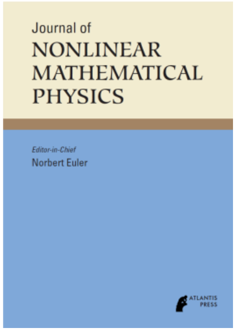

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CONSERVATION LAWS FOR THE SCHRÖDINGER–NEWTON EQUATIONS

G. GUBBIOTTI and M. C. NUCCI*

Dipartimento di Matematica e Informatica
Università di Perugia & INFN Sezione Perugia
06123 Perugia, Italy
**nucci@unipg.it*

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In this Letter a first-order Lagrangian for the Schrödinger–Newton equations is derived by modifying a second-order Lagrangian proposed by Christian [Exactly soluble sector of quantum gravity, *Phys. Rev. D* **56**(8) (1997) 4844–4877]. Then Noether's theorem is applied to the Lie point symmetries determined by Robertshaw and Tod [Lie point symmetries and an approximate solution for the Schrödinger–Newton equations, *Nonlinearity* **19**(7) (2006) 1507–1514] in order to find conservation laws of the Schrödinger–Newton equations.

Keywords: Schrödinger–Newton equations; calculus of variations; Noether's theorem.

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1. Introduction

The Schrödinger–Newton equations consist of a system of partial differential equations introduced by Penrose [9], and in dimensionless units[a] they are[b]:

$$\mathrm{i}\psi_t = -\Delta\psi + \phi\psi, \tag{1.2a}$$

$$\Delta\phi = |\psi|^2, \tag{1.2b}$$

where Δ is the Laplacian in $\mathbb{R}^3(x, y, z)$. The Lie point symmetry algebra admitted by the Schrödinger–Newton equations was determined by Robertshaw and Tod [10]. We rewrite

[a]For more details see Harrison [4].

[b]Including the equation satisfied by the complex conjugate ψ^* of the wave function ψ, i.e.

$$-\mathrm{i}\psi_t^* = -\Delta\psi^* + \phi\psi^*. \tag{1.1}$$

the Schrödinger–Newton equations as follows:

$$\mathrm{i}u_t = -\Delta u + uw, \tag{1.3a}$$

$$-\mathrm{i}v_t = -\Delta v + vw, \tag{1.3b}$$

$$\Delta w = uv. \tag{1.3c}$$

With $\psi = u$, $\psi^* = v$ and $\phi = w$. The Lie point symmetries found in [10] are:

- scaling:

$$\boldsymbol{v}_1 = 2t\partial_t + x\partial_x + y\partial_y + z\partial_z - 2u\partial_u - 2v\partial_v - 2w\partial_w, \tag{1.4}$$

- spatial rotations:

$$\boldsymbol{v}_2 = y\partial_x - x\partial_y, \tag{1.5a}$$

$$\boldsymbol{v}_3 = z\partial_x - x\partial_z, \tag{1.5b}$$

$$\boldsymbol{v}_4 = z\partial_y - y\partial_z, \tag{1.5c}$$

- translation in time and in all spatial directions:

$$\boldsymbol{v}_5 = \partial_t, \tag{1.6a}$$

$$\boldsymbol{v}_6 = \partial_x, \tag{1.6b}$$

$$\boldsymbol{v}_7 = \partial_y, \tag{1.6c}$$

$$\boldsymbol{v}_8 = \partial_z, \tag{1.6d}$$

- phase change in the wave function:

$$\boldsymbol{v}_9(\Omega) = \mathrm{i}\Omega(t)(u\partial_u - v\partial_v) - \Omega'(t)\partial_w, \tag{1.7}$$

- generalized Galilean group:

$$\boldsymbol{v}_{10}(a_1) = a_1(t)\partial_x + \frac{\mathrm{i}}{2}a_1'(t)(u\partial_u - v\partial_v) - \frac{1}{2}a_1''(t)\partial_w, \tag{1.8a}$$

$$\boldsymbol{v}_{11}(a_2) = a_2(t)\partial_y + \frac{\mathrm{i}}{2}a_2'(t)(u\partial_u - v\partial_v) - \frac{1}{2}a_2''(t)\partial_w, \tag{1.8b}$$

$$\boldsymbol{v}_{12}(a_3) = a_3(t)\partial_z + \frac{\mathrm{i}}{2}a_3'(t)(u\partial_u - v\partial_v) - \frac{1}{2}a_3''(t)\partial_w. \tag{1.8c}$$

2. First-Order Variational Formulation

Several variational formulations were proposed for the Schrödinger–Newton equations. At first they were recovered in the frame of the field theory by Kibble and Randjbar-Daemi [5] using an Hamiltonian formalism. Other variational formulations were given by Christian [2] and Diósi [3]. In his Ph.D. thesis Harrison [4], following Tod [11], proposed to derive the

Schrödinger–Newton equations (1.3) by means of the functional:

$$S_1[u,v] = \int_{t_0}^{+\infty} \iiint_{\mathbb{R}^3} \left[(\text{grad } u, \text{grad } v) + \frac{1}{2} wuv + \frac{\mathrm{i}}{2}(uv_t - vu_t) \right] \mathrm{d}^3\boldsymbol{x}\mathrm{d}t, \tag{2.1}$$

with the condition $\Delta w = uv$ implied. Following Tod [11] he later proposed to solve the equation for w with the usual method of the Green function in $\mathbb{R}^3$:

$$w(\boldsymbol{x},t) = \frac{-1}{4\pi} \iiint_{\mathbb{R}^3} \frac{u(\boldsymbol{y},t)v(\boldsymbol{y},t)}{|\boldsymbol{x}-\boldsymbol{y}|} \mathrm{d}^3\boldsymbol{y}. \tag{2.2}$$

However this leads to the functional:

$$\begin{aligned} S_2[u,v] = {} & \int_{t_0}^{+\infty} \iiint_{\mathbb{R}^3} \left[(\text{grad } u, \text{grad } v) - uv \frac{1}{8\pi} \iiint_{\mathbb{R}^3} \frac{u(\boldsymbol{y},t)v(\boldsymbol{y},t)}{|\boldsymbol{x}-\boldsymbol{y}|} \mathrm{d}^3\boldsymbol{y} \right] \mathrm{d}^3\boldsymbol{x}\mathrm{d}t \\ & + \int_{t_0}^{+\infty} \iiint_{\mathbb{R}^3} \left[\frac{\mathrm{i}}{2}(uv_t - vu_t) \right] \mathrm{d}^3\boldsymbol{x}\mathrm{d}t, \end{aligned} \tag{2.3}$$

that is nonlocal, and therefore the usual rules of Calculus of Variation do not hold [1].

Instead we take into consideration a paper by Christian [2] where the Schrödinger–Newton equations were derived from the following variational principle:

$$\hat{S}[u,v,w] = \int_{t_0}^{+\infty} \iiint_{\mathbb{R}^3} \left[\frac{1}{2} w\Delta w - (\text{grad } u, \text{grad } v) - \frac{\mathrm{i}}{2}(uv_t - u_t v) - wuv \right] \mathrm{d}^3\boldsymbol{x}\mathrm{d}t. \tag{2.4}$$

It was obtained by matching the Newton–Cartan theory and Quantum Mechanics. The functional (2.4) is not a first-order functional, but we note that using the general formula:

$$f\Delta g = \text{Div}(f \text{ grad } g) - (\text{grad } f, \text{grad } g), \quad f,g \in \mathcal{C}^2(\mathbb{R}^3,\mathbb{R}) \tag{2.5}$$

we may write:

$$w\Delta w = \text{Div}(w \text{ grad } w) - |\text{grad } w|^2, \tag{2.6}$$

and apply Gauss theorem to the integral of the first term in (2.6) in order to get:

$$\iiint_{\mathbb{R}^3} \text{Div}(w \text{ grad } w)\mathrm{d}^3\boldsymbol{x} = \lim_{R\to\infty} \iint_{S^2(R)} (w \text{ grad } w, \boldsymbol{n}_{S^2(R)})\mathrm{d}^2\boldsymbol{q}, \tag{2.7}$$

with $S^2(R) = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |x| = R\}$. This integral may be put equal to zero by assuming that the function w behaves like $1/r$ in a neighborhood of infinity if the usual boundary condition for the Poisson equation holds.

Thus we obtain[c] the following first-order variational principle:

$$\begin{aligned} S[u,v,w] = {} & \int_{t_0}^{+\infty} \iiint_{\mathbb{R}^3} \left[\frac{1}{2}|\text{grad } w|^2 + wuv \right] \mathrm{d}^3\boldsymbol{x}\mathrm{d}t \\ & + \int_{t_0}^{+\infty} \iiint_{\mathbb{R}^3} \left[(\text{grad } u, \text{grad } v) + \frac{\mathrm{i}}{2}(uv_t - u_t v) \right] \mathrm{d}^3\boldsymbol{x}\mathrm{d}t. \end{aligned} \tag{2.8}$$

It is easy to verify that (2.8) indeed yields the Schrödinger–Newton equations (1.3).

[c]Up to a multiplicative minus sign that does not effect the Euler–Lagrange equations.

3. Conservation Laws

Given a first-order variational principle in p independent variables $\boldsymbol{x} = (x^1, \ldots, x^p)$ and q dependent variables $\boldsymbol{u} = (u^1, \ldots, u^q)$ in some functional space of functions defined over a connected set A with smooth boundary:

$$S[\boldsymbol{u}] = \int \cdots \int_A L(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \mathrm{d}^p \boldsymbol{x} \tag{3.1}$$

a transformation group with infinitesimal generator given by a vector field $\boldsymbol{v}$ leaves the variational principle unchanged iff we can find a p-tuple $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{u}^{(1)}) = (B_1(\boldsymbol{x}, \boldsymbol{u}^{(1)}), \ldots, B_p(\boldsymbol{x}, \boldsymbol{u}^{(1)}))$ such that [8, Chap. 4]:

$$\mathrm{pr}^{(1)} \boldsymbol{v}(L) + L\, \mathrm{Div} \boldsymbol{\xi} = \mathrm{Div} \boldsymbol{B}. \tag{3.2}$$

Where $\boldsymbol{\xi} = (\xi^1, \ldots, \xi^p)$. In that case we call $\boldsymbol{v}$ a variational symmetry[d] for the variational principle (3.1).

We look for variational symmetries among the Lie point symmetries of the corresponding Euler–Lagrange equations.

A *conservation law* for a system of differential equations $\boldsymbol{\Theta}(\boldsymbol{x}, \boldsymbol{u}^{(n)}) = \boldsymbol{0}$ is a divergence expression vanishing identically along the solutions of the system:

$$\mathrm{Div} \boldsymbol{P} = 0. \tag{3.3}$$

The Schrödinger–Newton equations are a dynamical system such that their conservation laws can be put in the form [8]:

$$D_t T + \mathrm{Div} \boldsymbol{K} = 0 \tag{3.4}$$

along the solution of the system. The scalar $T(\boldsymbol{x}, t, \boldsymbol{u}^{(1)})$ is the *conserved density* and the vector $\boldsymbol{K}(\boldsymbol{x}, t, \boldsymbol{u}^{(1)})$ the *associated flux*.

A general theorem contained in Olver's book [8] tells us that if $\boldsymbol{K}(\boldsymbol{x}, t, \boldsymbol{u}(\boldsymbol{x})^{(n)}) \to 0$ for $\boldsymbol{x} \to \partial A$ then

$$\int \cdots \int_A T(\boldsymbol{x}, t, \boldsymbol{u}(\boldsymbol{x})^{(n)}) \mathrm{d}^p \boldsymbol{x} = \text{constant}. \tag{3.5}$$

A corollary to Noether's theorem to be found in her famous 1918-paper [6] states that given a variational symmetry $\boldsymbol{v}$ there is an explicit formula for the vector $\boldsymbol{P}$ in (3.3), i.e.:

$$P_i = \sum_{\alpha=1}^{q} \eta_\alpha \frac{\partial L}{\partial u_i^\alpha} + \xi^i L - B_i - \sum_{\alpha=1}^{q} \sum_{j=1}^{p} \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha}. \tag{3.6}$$

We wrote an *ad hoc* REDUCE interactive program that is based upon that by Nucci for finding Lie symmetries [7]. It verifies the condition (3.2) and then returns the conserved density and the associated flux given by Eq. (3.6). We found out that with respect to the

[d]In [8] Olver calls a vector field $\boldsymbol{v}$ a variational symmetry if $\boldsymbol{B} \equiv \boldsymbol{0}$ and a divergence variational symmetry if $\boldsymbol{B} \neq \boldsymbol{0}$. Since variational symmetries are a particular class of divergence variational symmetries we prefer to call the latter just variational symmetries.

functional (2.8) all the Lie point symmetries of the Schrödinger–Newton equations, except $\boldsymbol{v}_1$ as expected, are variational symmetries.

Introducing the shorthand notation:

$$\mathcal{E} = uvw + (\text{grad } u, \text{grad } v) + \frac{1}{2}|\text{grad } w|^2, \tag{3.7a}$$

$$\Pi^i = \frac{\mathrm{i}}{2}(u_i v - u v_i), \quad i = x, y, z, t, \tag{3.7b}$$

$$\Phi^{ij} = u_i v_j + u_j v_i + w_i w_j, \quad i, j = x, y, z, t, \tag{3.7c}$$

$$\Lambda^{ij} = x^j \Pi^i - x^i \Pi^j, \quad i, j = x, y, z, \tag{3.7d}$$

$$\varepsilon^i = \mathcal{E} - 2u_i v_i - w_i^2, \quad i = x, y, z, \tag{3.7e}$$

we find that:

- $\boldsymbol{v}_1$ is not a variational symmetry because the condition (3.2) is not satisfied,
- $\boldsymbol{v}_2$ is a variational symmetry with $\boldsymbol{B}_2 = \boldsymbol{0}$ and

$$T_2 = \Lambda^{xy}, \tag{3.8a}$$

$$\boldsymbol{K}_2 = (y(-\Pi^t + \varepsilon^x) + x\Phi^{xy}, x(\Pi^t - \varepsilon^y) - y\Phi^{xy}, x\Phi^{yz} - y\Phi^{xz}), \tag{3.8b}$$

- $\boldsymbol{v}_3$ is a variational symmetry with $\boldsymbol{B}_3 = \boldsymbol{0}$ and

$$T_3 = \Lambda^{xz}, \tag{3.9a}$$

$$\boldsymbol{K}_3 = (z(\varepsilon^x - \Pi^t) + x\Phi^{xz}, x\Phi^{yz} - z\Phi^{xy}, x(\Pi^t - \varepsilon^z) - z\Phi^{xz}), \tag{3.9b}$$

- $\boldsymbol{v}_4$ is a variational symmetry with $\boldsymbol{B}_4 = \boldsymbol{0}$ and

$$T_4 = \Lambda^{zy}, \tag{3.10a}$$

$$\boldsymbol{K}_4 = (y\Phi^{xz} - z\Phi^{xy}, z(\varepsilon^y - \Pi^t) + y\Phi^{xy}, y(\Pi^t - \varepsilon^z) - z\Phi^{yz}), \tag{3.10b}$$

- $\boldsymbol{v}_5$ is a variational symmetry with $\boldsymbol{B}_5 = \boldsymbol{0}$ and

$$T_5 = \mathcal{E}, \tag{3.11a}$$

$$\boldsymbol{K}_5 = (-\Phi^{tx}, -\Phi^{ty}, -\Phi^{tz}), \tag{3.11b}$$

- $\boldsymbol{v}_6$ is a variational symmetry with $\boldsymbol{B}_6 = \boldsymbol{0}$ and

$$T_6 = \Pi^x, \tag{3.12a}$$

$$\boldsymbol{K}_6 = (\varepsilon^x - \Pi^t, -\Phi^{xy}, -\Phi^{xz}), \tag{3.12b}$$

- $\boldsymbol{v}_7$ is a variational symmetry with $\boldsymbol{B}_7 = \boldsymbol{0}$ and

$$T_7 = \Pi^y, \tag{3.13a}$$

$$\boldsymbol{K}_7 = (-\Phi^{xy}, \varepsilon^y - \Pi^t, -\Phi^{yz}), \tag{3.13b}$$

- $\boldsymbol{v}_8$ is a variational symmetry with $\boldsymbol{B}_8 = \boldsymbol{0}$ and

$$T_8 = \Pi^z, \tag{3.14a}$$
$$\boldsymbol{K}_8 = (-\Phi^{xz}, -\Phi^{yz}, \varepsilon^z - \Pi^t), \tag{3.14b}$$

- $\boldsymbol{v}_9$ is a variational symmetry with $\boldsymbol{B}_9 = \boldsymbol{0}$ and

$$T_9 = \Omega(t)uv, \tag{3.15a}$$
$$\boldsymbol{K}_9 = \begin{pmatrix} 2\Omega(t)\Pi^x - \Omega'(t)w_x \\ 2\Omega(t)\Pi^y - \Omega'(t)w_y \\ 2\Omega(t)\Pi^z - \Omega'(t)w_z \end{pmatrix}^t, \tag{3.15b}$$

- $\boldsymbol{v}_{10}$ is a variational symmetry with $\boldsymbol{B}_{10} = (0, -a_1''(t)w/2, 0, 0)$ and

$$T_{10} = a_1(t)\Pi^x + \frac{1}{2}a_1'(t)uvx, \tag{3.16a}$$
$$\boldsymbol{K}_{10} = \begin{pmatrix} a_1(t)(\varepsilon^x - \Pi^t) - a_1'(t)x\Pi^x + \frac{1}{2}a_1''(t)(w - w_x x) \\ -a_1(t)\Phi^{xy} - a_1'(t)x\Pi^y - \frac{1}{2}a_1''(t)w_y x \\ -a_1(t)\Phi^{xz} - a_1'(t)x\Pi^z - \frac{1}{2}a_1''(t)w_z x \end{pmatrix}^t, \tag{3.16b}$$

- $\boldsymbol{v}_{11}$ is a variational symmetry with $\boldsymbol{B}_{11} = (0, 0, -a_2''(t)w/2, 0)$ and

$$T_{11} = a_2(t)\Pi^y + \frac{1}{2}a_2'(t)uvy, \tag{3.17a}$$
$$\boldsymbol{K}_{11} = \begin{pmatrix} -a_2(t)\Phi^{xy} - a_2'(t)y\Pi^x - \frac{1}{2}a_2''(t)w_x y \\ a_2(t)(\varepsilon^y - \Pi^t) - a_2'(t)y\Pi^y + \frac{1}{2}a_2''(t)(w - w_y y) \\ -a_2(t)\Phi^{yz} - a_2'(t)y\Pi^z - \frac{1}{2}a_1''(t)w_z y \end{pmatrix}^t, \tag{3.17b}$$

- $\boldsymbol{v}_{12}$ is a variational symmetry with $\boldsymbol{B}_{12} = (0, 0, 0, -a_3''(t)w/2)$ and

$$T_{12} = a_3(t)\Pi^z + \frac{1}{2}a_3'(t)uvz, \tag{3.18a}$$
$$\boldsymbol{K}_{12} = \begin{pmatrix} -a_3(t)\Phi^{xz} - a_3'(t)z\Pi^x - \frac{1}{2}a_3''(t)w_x z, \\ -a_3(t)\Phi^{yz} - a_3'(t)y\Pi^y - \frac{1}{2}a_3''(t)w_y z, \\ a_3(t)(\varepsilon^z - \Pi^t) - a_3'(t)z\Pi^z + \frac{1}{2}a_3''(t)(w - w_z z) \end{pmatrix}^t. \tag{3.18b}$$

Assuming the boundary conditions $u, v, w \xrightarrow{|\boldsymbol{x}|\to\infty} 0$ then Eq. (3.5) yields the following conserved quantities of the Schrödinger–Newton equations:

- energy:

$$E = \iiint_{\mathbb{R}^3} \mathcal{E} \mathrm{d}^3\boldsymbol{x}, \tag{3.19}$$

- angular momenta:

$$L^{ij} = \iiint_{\mathbb{R}^3} \Lambda^{ij} \mathrm{d}^3\boldsymbol{x}, \quad i, j = x, y, z, \tag{3.20}$$

- linear momenta:

$$p^i = \iiint_{\mathbb{R}^3} \Pi^i \mathrm{d}^3\boldsymbol{x}, \quad i = x, y, z, \tag{3.21}$$

- generalized probability:

$$P_g(\Omega(t)) = \iiint_{\mathbb{R}^3} \Omega(t) uv \mathrm{d}^3\boldsymbol{x} \tag{3.22}$$

  that becomes the usual probability if $\Omega(t) = 1$, i.e.

$$P = \iiint_{\mathbb{R}^3} uv \mathrm{d}^3\boldsymbol{x}, \tag{3.23}$$

- generalized linear momenta:

$$h^i(a_i(t)) = \iiint_{\mathbb{R}^3} \left(a_i(t)\Pi^i + \frac{1}{2} a_i'(t) uvx^i \right) \mathrm{d}^3\boldsymbol{x}, \quad i = x, y, z. \tag{3.24}$$

The conservation of energy[e] (3.19), the angular (3.20) and linear momenta (3.21) and the usual probability (3.23) were recovered by Harrison [4] but without any consideration of symmetries. As far as we know the generalized probability (3.22) and generalized linear momenta (3.24) are new conserved quantities of the Schrödinger–Newton equations.

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