{w}_k \mathbf{K}^{(\text{up})}(n, n+1) + O(z^{-1}), \quad z \rightarrow \infty,\end{aligned}\tag{2.11d}$$

where we used  $\mathbf{L}^{(\text{up})}(n, n) = I_N$  and (2.8b) to derive (2.11). As a result, we have shown that  $\mathbf{a}(z)$ ,  $\bar{\mathbf{a}}(z)$ ,  $\bar{\mathbf{c}}(z)$  and  $\mathbf{c}(z)$  are discrete Fourier transforms of  $\ell^1$ -sequences with the correct analyticity properties. Further,

$$\bar{\mathbf{c}}(0) = \lim_{n \rightarrow -\infty} \bar{\mathbf{K}}^{(\text{up})}(n, n) = \left[ \prod_{k=-\infty}^{\infty} (I_N - \mathbf{u}_k \mathbf{w}_k) \right]^{-1},\tag{2.12a}$$

while (2.11d) implies that

$$\mathbf{c}(\infty) = \lim_{n \rightarrow -\infty} \mathbf{K}^{(\text{dn})}(n, n) = \left[ \prod_{k=-\infty}^{\infty} (I_M - \mathbf{w}_k \mathbf{u}_k) \right]^{-1}.\tag{2.12b}$$

In (2.12a) the infinite product is ordered  $\dots (I_N - \mathbf{u}_{k+1} \mathbf{w}_{k+1})(I_N - \mathbf{u}_k \mathbf{w}_k) \dots$ , whereas in (2.12b) it is ordered  $\dots (I_M - \mathbf{w}_{k+1} \mathbf{u}_{k+1})(I_M - \mathbf{w}_k \mathbf{u}_k) \dots$ . Consequently, the inverses of  $\bar{\mathbf{c}}(0)$  and  $\mathbf{c}(\infty)$  exist.

Using (2.4) we obtain

$$\mathbf{b}(z) = \sum_{k=-\infty}^{\infty} z^{2k+1} \mathbf{w}_k M_k^{(\text{up})}(z) = \sum_{k=-\infty}^{\infty} z^{k+1} \mathbf{w}_k \phi_k^{(\text{up})}(z),\tag{2.13a}$$

$$\bar{\mathbf{b}}(z) = \sum_{k=-\infty}^{\infty} z^{-2k-1} \mathbf{u}_k \bar{M}_k^{(\text{dn})}(z) = \sum_{k=-\infty}^{\infty} z^{-k-1} \mathbf{u}_k \bar{\phi}_k^{(\text{dn})}(z),\tag{2.13b}$$

$$\bar{\mathbf{d}}(z) = - \sum_{k=-\infty}^{\infty} z^{2k+1} \mathbf{w}_k \bar{N}_k^{(\text{up})}(z) = - \sum_{k=-\infty}^{\infty} z^{k+1} \mathbf{w}_k \bar{\psi}_k^{(\text{up})}(z),\tag{2.13c}$$

$$\mathbf{d}(z) = - \sum_{k=-\infty}^{\infty} z^{-2k-1} \mathbf{u}_k M_k^{(\text{dn})}(z) = - \sum_{k=-\infty}^{\infty} z^{-k-1} \mathbf{u}_k \psi_k^{(\text{dn})}(z).\tag{2.13d}$$

Equations (2.13) are valid if  $|z| = 1$ . Because of (1.3), they show that  $\mathbf{b}(z)$ ,  $\bar{\mathbf{b}}(z)$ ,  $\bar{\mathbf{d}}(z)$  and  $\mathbf{d}(z)$  are discrete Fourier transforms of  $\ell^1$ -sequences.  $\square$

### 3. Marchenko Equations

In this section we write down the alternative Marchenko equations in terms of the scattering data. The corresponding Marchenko kernels are the sums of two contributions, one derived

from the Fourier coefficients of a reflection coefficient and the other derived from the poles of a transmission coefficient and so-called norming constants. In the second half of this section we shall exploit the invariance of the discrete Zakharov–Shabat spectrum under the sign inversion  $z \mapsto -z$  to reduce the number of quantities to be computed by a factor of two.

Assuming there are no spectral singularities, we write the reflection coefficients as the absolutely convergent Fourier series

$$\rho(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\rho}(s), \quad \bar{\rho}(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\bar{\rho}}(s), \quad (3.1a)$$

$$\bar{\ell}(z) = \sum_{s=-\infty}^{\infty} z^s \hat{\bar{\ell}}(s), \quad \ell(z) = \sum_{s=-\infty}^{\infty} z^{-s} \hat{\ell}(s). \quad (3.1b)$$

Under the condition that the poles of the transmission coefficients are all simple, we define the Marchenko kernels

$$\mathbf{F}(j) = \hat{\ell}(j) + \sum_k \zeta_k^{j-1} \mathbf{C}_k, \quad \bar{\mathbf{F}}(j) = \hat{\bar{\ell}}(j) - \sum_k \bar{\zeta}_k^{-(j+1)} \bar{\mathbf{C}}_k. \quad (3.2)$$

Here  $\zeta_k$ , with  $|\zeta_k| > 1$ , are the finitely many simple poles of  $\mathbf{t}_r(z)$  and  $\mathbf{t}_l(z)$ , whereas  $\bar{\zeta}_k$ , with  $0 < |\bar{\zeta}_k| < 1$ , are the finitely many simple poles of  $\bar{\mathbf{t}}_l(z)$  and  $\bar{\mathbf{t}}_r(z)$ . The quantities  $\mathbf{C}_k$  and  $\bar{\mathbf{C}}_k$  are called the *norming constants*. Using the Kronecker delta  $\delta_{nj}$ , the Marchenko equations are then given by

$$\mathbf{L}(n, j) = \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix} (I_N - \mathbf{u}_n \mathbf{w}_n)^{-1} \delta_{nj} - \sum_{j'=-\infty}^n \bar{\mathbf{L}}(n, j') \bar{\mathbf{F}}(j' + j), \quad (3.3a)$$

$$\bar{\mathbf{L}}(n, j) = \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix} (I_M - \mathbf{w}_n \mathbf{u}_n)^{-1} \delta_{nj} - \sum_{j'=-\infty}^n \mathbf{L}(n, j') \mathbf{F}(j' + j), \quad (3.3b)$$

where  $j \leq n$  [See Appendix B]. The potentials can then be expressed in terms of the solutions to (3.3) as follows:

$$\mathbf{u}_n = \bar{\mathbf{L}}^{(\text{up})}(n+1, n), \quad (3.4a)$$

$$\mathbf{w}_n = \mathbf{L}^{(\text{dn})}(n+1, n). \quad (3.4b)$$

In the focusing case the Marchenko equations are easily seen to be uniquely solvable.

If the transmission coefficients have multiple poles, the Marchenko equations (3.3) and the expressions (3.4) for the potentials in terms of their solutions do not change. The bound state terms in (3.2) become much more complicated, because each pole term gets replaced by a number of terms equal to the corresponding pole order [cf. (4.1) below].

It is easily verified that, for each solution  $\mathbf{v}_n(z)$  of (1.2), also  $\tilde{\mathbf{v}}_n(z) = (-1)^n \mathbf{J} \mathbf{v}_n(-z)$  is a solution of (1.2). As a result, we get for the Jost functions and transition coefficient matrices the sign inversion symmetries

$$\begin{aligned} \begin{pmatrix} \bar{\psi}_n(-z) & \psi_n(-z) \end{pmatrix} &= (-1)^n \mathbf{J} \begin{pmatrix} \bar{\psi}_n(z) & \psi_n(z) \end{pmatrix} \mathbf{J}, \\ \begin{pmatrix} \bar{\phi}_n(-z) & \phi_n(-z) \end{pmatrix} &= (-1)^n \mathbf{J} \begin{pmatrix} \bar{\phi}_n(z) & \phi_n(z) \end{pmatrix} \mathbf{J}, \\ \mathbf{T}(-z) &= \mathbf{J} \mathbf{T}(z) \mathbf{J}, \quad \bar{\mathbf{T}}(-z) = \mathbf{J} \bar{\mathbf{T}}(z) \mathbf{J}, \end{aligned}$$

so that the transmission coefficients are even functions of  $z$  (and hence the discrete Zakharov–Shabat spectrum is invariant under sign inversion) and the reflection coefficients are odd functions of  $z$ . Therefore the functions  $\hat{\rho}(s)$ ,  $\hat{\bar{\rho}}(s)$ ,  $\hat{\ell}(z)$  and  $\hat{\bar{\ell}}(z)$  appearing in (3.1) vanish if  $s$  is even. Using (2.4) together with the sign inversion symmetry of the Jost functions, we get

$$\begin{aligned} (\bar{\mathbf{K}}(n, j) \quad \mathbf{K}(n, j)) &= (-1)^{j-n} J (\bar{\mathbf{K}}(n, j) \quad \mathbf{K}(n, j)) J, \\ (\mathbf{L}(n, j) \quad \bar{\mathbf{L}}(n, j)) &= (-1)^{n-j} J (\mathbf{L}(n, j) \quad \bar{\mathbf{L}}(n, j)) J. \end{aligned}$$

Therefore,  $\mathbf{L}^{(\text{dn})}(n, j)$  and  $\bar{\mathbf{L}}^{(\text{up})}(n, j)$  vanish if  $j - n$  is even, while  $\mathbf{L}^{(\text{up})}(n, j)$  and  $\bar{\mathbf{L}}^{(\text{dn})}(n, j)$  vanish if  $j - n$  is odd. From these symmetry properties we see that the Marchenko kernels  $\mathbf{F}(s)$  and  $\bar{\mathbf{F}}(s)$  vanish if  $s$  is even.

Breaking up the Marchenko equations (3.3) for quantities like  $\mathbf{L}(n, j)$  into separate equations for quantities like  $\mathbf{L}^{(\text{up})}(n, j)$  and  $\mathbf{L}^{(\text{dn})}(n, j)$  and executing one iteration of each resulting coupled pair of equations in order to get them decoupled, we arrive at so-called uncoupled Marchenko equations whose Marchenko kernels have the form

$$\mathbb{L}(j, j') \stackrel{\text{def}}{=} \sum_{j''=-\infty}^{n-1} \mathbf{F}_1(j' + j'') \mathbf{F}_2(j'' + j).$$

These kernels  $\mathbb{L}(j, j')$  vanish if one of  $j, j'$  is even and the other is odd. As a result, the uncoupled Marchenko equations can be decoupled further.

Let us now write (3.3) in the form

$$\mathbf{L}(n, j) = - \begin{pmatrix} 0_{NM} \\ I_M \end{pmatrix} \bar{\mathbf{F}}(n + j) - \sum_{j'=-\infty}^{n-1} \bar{\mathbf{L}}(n, j') \bar{\mathbf{F}}(j' + j), \tag{3.5a}$$

$$\bar{\mathbf{L}}(n, j) = - \begin{pmatrix} I_N \\ 0_{MN} \end{pmatrix} \mathbf{F}(n + j) - \sum_{j'=-\infty}^{n-1} \mathbf{L}(n, j') \mathbf{F}(j' + j), \tag{3.5b}$$

where  $j \geq n + 1$ . Then the potentials are given by

$$\mathbf{u}_n = \bar{\mathbf{L}}^{(\text{up})}(n + 1, n), \quad \mathbf{w}_n = \mathbf{L}^{(\text{dn})}(n + 1, n). \tag{3.6}$$

Let us decouple (3.5) further as follows:

$$\begin{aligned} \mathbf{L}^{(\text{up})}(n, n - 2\sigma) &= \sum_{\sigma''=0}^{\infty} \mathbf{F}(2[n - \sigma''] - 1) \bar{\mathbf{F}}(2[n - \sigma'' - \sigma] - 1) \\ &\quad + \sum_{\sigma'=1}^{\infty} \mathbf{L}^{(\text{up})}(n, n - 2\sigma') \\ &\quad \times \sum_{\sigma''=0}^{\infty} \mathbf{F}(2[n - \sigma' - \sigma''] - 1) \bar{\mathbf{F}}(2[n - \sigma'' - \sigma] - 1), \end{aligned} \tag{3.7a}$$

$$\begin{aligned} \bar{\mathbf{L}}^{(\text{up})}(n, n - 2\sigma - 1) &= -\mathbf{F}(2[n - \sigma] - 1) \\ &+ \sum_{\sigma'=0}^{\infty} \bar{\mathbf{L}}^{(\text{up})}(n, n - 2\sigma' - 1) \\ &\times \sum_{\sigma''=1}^{\infty} \bar{\mathbf{F}}(2[n - \sigma' - \sigma''] - 1) \mathbf{F}(2[n - \sigma'' - \sigma] - 1), \end{aligned} \quad (3.7b)$$

$$\begin{aligned} \mathbf{L}^{(\text{dn})}(n, n - 2\sigma - 1) &= -\bar{\mathbf{F}}(2[n - \sigma] - 1) \\ &+ \sum_{\sigma'=0}^{\infty} \mathbf{L}^{(\text{dn})}(n, n - 2\sigma' - 1) \\ &\times \sum_{\sigma''=1}^{\infty} \mathbf{F}(2[n - \sigma' - \sigma''] - 1) \bar{\mathbf{F}}(2[n - \sigma'' - \sigma] - 1), \end{aligned} \quad (3.7c)$$

$$\begin{aligned} \bar{\mathbf{L}}^{(\text{dn})}(n, n - 2\sigma) &= \sum_{\sigma''=0}^{\infty} \bar{\mathbf{F}}(2[n - \sigma''] - 1) \mathbf{F}(2[n - \sigma'' - \sigma] - 1) \\ &+ \sum_{\sigma'=1}^{\infty} \bar{\mathbf{L}}^{(\text{dn})}(n, n - 2\sigma') \\ &\times \sum_{\sigma''=0}^{\infty} \bar{\mathbf{F}}(2[n - \sigma' - \sigma''] - 1) \mathbf{F}(2[n - \sigma'' - \sigma] - 1). \end{aligned} \quad (3.7d)$$

Equations (3.7a) and (3.7d) are valid for  $\sigma \geq 1$ , whereas (3.7b) and (3.7c) are valid for  $\sigma \geq 0$ . This distinction in the ranges of the summation index  $\sigma$  is to bear in mind when deriving exact solutions of (1.1).

#### 4. IDNLS Solutions in Terms of Matrix Triplets

In this section we write the solutions of the Marchenko equations in terms of suitable matrix triplets if the reflection coefficients vanish. Once the time evolution of the scattering data has been taken into account as well as the maximal reduction of the Marchenko equations, we quickly arrive at explicit IDNLS solutions.

Using two matrix triplets, we generalize the expressions (3.2) for the Marchenko kernels as follows:

$$\mathbf{F}(j) = \hat{\ell}(j) + CA^{j-1}B, \quad \bar{\mathbf{F}}(j) = \hat{\bar{\ell}}(j) + \bar{C}\bar{A}^{-(j+1)}\bar{B}, \quad (4.1)$$

where the triplets  $(A, B, C)$  and  $(\bar{A}, \bar{B}, \bar{C})$  have the following properties:

- (i)  $A$ ,  $B$  and  $C$  are  $p \times p$ ,  $p \times N$  and  $M \times p$  matrices, respectively, and  $A$  is a matrix having only eigenvalues of modulus larger than one, and
- (ii)  $\bar{A}$ ,  $\bar{B}$  and  $\bar{C}$  are  $\bar{p} \times \bar{p}$ ,  $\bar{p} \times M$  and  $N \times \bar{p}$  matrices, respectively, and  $\bar{A}$  is a nonsingular matrix which has only eigenvalues of modulus less than one.

If the poles of the transmission coefficients are all simple, we can recover the original expressions (3.2) by taking

$$A = \text{diag}(\zeta_1, \dots, \zeta_p), \quad B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad C = (\mathbf{C}_1 \quad \dots \quad \mathbf{C}_p), \quad (4.2a)$$

$$\bar{A} = \text{diag}(\bar{\zeta}_1, \dots, \bar{\zeta}_{\bar{p}}), \quad \bar{B} = \begin{pmatrix} \bar{C}_{r1} \\ \vdots \\ \bar{C}_{r\bar{p}} \end{pmatrix}, \quad \bar{C} = (-1 \quad \dots \quad -1), \quad (4.2b)$$

where the norming constants are encoded by one of  $C$  or  $\bar{B}$ . Here  $\bar{p}$  is the number of poles of  $\bar{\mathbf{t}}_l(z)$  or  $\bar{\mathbf{t}}_r(z)$ .

Because the Marchenko kernels  $\mathbf{F}(s)$  and  $\bar{\mathbf{F}}(s)$  vanish if  $s$  is even, we need to restrict the generality of the above matrix triplets by writing

$$A = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A} \end{pmatrix}, \quad B = \begin{pmatrix} \mathcal{B} \\ \mathcal{B} \end{pmatrix}, \quad C = (\mathcal{C} \quad \mathcal{C}), \quad (4.3a)$$

$$\bar{A} = \begin{pmatrix} \bar{\mathcal{A}} & 0 \\ 0 & -\bar{\mathcal{A}} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{\mathcal{B}} \\ \bar{\mathcal{B}} \end{pmatrix}, \quad \bar{C} = (\bar{\mathcal{C}} \quad \bar{\mathcal{C}}). \quad (4.3b)$$

If the transmission coefficients only have simple poles, then the triplets (4.2) can be made to correspond to (4.3) by properly ordering the poles, because norming constants corresponding to  $\pm$  pairs of poles coincide. Consequently,

$$\mathbf{F}(j) = \hat{\ell}(j) + [1 + (-1)^{j+1}] \mathcal{C} \mathcal{A}^{j-1} \mathcal{B}, \quad (4.4a)$$

$$\bar{\mathbf{F}}(j) = \hat{\ell}(j) + [1 + (-1)^{j+1}] \bar{\mathcal{C}} \bar{\mathcal{A}}^{-(j+1)} \bar{\mathcal{B}}. \quad (4.4b)$$

In the focusing case, we have the following conjugation symmetry relations for the Marchenko kernels:

$$\bar{\mathbf{F}}(j) = -\mathbf{F}(j)^\dagger. \quad (4.5)$$

Equations (4.5) lead to uniquely solvable Marchenko equations and focusing potentials. Relating the matrix triplets to each other in the following way:

$$\bar{\mathcal{A}} = \mathcal{A}^{\dagger-1}, \quad \bar{\mathcal{B}} = \mathcal{A}^{\dagger-1} \mathcal{C}^\dagger, \quad \bar{\mathcal{C}} = -\mathcal{B}^\dagger \mathcal{A}^{\dagger-1}, \quad (4.6)$$

we get Marchenko kernels of the form (4.1) that satisfy the conjugation symmetry relation (4.5).

Let us return to the general case. Put

$$\mathcal{Q} = \sum_{\sigma=0}^{\infty} \bar{\mathcal{A}}^{2\sigma} \bar{\mathcal{B}} \mathcal{C} \mathcal{A}^{-2\sigma}, \quad \mathcal{N} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{-2\sigma} \mathcal{B} \bar{\mathcal{C}} \bar{\mathcal{A}}^{2\sigma}. \quad (4.7)$$

Because the spectral radii of  $\mathcal{A}^{-1}$  and  $\bar{\mathcal{A}}$  are strictly less than one, the series in (4.7) are absolutely convergent. It is immediate that  $\mathcal{Q}$  and  $\mathcal{N}$  are the unique solutions of the matrix

equations (cf. [12, Theorem 18.1], using that  $\mathcal{A}^2$  and  $\bar{\mathcal{A}}^2$  do not have eigenvalues in common)

$$\mathcal{Q} - \bar{\mathcal{A}}^2 \mathcal{Q} \mathcal{A}^{-2} = \bar{\mathcal{B}} \mathcal{C}, \quad \mathcal{N} - \mathcal{A}^{-2} \mathcal{N} \bar{\mathcal{A}}^2 = \mathcal{B} \bar{\mathcal{C}}.$$

Suppose the Marchenko kernels are given by (4.4), where the reflection coefficients vanish. In other words, assume the Marchenko kernels to be given in terms of suitable matrix triplets. Then each of the four uncoupled Marchenko equations (3.7) can be solved by separation of variables, using well-known techniques detailed in [5, 9]. The results will be listed in Proposition 4.1 and Theorems 4.2 and 4.3. If we employ a matrix triplet  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ , we end up solving the Marchenko equations (3.7) and deriving the potentials with the help of (3.6).

By separation of variables and using the Marchenko equations (3.7) we easily derive the following result.

**Proposition 4.1.** *Suppose the Marchenko kernels are given by (4.4), where the reflection coefficients vanish. Then the Marchenko equations (3.7) have the solutions*

$$\begin{aligned} \mathbf{L}^{(\text{up})}(n, n - 2\sigma) &= 4\mathcal{C}[I - 4\mathcal{A}^{2(n-1)}\mathcal{N}\bar{\mathcal{A}}^{-2(n-1)}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{A}^{2(n-1)}\mathcal{N}\bar{\mathcal{A}}^{-2(n-\sigma)}\bar{\mathcal{B}}, \\ \bar{\mathbf{L}}^{(\text{up})}(n, n - 2\sigma - 1) &= -2\mathcal{C}\mathcal{A}^2[I - 4\mathcal{A}^{2(n-2)}\mathcal{N}\bar{\mathcal{A}}^{-2(n-1)}\mathcal{Q}]^{-1}\mathcal{A}^{2(n-\sigma-2)}\mathcal{B}, \\ \mathbf{L}^{(\text{dn})}(n, n - 2\sigma - 1) &= -2\bar{\mathcal{C}}\bar{\mathcal{A}}^{-2}[I - 4\bar{\mathcal{A}}^{-2(n-1)}\mathcal{Q}\mathcal{A}^{2(n-2)}\mathcal{N}]^{-1}\bar{\mathcal{A}}^{-2(n-\sigma-1)}\bar{\mathcal{B}}, \\ \bar{\mathbf{L}}^{(\text{dn})}(n, n - 2\sigma) &= 4\bar{\mathcal{C}}[I - 4\bar{\mathcal{A}}^{-2n}\mathcal{Q}\mathcal{A}^{2(n-2)}\mathcal{N}\bar{\mathcal{A}}^2]^{-1}\bar{\mathcal{A}}^{-2n}\mathcal{Q}\mathcal{A}^{2(n-\sigma-1)}\mathcal{B}, \end{aligned}$$

provided the matrix inverses appearing in these expressions exist. In this case the potentials are given by

$$\begin{aligned} \mathbf{u}_n &= -2\mathcal{C}[\mathcal{A}^{-2n} - 4\mathcal{N}\bar{\mathcal{A}}^{-2n}\mathcal{Q}\mathcal{A}^{-2}]^{-1}\mathcal{B}, \\ \mathbf{w}_n &= -2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{2(n+1)} - 4\mathcal{Q}\mathcal{A}^{2(n-1)}\mathcal{N}\bar{\mathcal{A}}^2]^{-1}\bar{\mathcal{B}}. \end{aligned}$$

Let us now take into account the time dependence of the scattering data. Then for odd  $j$  the Marchenko kernels (4.4) are to be modified as follows:

$$\mathbf{F}(j; \tau) = \frac{1}{2\pi i} \oint dz z^{j-1} \ell(z) e^{-i\tau(z-z^{-1})^2} + 2\mathcal{C}\mathcal{A}^{j-1} e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mathcal{B}, \quad (4.8a)$$

$$\bar{\mathbf{F}}(j; \tau) = \frac{1}{2\pi i} \oint dz \frac{\bar{\ell}(z)}{z^{j+1}} e^{i\tau(z-z^{-1})^2} + 2\bar{\mathcal{C}} e^{i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2} \bar{\mathcal{A}}^{-(j+1)} \bar{\mathcal{B}}, \quad (4.8b)$$

where the contour integration is performed over the unit circle.

We now easily arrive at the following main theorem.

**Theorem 4.2.** *Suppose the Marchenko kernels are given by (4.8), where the reflection coefficients vanish. Then the integrable discrete nonlinear Schrödinger solutions are given by*

$$\mathbf{u}_n(\tau) = -2\mathcal{C}[\mathcal{A}^{-2n} e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} - 4\mathcal{N} e^{i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2} \bar{\mathcal{A}}^{-2n} \mathcal{Q} \mathcal{A}^{-2}]^{-1} \mathcal{B}, \quad (4.9a)$$

$$\mathbf{w}_n(\tau) = -2\bar{\mathcal{C}}[\bar{\mathcal{A}}^{2(n+1)} e^{-i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2} - 4\mathcal{Q} \mathcal{A}^{2(n-1)} e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mathcal{N} \bar{\mathcal{A}}^2]^{-1} \bar{\mathcal{B}}, \quad (4.9b)$$

provided the matrix inverses in these expressions exist.

**Proof.** It is sufficient to prove Theorem 4.2 for  $\tau = 0$  and then to make the following changes in the final result:

$$\mathcal{B} \mapsto e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mathcal{B}, \quad \bar{\mathcal{C}} \mapsto \bar{\mathcal{C}} e^{i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2}, \quad \mathcal{N} \mapsto e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mathcal{N} e^{i\tau(\bar{\mathcal{A}}-\bar{\mathcal{A}}^{-1})^2},$$

whereas  $\mathcal{A}$ ,  $\mathcal{C}$ ,  $\bar{\mathcal{B}}$  and  $\mathcal{Q}$  remain unchanged.  $\square$

Let us return to the focusing case. Define the non-negative selfadjoint matrices

$$\mathcal{Q} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{\dagger-2\sigma} \mathcal{C}^{\dagger} \mathcal{C} \mathcal{A}^{-2\sigma}, \quad \mathcal{N} = \sum_{\sigma=0}^{\infty} \mathcal{A}^{-2\sigma} \mathcal{B} \mathcal{B}^{\dagger} \mathcal{A}^{\dagger-2\sigma}, \quad (4.10)$$

so that

$$\mathcal{Q} = \mathcal{A}^{\dagger-1} \mathcal{Q}, \quad \mathcal{N} = -\mathcal{N} \mathcal{A}^{\dagger-1}.$$

Then  $\mathcal{Q}$  and  $\mathcal{N}$  are the unique solutions of the Stein equations [12]

$$\mathcal{Q} - \mathcal{A}^{\dagger-2} \mathcal{Q} \mathcal{A}^{-2} = \mathcal{C}^{\dagger} \mathcal{C}, \quad \mathcal{N} - \mathcal{A}^{-2} \mathcal{N} \mathcal{A}^{\dagger-2} = \mathcal{B} \mathcal{B}^{\dagger}. \quad (4.11)$$

Using (4.6) and (4.10) we now specialize Theorem 4.2 to the focusing case as follows.

**Theorem 4.3 (Focusing case).** *Let the Marchenko kernel be given by*

$$\mathbf{F}(2s-1) = -\bar{\mathbf{F}}(2s-1)^{\dagger} = 2\mathcal{C} \mathcal{A}^{2(s-1)} e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mathcal{B}.$$

Then the potential is given by

$$\mathbf{u}_n(\tau) = -2\mathcal{C}[\mathcal{A}^{-2n} e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} + 4\mathcal{N} \mathcal{A}^{\dagger 2(n-1)} e^{i\tau(\mathcal{A}^{\dagger}-\mathcal{A}^{\dagger-1})^2} \mathcal{Q} \mathcal{A}^{-2}]^{-1} \mathcal{B}, \quad (4.12)$$

where  $\mathbf{w}_n(\tau)$  satisfies

$$\mathbf{w}_n(\tau) = -\mathbf{u}_n(\tau)^{\dagger}, \quad n \in \mathbb{Z}.$$

**Proof.** Let us define the additional triplet  $(\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{C}})$  by (4.6). Then (4.12) follows immediately from (4.9a) and (4.10). Using (4.9b), (4.6) and (4.10) we derive

$$\mathbf{w}_n(\tau) = 2\mathcal{B}^{\dagger}[\mathcal{A}^{\dagger-2n} e^{-i\tau(\mathcal{A}^{\dagger}-\mathcal{A}^{\dagger-1})^2} + 4\mathcal{Q} \mathcal{A}^{2(n-1)} e^{-i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mathcal{N} \mathcal{A}^{\dagger-2}]^{-1} \mathcal{C}^{\dagger}, \quad (4.13)$$

which implies (4.12).  $\square$

## 5. Continuous Matrix NLS Limit

In this section we prove that the matrix IDNLS solutions obtained converge to solutions of the continuous matrix NLS equations as the discretization step size vanishes. We restrict ourselves to the focusing case while pointing out that in general the convergence proof is not essentially different.



First of all, we observe that

$$i\frac{d}{dt}\mathbf{U}_n = \frac{\mathbf{U}_{n+1} - 2\mathbf{U}_n + \mathbf{U}_{n-1}}{h^2} - \mathbf{U}_{n+1}\mathbf{W}_n\mathbf{U}_n - \mathbf{U}_n\mathbf{W}_n\mathbf{U}_{n-1}, \quad (5.1a)$$

$$-i\frac{d}{dt}\mathbf{W}_n = \frac{\mathbf{W}_{n+1} - 2\mathbf{W}_n + \mathbf{W}_{n-1}}{h^2} - \mathbf{W}_{n+1}\mathbf{U}_n\mathbf{W}_n - \mathbf{W}_n\mathbf{U}_n\mathbf{W}_{n-1}, \quad (5.1b)$$

is the discretization of the continuous matrix Zakharov–Shabat system

$$i\mathbf{U}_t = \mathbf{U}_{xx} - 2\mathbf{U}\mathbf{W}\mathbf{U}, \quad (5.2a)$$

$$-i\mathbf{W}_t = \mathbf{W}_{xx} - 2\mathbf{W}\mathbf{U}\mathbf{W}, \quad (5.2b)$$

obtained by using the following finite differencing scheme:

$$\mathbf{U}_n(t) = \mathbf{U}(nh, t), \quad \mathbf{W}_n(t) = \mathbf{W}(nh, t).$$

Equations (5.2) reduce to (1.1) when employing the following rescaling [4, Eq. (3.2.6)]:

$$\mathbf{u}_n = h\mathbf{U}_n, \quad \mathbf{w}_n = h\mathbf{W}_n, \quad \tau = (t/h^2).$$

Taking into account (4.12) where all of the eigenvalues of  $\mathcal{A}$  have modulus larger than one, we rescale the matrix triplet  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  as follows:

$$\mathcal{A} \mapsto e^{h\mathcal{A}}, \quad \mathcal{B} \mapsto \sqrt{h}\mathcal{B}, \quad \mathcal{C} \mapsto \sqrt{h}\mathcal{C}. \quad (5.3)$$

Here the new triplet  $(\mathbf{A}, \mathbf{B}, \mathbf{C})$  is such that  $\mathbf{A}$  has only eigenvalues with positive real part. We easily get

$$e^{i\tau(\mathcal{A}-\mathcal{A}^{-1})^2} \mapsto e^{4it\left(\frac{\sinh(h\mathcal{A})}{h}\right)^2} \sim e^{4it\mathbf{A}^2} \quad \text{as } h \rightarrow 0^+. \quad (5.4)$$

Moreover, using the trapezoid rule

$$\int_0^\infty dx F(x) \sim h \left\{ \frac{1}{2}F(0) + \sum_{j=1}^\infty F(jh) \right\}$$

valid for  $C^2$ -functions belonging to  $L^1(\mathbb{R}^+)$ , we see that

$$\begin{aligned} \mathbf{Q} &= \sum_{j=0}^\infty \mathcal{A}^{\dagger-2j} \mathcal{C}^\dagger \mathcal{C} \mathcal{A}^{-2j} \mapsto h \sum_{j=0}^\infty e^{-2jh\mathbf{A}^\dagger} \mathbf{C}^\dagger \mathbf{C} e^{-2jh\mathbf{A}} \\ &\simeq \int_0^\infty dz e^{-2z\mathbf{A}^\dagger} \mathbf{C}^\dagger \mathbf{C} e^{-2z\mathbf{A}} = \frac{1}{2}\mathbf{Q}, \end{aligned} \quad (5.5a)$$

$$\begin{aligned} \mathbf{N} &= \sum_{j=0}^\infty \mathcal{A}^{-2j} \mathcal{B}\mathcal{B}^\dagger \mathcal{A}^{\dagger-2j} \mapsto h \sum_{j=0}^\infty e^{-2jh\mathbf{A}} \mathbf{B}\mathbf{B}^\dagger e^{-2jh\mathbf{A}^\dagger} \\ &\simeq \int_0^\infty dz e^{-2z\mathbf{A}} \mathbf{B}\mathbf{B}^\dagger e^{-2z\mathbf{A}^\dagger} = \frac{1}{2}\mathbf{N}, \end{aligned} \quad (5.5b)$$

where

$$Q = \int_0^\infty dz e^{-z\mathbf{A}^\dagger} \mathbf{C}^\dagger \mathbf{C} e^{-z\mathbf{A}}, \quad N = \int_0^\infty dz e^{-z\mathbf{A}} \mathbf{B} \mathbf{B}^\dagger e^{-z\mathbf{A}^\dagger}.$$

Applying the rescaling (5.3) to (4.12) we get

$$\begin{aligned} \mathbf{U}_n(t) &= -\frac{2}{h} \mathbf{C} [\mathbf{A}^{-2n} e^{ih^{-2}t(\mathbf{A}-\mathbf{A}^{-1})^2} + 4N\mathbf{A}^{\dagger 2(n-1)} e^{ih^{-2}t(\mathbf{A}^\dagger-\mathbf{A}^{\dagger-1})^2} \mathbf{Q} \mathbf{A}^{-2}]^{-1} \mathbf{B} \\ &= -2\mathbf{C} \left[ e^{-2nh\mathbf{A}} e^{4it\left(\frac{\sinh(h\mathbf{A})}{h}\right)^2} + 4\frac{N}{2} e^{2nh\mathbf{A}^\dagger} e^{-2h\mathbf{A}^\dagger} e^{4it\left(\frac{\sinh(h\mathbf{A}^\dagger)}{h}\right)^2} \frac{\mathbf{Q}}{2} e^{-2h\mathbf{A}} \right]^{-1} \mathbf{B} \end{aligned} \quad (5.6)$$

With the help of (5.4) and (5.5) we calculate the limit of (5.6) as the step size  $h$  goes to zero while  $x = nh$ , obtaining

$$\mathbf{U}_n \sim -2\mathbf{C} [e^{-2x\mathbf{A}} e^{4it\mathbf{A}^2} + N e^{2x\mathbf{A}^\dagger} e^{4it\mathbf{A}^{\dagger 2}} \mathbf{Q}]^{-1} \mathbf{B}. \quad (5.7)$$

In [5, 9, 10], explicit solutions of the continuous focusing matrix NLS equation have been derived by solving the Marchenko equations by separation of variables, leading to the solution formula (5.7).

## 6. Examples

In this section we work out two illustrative examples. The first one regards multipole solutions which can be easily treated by using the  $(A, B, C)$  method discussed in this paper. The second one involves the physically relevant (see, for example, [15, 16]) Manakov case, where  $N = 2$  and  $M = 1$ .

**Example 6.1 (Nonscalar multipole solutions).** In the focusing case we consider the triplet  $(\mathcal{A}, \mathcal{B}, \mathcal{C})$  defined by

$$\mathcal{A} = \begin{pmatrix} 1-i & -1 \\ 0 & 1-i \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}.$$

Then the solutions of the corresponding Stein equations (4.11) are given by

$$\mathcal{Q} = \begin{pmatrix} 14-14i & \frac{70}{3}-14i \\ \frac{52}{3}-\frac{122}{3}i & \frac{460}{9}-\frac{566}{9}i \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} -\frac{236}{27} + \frac{326}{27}i & -\frac{46}{9} + \frac{10}{3}i \\ -\frac{10}{3} + \frac{70}{9}i & -\frac{8}{3} + \frac{8}{3}i \end{pmatrix}.$$

It is not a good idea to write explicitly the  $2 \times 2$  solutions of (1.1) by unzipping (4.12) and (4.13), because the expressions obtained are very long and take two to three pages! However, by using Mathematica to compute  $\mathbf{u}_n(\tau)$  and  $\mathbf{w}_n(\tau)$  by means of (4.12) and (4.13) for various values of  $\tau$  and substituting the result into (1.1), we have found our solutions









