Spectral Zeta Functions of a 1D Schrödinger Problem

Joe Watkins

SPECTRAL ZETA FUNCTIONS
OF A 1D SCHRÖDINGER PROBLEM

JOE WATKINS

SMSAS, University of Kent
Canterbury, Kent, CT2 7NF, UK
jw352@kent.ac.uk

We study the spectral zeta functions associated to the radial Schrödinger problem with potential $V(x) = x^{2M} + \alpha x^{M-1} + (\lambda^2 - 1/4)/x^2$. After directly computing some of the zeta functions, we use the quantum Wronskian equation to give sum rules between them, allowing for instances where the explicit form of the zeta functions can be simplified. An immediate application of this work is to derive functional relations and identities involving hypergeometric series, allowing for known identities to be found as instances of more general results. Our work is then extended to a class of related $\mathcal{PT}$-symmetric eigenvalue problems. Using the fused quantum Wronskian, we give a simple method for indirectly calculating the associated spectral zeta functions. This method is then applied to calculate the nonlocal integrals of motion G_n which appear in an associated integrable quantum field theory.

Keywords: Spectral zeta; hypergeometric series; $\mathcal{PT}$-symmetric; integrals of motion.

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1. Introduction

A common set of models to be studied in quantum mechanics are the anharmonic oscillators defined by the Schrödinger equation

$$-\psi'' + x^{2M}\psi = E\psi. \tag{1.1}$$

Any eigenfunction must obey the requirement that $\psi \to 0$ as $x \to +\infty$ and additionally satisfy either the Dirichlet or Neumann conditions at the origin. The anharmonic oscillators include the two exactly-solvable cases of the harmonic oscillator ($M = 1$) and infinite square well ($M = \infty$). For these two models the eigenvalues can be found in closed form, whereas for general M numerical methods must be used. The Dirichlet eigenvalues $\{E_k^-\}$ and Neumann

eigenvalues $\{E_k^+\}$ are associated to the spectral zeta function

$$Z_\mp(s, M) \equiv \sum_{k=0}^\infty \frac{1}{(E_k^\mp)^s}.$$ (1.2)

WKB estimates show that $E_k^\mp \propto k^{\frac{2M}{M+1}}$ as $k \to \infty$ and hence $Z_\mp(s)$ converges for $\Re(s) > (M+1)/2M$. In the two exactly-solvable cases $Z_\mp(s)$ can be given in terms of the Riemann zeta function. However $Z_\mp(s)$ can also be calculated for $s \in \mathbb{N}$, the most concise example being given by[a]

$$Z_\mp(1) = \frac{\sigma^{2-2\sigma}\Gamma(\sigma(1 \pm \frac{1}{2}))\Gamma(\sigma)\Gamma(\frac{1}{2} - \sigma)}{\sqrt{\pi}\Gamma(1 - \sigma(1 \mp \frac{1}{2}))},$$ (1.3)

where $M > 1$ and

$$\sigma \equiv \frac{1}{M+1}.$$ (1.4)

The zeta functions $Z_\mp(2)$ can also be calculated [32], although these functions involve a $_5F_4$ hypergeometric series and no example of a closed form expression has been given.

There has been much study related to these particular spectral zeta functions [16, 31–34], with one significant result being the determination of "sum rules" which relate together different $Z_\mp(s)$ when $s \in \mathbb{N}$ [32]. An elegant example of this is found for the quartic anharmonic oscillator, the first few sum rules being

$$Z_+(1) = 2Z_-(1),$$
$$2Z_+(2) = Z_-(2) + 3Z_-(1)^2,$$
$$2Z_+(3) = 9Z_-(1)^3 - Z_-(1)^2 - Z_-(3).$$

The techniques for obtaining the sum rules, which will be covered later, are the consequence of a wider study known as exact quantization, a review of many such problems being found in [35]. For a review of the physical applications of spectral zeta functions in general, we direct the reader toward [22].

We will consider a generalization of (1.1) given by the Schrödinger equation

$$-\psi'' + \left(x^{2M} + \alpha x^{M-1} + \frac{\lambda^2 - \frac{1}{4}}{x^2}\right)\psi = E\psi$$ (1.5)

with boundary conditions on the half-line. The eigenfunction criteria is that $\psi \to 0$ as $x \to +\infty$, with the wavefunction having either of the behaviors $\psi_- \sim x^{\frac{1}{2}+\lambda}$ or $\psi_+ \sim x^{\frac{1}{2}-\lambda}$ as $x \to 0$. Such eigenvalue problems have a long and rich history and have been investigated in many contexts such as quasi-exact solvability [14, 30], integrable models [9, 21, 23] and $\mathcal{PT}$-symmetry [12, 13, 17, 18].

The two boundary behaviors ψ_- and ψ_+ respectively define two sets of spectra $\{E_k^-\}$ and $\{E_k^+\}$, which we say are the eigenvalues of the "regular" and "irregular" radial problem [27]. To these spectra we associate the zeta functions $Z_\mp(s)$, which for $\lambda = 1/2$ are the functions

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[a]Calculations for two related zeta functions were originally given in [16, 32].

defined in (1.2). However for general λ the regular and irregular problem are related by the analytic continuation $\lambda \to -\lambda$ [21], giving the zeta-function identity

$$Z_+(s, \lambda) = Z_-(s, -\lambda). \tag{1.6}$$

Throughout λ will be restricted from taking the values

$$\lambda = \pm \frac{1}{2}((2m_1 + 1)(M + 1) + \alpha) \quad \text{and} \quad \lambda = \pm \left(m_2 + \frac{m_3}{2}(M + 1) \right) \tag{1.7}$$

where $m_1, m_2, m_3 \in \mathbb{Z}^+$, although when $\alpha = 0$ the condition becomes $2m_3 \in \mathbb{Z}^+$ [21]. The first restriction excludes the possibility that any $E_k^{\mp}$ could be equal to zero and the second restriction is to ensure linear independence of the ψ_- and ψ_+ [21], a condition which is necessary for later work. Finally, as the eigenvalues are known to have large-k behavior $E_k^{\mp} \propto k^{\frac{2M}{M+1}}$, the associated spectral zeta functions $Z_{\mp}(s)$ will converge when $\Re(s) > (M + 1)/2M$. As we will be studying $Z_{\mp}(s)$ for $s \geq 1$, we also restrict $M > 1$ throughout.

This paper will be divided into two sections and will investigate the radial eigenvalue problems (1.5). In Sec. 2 we use the well-known method involving Green's functions to calculate $Z_{\mp}(s)$ when $s \in \mathbb{N}$. Then we will use the quantum Wronskian equation to derive appropriate sum rules and give examples where $Z_+(2)$ can be written in closed form, including examples involving the anharmonic oscillators. Similar techniques are then used to derive functional relations for two different hypergeometric series which appear in the expressions for $Z_{\mp}(1)$ and $Z_{\mp}(2)$. These functional relations can recover many specific, known properties of hypergeometric series, as well as providing new identities. In Sec. 3 we will extend our work to a class of eigenvalue problems commonly investigated in $\mathcal{PT}$-symmetric quantum mechanics. The zeta functions $\mathcal{Z}_K(s)$ of these problems are studied for $s \in \mathbb{N}$ and found to be constructible by sum rules involving $Z_{\mp}(s)$. These expressions for $\mathcal{Z}_K(s)$ are then used in conjunction with the ODE/IM correspondence [8, 19, 21] to calculate the vacuum nonlocal integrals of motion G_n which appear in a related quantum integrable field theory [4, 5].

2. Radial Schrödinger Problems

Given that the eigenvalues $\{E_k^{\mp}\}$ have no known closed form for general M, naively we might expect that associated spectral zeta functions $Z_{\mp}(s)$ cannot be given explicitly, except in the exactly-solvable cases. However when $s \in \mathbb{N}$ the zeta functions are computed using the process given in [32]. Although $Z_{\mp}(s)$ can be defined by analytic continuation when $s \in \mathbb{Z}^-$ [33], generally there is no known method to calculate their explicit forms when s takes general values. Therefore we elect to consider $Z_{\mp}(s)$ only when $s \in \mathbb{N}$, which we notate $Z_{\mp}(n)$.

For a Hermitian eigenvalue problem the eigenfunctions are complete and the Green's function is written as

$$R(E; x, x') = \sum_{k=0}^{\infty} \frac{\psi_k(x)\psi_k^*(x')}{E_k^- - E}, \tag{2.1}$$

where $\{E_k^-\}$ are the associated eigenvalues. As the irregular problem is generally non-Hermitian, the eigenfunctions are not guaranteed to be complete and consequently the

Green's function cannot be written as in (2.1). Instead $Z_+(n)$ is defined by analytic continuation as in (1.6), which is valid except for λ as in (1.7).

As the eigenfunctions in the Hermitian case are necessarily orthonormal, the zeta functions are calculated by

$$Z_-(n) = \int_{\mathbb{R}^+} R(0; x_1, x_2) R(0; x_2, x_3) \cdots R(0; x_n, x_1) dx,$$

where $\mathbb{R}^+$ denotes the integration over all positive space in n dimensions. To evaluate this repeated integral, $R(0; x, x')$ is decomposed as a combination of ψ_L and ψ_R, two linearly-independent wavefunctions which solve the Schrödinger equation when $E = 0$. For a radial problem the two wavefunctions have specific asymptotic behaviors; ψ_L must obey the boundary conditions at the origin and we require that $\psi_R \to 0$ as $x \to +\infty$. The Green's function is written as

$$R(0; x, x') = \frac{1}{\mathcal{W}} \psi_L(x_<) \psi_R(x_>),$$

where $x_< \equiv \min(x, x')$, $x_> \equiv \max(x, x')$ and $\mathcal{W} \equiv \psi_R \psi_L' - \psi_R' \psi_L$, the Wronskian of the two solutions. Thus

$$Z_-(n) = \frac{n!}{\mathcal{W}^n} \int_{0 < x_1 < x_2 < \cdots < x_n < \infty} \psi_L(x_1) \psi_R(x_n) \prod_{i=1}^{n-1} \psi_L(x_i) \psi_R(x_{i+1}) dx_1 dx_2 \cdots dx_n. \quad (2.2)$$

Now we specialize to the regular radial problem (1.5), which is solved for $E = 0$ by

$$\psi_L = x^{\frac{\sigma-1}{2\sigma}} M_{-\frac{\sigma\alpha}{2}, \sigma\lambda}(2\sigma x^{\frac{1}{\sigma}}) \quad \text{and} \quad \psi_R = x^{\frac{\sigma-1}{2\sigma}} W_{-\frac{\sigma\alpha}{2}, -\sigma\lambda}(2\sigma x^{\frac{1}{\sigma}}).$$

Here M and W are the Whittaker functions [25] and the Wronskian of the two solutions is given by

$$\mathcal{W}[\psi_R, \psi_L] = \frac{2\Gamma(1 + 2\sigma\lambda)}{\Gamma(\frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma\lambda)}.$$

After making the change of variables $p = 2\sigma x_1^{\frac{1}{\sigma}}$ and $q = 2\sigma x_2^{\frac{1}{\sigma}}$, we find that

$$Z_-(1) = \frac{\sigma^{2-2\sigma}\Gamma(\frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma\lambda)}{4^\sigma \Gamma(1 + 2\sigma\lambda)} \int_0^\infty p^{2\sigma-2} W_{-\frac{\sigma\alpha}{2}, -\lambda\sigma}(p) M_{-\frac{\sigma\alpha}{2}, \lambda\sigma}(p) dp \qquad (2.3a)$$

and

$$Z_-(2) = \frac{2\sigma^{4-4\sigma}\Gamma^2(\frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma\lambda)}{16^\sigma \Gamma^2(1 + 2\sigma\lambda)} \int_0^\infty q^{2\sigma-2} W^2_{-\frac{\sigma\alpha}{2}, -\lambda\sigma}(q) \int_0^q p^{2\sigma-2} M^2_{-\frac{\sigma\alpha}{2}, \lambda\sigma}(p) dp \, dq.$$

$$(2.3b)$$

The evaluation of these integrals requires known identities from [25, (7.625)]. The integral (2.3a) can be calculated immediately and found to converge for $\sigma > 1/2$, reproducing the restriction $M > 1$ which was imposed earlier. Completing the integration in (2.3b) is more complicated and can be handled in a similar way to the calculation for the anharmonic

operators as given in [32]. Completing the integrations gives the first two zeta functions

$$Z_-(1) = \frac{\sigma^{2-2\sigma}\Gamma(\frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma\lambda)\Gamma(2\sigma(1+\lambda))\Gamma(2\sigma)}{4^{\sigma}\Gamma(1+2\sigma\lambda)\Gamma(\frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma(2+\lambda))}\,_3F_2\left(\begin{array}{c} \frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma\lambda, 2\sigma(1+\lambda), 2\sigma \\ 1 + 2\sigma\lambda, \frac{1}{2} + \frac{\sigma\alpha}{2} + \sigma(2+\lambda) \end{array}\right)$$

(2.4a)

and

$$Z_-(2) = \frac{\sigma^{4-4\sigma}}{16^{\sigma}} \sum_{l,m,n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \frac{\sigma\alpha}{2} + \lambda\sigma + m)\Gamma(\frac{1}{2} + \frac{\sigma\alpha}{2} + \lambda\sigma + n)}{\Gamma(1+2\sigma\lambda+m)\Gamma(1+2\sigma\lambda+n)m!n!} \frac{1}{(m+n+l+2\sigma(1+\lambda))_{l+1}}$$

$$\times G_{33}^{22}\left(\begin{array}{c} \frac{1}{2} - \sigma\lambda, \frac{1}{2} + \sigma\lambda, l+m+n+\frac{\sigma\alpha}{2} + 2\sigma(2+\lambda) \\ l+m+n+\sigma(4+\lambda)-\frac{1}{2}, l+m+n+\sigma(4+3\lambda)-\frac{1}{2}, -\frac{\sigma\alpha}{2} \end{array}\right),$$

(2.4b)

where $(a)_b$ is the Pochhammer symbol, $_pF_q$ is the generalized hypergeometric series and G is the Meijer-G function [25]. The latter two functions are both evaluated at $x = 1$.

By setting $\alpha = 0$, the Whittaker functions in (2.3a) and (2.3b) are replaced by Bessel functions and the integration techniques directly follow those in [32]. The zeta functions then take on the neater forms

$$Z_-(1, \alpha = 0) = \frac{\sigma^{2-2\sigma}\Gamma(\sigma(1+\lambda))\Gamma(\sigma)\Gamma(\frac{1}{2}-\sigma)}{4\sqrt{\pi}\Gamma(1-\sigma(1-\lambda))}$$

(2.5a)

and

$$Z_-(2, \alpha = 0) = \frac{\sqrt{\pi}\sigma^{3-4\sigma}}{4^{1+\sigma\lambda}(1+\lambda)} \frac{\Gamma(2\sigma(1+\lambda))\Gamma(\sigma(2+\lambda))\Gamma(2\sigma)}{\Gamma^2(1+\sigma\lambda)\Gamma(\frac{1}{2}+\sigma(2+\lambda))}$$

$$\times {}_5F_4\left(\begin{array}{c} \frac{1}{2} + \sigma\lambda, 2\sigma(1+\lambda), \sigma(2+\lambda), 2\sigma, \sigma(1+\lambda) \\ 1+\sigma\lambda, 1+2\sigma\lambda, \frac{1}{2}+\sigma(2+\lambda), 1+\sigma(1+\lambda) \end{array}\right).$$

(2.5b)

Setting $\alpha = 0$ in (2.4a) also recovers (2.5a) directly by Dixon's theorem.

2.1. Radial sum rules and simplifications ($\alpha = 0$)

For $M > 1$ the growth rate of the eigenvalues is sufficient to ensure the convergence of the spectral determinants

$$D_-(E, M, \alpha, \lambda) \equiv D_-(0) \prod_{k=0}^{\infty} \left(1 - \frac{E}{E_k^-}\right).$$

(2.6)

Here $D_-(0)$ is a regularizing prefactor designed to vanish when some $E_k^- = 0$. By analytic continuation we define $D_+(E, \lambda) \equiv D_-(E, -\lambda)$ except for the values of λ given in (1.7).

In addition to the analytic continuation relating $D_-(E)$ and $D_+(E)$, there is the quantum Wronskian equation

$$2\lambda\omega^{\frac{\alpha}{2}} = \overline{\omega}^{\lambda}D_-(\overline{\omega}E, -i\alpha)D_+(\omega E, i\alpha) - \omega^{\lambda}D_-(\omega E, i\alpha)D_+(\overline{\omega}E, -i\alpha),$$

(2.7)

where

$$\omega \equiv \exp(i\pi\sigma).$$

(2.8)

The sets of eigenvalues $\{E_k^-\}$ and $\{E_k^+\}$, having no relationship except at (1.7), are thus related to each other by a simple expression involving their associated spectral determinants.

Two clarifications need to be made regarding the quantum Wronskian equation. First is to note that when $\alpha \neq 0$ the spectral determinants in (2.7) refer to differential equations with imaginary coupling constants on the x^{M-1} term, a problem which is remedied in [35]. Secondly, for general α, the quantum Wronskian links together four different spectral determinants, whereas setting $\alpha = 0$ reduces this number to two. For this latter case the subsequent results are much less complicated and so $\alpha = 0$ is fixed for the remainder of this section.

Using (2.7), we can derive sum rules relating together different $Z_{\mp}(n)$. For small E the spectral determinants can be written in terms of their zeta functions [33] as

$$D_{\mp}(E) = D_{\mp}(0) \exp\left(-\sum_{n=1}^{\infty} \frac{Z_{\mp}(n)}{n} E^n\right). \tag{2.9}$$

Using the technique of [32], (2.9) is substituted into (2.7) and the coefficients of the powers of E are compared to obtain the radial sum rules. The first few of these are

$$0 = N_1 Z_-(1) + N_{-1} Z_+(1), \tag{2.10a}$$

$$0 = N_2 Z_-(2) + N_{-2} Z_+(2) + (N_1^2 - N_2)(Z_+(1) - Z_-(1))^2, \tag{2.10b}$$

$$0 = N_3 Z_-(3) + N_{-3} Z_+(3) - \frac{3}{2}(N_3 - N_2 N_1)(Z_+(2) - Z_-(2))(Z_+(1) - Z_-(1))$$

$$- \frac{1}{2}\left(N_3 - 3N_2 N_1 + 2N_1^3\right)(Z_+(1) - Z_-(1))^3, \tag{2.10c}$$

where

$$N_a \equiv \sin(\pi\sigma(\lambda + a))\csc(\pi\sigma\lambda).$$

Under certain parameter choices, (2.10) allows for $Z_+(n)$ to be given in terms of spectral zeta functions with a simpler form. As an example, the choice $\sigma(\lambda + 2) = m \in \mathbb{N}$ implies that $N_2 = 0$. At such a point (2.10b) does not include $Z_-(2)$ and hence, for $m \in \mathbb{N}$, we find

$$Z_+\left(2, \sigma = \frac{m}{(\lambda + 2)}\right) = -\frac{\sigma^{4-4\sigma}\Gamma^4(\sigma)\Gamma^2(1-2\sigma)\sin^3(2\pi\sigma)\csc^2(3\pi\sigma)}{2^{4-4\sigma}\Gamma^2(1-3\sigma+m)\Gamma^2(1+\sigma-m)\sin(4\pi\sigma)}. \tag{2.11}$$

This expression is much less complicated than the general form for $Z_{\mp}(2)$ given in (2.5b), reducing the $_5F_4$ hypergeometric series to a product of gamma and trigonometric functions.

Example 2.1. The anharmonic cubic oscillator has the zeta values

$$Z_+(2) = \frac{8(\sqrt{5}-1)\pi^4}{5^{\frac{17}{5}}\Gamma^4(\frac{4}{5})\Gamma^2(\frac{3}{5})}$$

and

$$Z_-(3) = \frac{2^{\frac{14}{5}}3\pi^{\frac{9}{2}}\Gamma(\frac{7}{10})}{5^{\frac{23}{5}}\Gamma^5(\frac{4}{5})\Gamma^2(\frac{9}{10})} - \frac{32\pi^6}{5^{\frac{51}{10}}\Gamma^6(\frac{4}{5})\Gamma^3(\frac{3}{5})} - \frac{2\pi\sqrt{5-2\sqrt{5}}}{5^{\frac{21}{10}}\Gamma(\frac{3}{5})} \times {}_4F_3\left(\begin{matrix} \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, 1 \\ \frac{7}{5}, \frac{3}{2}, \frac{8}{5} \end{matrix}\right). \tag{2.12}$$

In addition to finding closed form expressions for $Z_{\mp}(n)$, there is a reason to investigate the composite zeta functions

$$Z(s) \equiv Z_+(s) + Z_-(s) \quad \text{and} \quad \tilde{Z}(s) \equiv Z_+(s) - Z_-(s). \tag{2.13}$$

For the potential $V(x) = |x|^{2M}$ with eigenfunction condition $\psi \in L^2(\mathbb{R})$ and eigenvalues $\{E_k\}$, $Z(s)$ and $\tilde{Z}(s)$ are respectively the "full" and "skew" zeta functions, defined as

$$Z(s) \equiv \sum_{k=0}^{\infty} \frac{1}{(E_k)^s} \quad \text{and} \quad \tilde{Z}(s) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{(E_k)^s}.$$

For the anharmonic oscillators, these definitions correspond to (2.13) as the spectrum $\{E_k\}$ is the interlacing union of $\{E_k^-\}$ and $\{E_k^+\}$. For general α and λ this cannot be guaranteed.

Although generally used as a convenient notation, simplifications in the explicit forms of $Z(n)$ and $\tilde{Z}(n)$ will be of use later. The radial sum rules are used to find reduced forms for $Z(2)$ and $\tilde{Z}(2)$ by realizing points where $N_2 = \pm N_{-2}$. For $m \in \mathbb{Z}^+$ we find

$$Z\left(2, \sigma = \frac{(2m-1)}{2\lambda}\right) = -\frac{\pi^4 \sigma^{4-4\sigma} \Gamma^2(1-2\sigma) \sec^2(\pi\sigma) \sec(2\pi\sigma)}{4^{1-2\sigma} \Gamma^4(1-\sigma) \Gamma^2(\frac{3}{2} - \sigma - m) \Gamma^2(\frac{1}{2} - \sigma + m)} \tag{2.14}$$

and

$$\tilde{Z}(2, \sigma = 1/4) = \frac{\pi^5 \sec^2(\frac{\pi\lambda}{2}) \tan(\frac{\pi\lambda}{4})}{64 \Gamma^4(\frac{3}{4}) \Gamma^2(\frac{3+\lambda}{4}) \Gamma^2(\frac{3-\lambda}{4})}. \tag{2.15}$$

There are no simplifications for $Z(2, \lambda = 1/2)$ and no general simplifications for $\tilde{Z}(2)$, as λ would conflict with (1.7).

Example 2.2. The anharmonic sextic oscillator has the zeta value

$$\tilde{Z}(2) = \frac{(\sqrt{2} - 1)\pi^5}{32 \Gamma^4(\frac{3}{4}) \Gamma^2(\frac{7}{8}) \Gamma^2(\frac{5}{8})}.$$

2.2. *Functional relations between hypergeometric series* ($\alpha = 0$)

While the radial sum rules in (2.10) are useful for determining properties of $Z_{\mp}(n)$, they can also be used to determine properties of the special functions appearing in the explicit forms for $Z_{\mp}(n)$, as in (2.5). There are two principle reasons that such derivations can be made. First is the analytic continuation (1.6) and the second is that the radial sum rules can be written as

$$Z_+(1) = -\frac{N_1}{N_{-1}} Z_-(1), \tag{2.16a}$$

$$Z_+(2) = -\frac{N_2}{N_{-2}} Z_-(2) + \left(\frac{N_2}{N_{-2}} - \frac{2N_1}{N_{-1}N_{-2}} + \frac{N_1^2}{N_{-1}^2}\right) Z_-(1)^2. \tag{2.16b}$$

These identities implicitly specify functional relations for the special functions appearing in (2.5). By substituting (2.5a) into (2.16a) and using (1.6), we recover the identity

$$\frac{\Gamma(\sigma(1-\lambda))}{\Gamma(1 - \sigma(1+\lambda))} = \frac{\Gamma(\sigma(1+\lambda)) \sin(\pi\sigma(1+\lambda))}{\Gamma(1 - \sigma(1-\lambda)) \sin(\pi\sigma(1-\lambda))},$$

which is effectively two copies of the Euler reflection formula for the gamma function.

The next application is to the zeta functions $Z_{\mp}(2)$, the explicit forms involving a $_5F_4$ hypergeometric series as in (2.5b). Therefore we expect to derive properties of the function

$$\mathcal{F}(\lambda) \equiv {}_5F_4 \left(\begin{array}{c} \frac{1}{2} + \sigma\lambda, 2\sigma(1 + \lambda), \sigma(2 + \lambda), 2\sigma, \sigma(1 + \lambda) \\ 1 + \sigma\lambda, 1 + 2\sigma\lambda, \frac{1}{2} + \sigma(2 + \lambda), 1 + \sigma(1 + \lambda) \end{array} \right). \tag{2.17}$$

Substituting $Z_-(1)$ and $Z_{\mp}(2)$ into (2.16b) gives the functional relation

$$\frac{\sin(\pi\sigma(\lambda + 2))4^{-\sigma\lambda}\Gamma(2\sigma(1 + \lambda))\Gamma(\sigma(2 + \lambda))}{\sin(\pi\sigma(\lambda - 2))(1 + \lambda)\Gamma^2(1 + \sigma\lambda)\Gamma(\frac{1}{2} + \sigma(2 + \lambda))}\mathcal{F}(\lambda)$$

$$= \frac{4^{\sigma\lambda}\Gamma(2\sigma(1 - \lambda))\Gamma(\sigma(2 - \lambda))}{(\lambda - 1)\Gamma^2(1 - \sigma\lambda)\Gamma(\frac{1}{2} + \sigma(2 - \lambda))}\mathcal{F}(-\lambda)$$

$$+ \frac{4^{2\sigma-1}\sigma\pi^{\frac{3}{2}}\Gamma^2(1 - 2\sigma)\Gamma^2(\sigma(1 + \lambda))}{\Gamma^4(1 - \sigma)\Gamma^2(1 - \sigma(1 - \lambda))\Gamma(2\sigma)\sin^2(\pi\sigma)}$$

$$\times \left(\frac{\sin^2(\pi\sigma(1 + \lambda))}{\sin^2(\pi\sigma(1 - \lambda))} - \frac{\sin(\pi\sigma(2 + \lambda))}{\sin(\pi\sigma(2 - \lambda))} - \frac{2\sin(\pi\sigma(1 + \lambda))\sin(\pi\sigma\lambda)}{\sin(\pi\sigma(1 - \lambda))\sin(\pi\sigma(2 - \lambda))} \right). \tag{2.18}$$

Furthermore, numerical testing indicates that this identity appears to hold when $0 < \Re(\sigma) < 3/4$, although our result only applies when $\sigma \in \mathbb{R}$.

By choosing σ and λ to take special values — precisely those which give simple forms for $Z_+(2)$ — we recover further properties of $\mathcal{F}$. For example the simplification (2.11) gives, for $m \in \mathbb{N}$, the identity

$$\mathcal{F}\left(\frac{2 - m}{\sigma} \right)$$

$$= \frac{(m - 3\sigma)\Gamma^4(\sigma)\Gamma^2(1 + 2\sigma - m)\Gamma^2(1 - 2\sigma)\Gamma(\frac{1}{2} + 4\sigma - m)\sin^3(2\pi\sigma)\csc^2(3\pi\sigma)}{\sqrt{\pi}4^{m+1-4\sigma}\Gamma^2(1 - 3\sigma + m)\Gamma^2(1 + \sigma - m)\Gamma(2\sigma)\Gamma(6\sigma - 2m)\Gamma(4\sigma - m)\sin(4\pi\sigma)}.$$

When $m = 1$ this identity is verified by Dixon's theorem but we have been unable to find any known identities accounting for general $m \in \mathbb{N}$. While $\mathcal{F}$ can be rewritten in terms of $_4F_3$ hypergeometric series [32], this appears to provide no extra information and the functional identity (2.18) is not apparent.

The exact form of $Z_-(3)$ is known to contain Appell series [32], functional relations for which being obtained by the above methods. We would then expect relationships between two different Appell series in terms of $_5F_4$ hypergeometric series.

2.3. *Functional relations between hypergeometric series* ($\alpha \neq 0$)

The techniques used in Sec. 2.2 also apply when $\alpha \neq 0$. In this situation the sum rules found from (2.7) are more complicated, featuring four separate zeta functions. Substituting (2.9) into (2.7), the coefficient of E in the subsequent expansion gives, after the transformation

$\alpha \to i\alpha$,

$$D_-(0,\alpha)D_+(0,-\alpha)\left(\omega^{1-\lambda}Z_+(1,-\alpha) + \omega^{-(1+\lambda)}Z_-(1,\alpha)\right)$$

$$= D_-(0,-\alpha)D_+(0,\alpha)(\omega^{\lambda-1}Z_+(1,\alpha) + \omega^{1+\lambda}Z_-(1,-\alpha)), \qquad (2.19)$$

where $D_\mp(0)$ is given in [18] as

$$D_-(0,\alpha) \propto (2\sigma)^{\frac{\alpha\sigma}{2}-\sigma\lambda-1/2}\frac{\Gamma(1+2\sigma\lambda)}{\Gamma(\frac{1}{2}+\frac{\alpha\sigma}{2}+\sigma\lambda)}. \qquad (2.20)$$

The explicit form for $Z_-(1)$ features a $_3F_2$ hypergeometric series as in (2.4a). Therefore (2.19) is implicitly a four-term functional relation involving this function. Splitting into real and imaginary parts, (2.19) is actually two independent functional relations satisfied by the $_3F_2$ hypergeometric series. We define the function

$$\mathcal{G}(\alpha,\lambda) \equiv \frac{1}{\Gamma(\frac{1}{2}+2\sigma+\frac{\sigma\alpha}{2}+\sigma\lambda)\Gamma(\frac{1}{2}-\frac{\sigma\alpha}{2}-\sigma\lambda)}{_3F_2}\begin{pmatrix}\frac{1}{2}+\frac{\sigma\alpha}{2}+\sigma\lambda, 2\sigma(1+\lambda), 2\sigma\\ 1+2\sigma\lambda, \frac{1}{2}+\frac{\sigma\alpha}{2}+\sigma(2+\lambda)\end{pmatrix},$$
$$(2.21)$$

where the hypergeometric series converges for $\Re(\sigma) < 1/2$. Substituting (2.4a) and (2.20) into (2.19) gives, after taking real and imaginary parts, the pair of four-term relations

$$\mathcal{G}(\alpha,\lambda) + \mathcal{G}(-\alpha,\lambda)$$

$$= \frac{\sin(\pi\sigma(1-\lambda))\Gamma(1+2\sigma\lambda)\Gamma(2\sigma(1-\lambda))}{\sin(\pi\sigma(1+\lambda))\Gamma(1-2\sigma\lambda)\Gamma(2\sigma(1+\lambda))}(\mathcal{G}(\alpha,-\lambda) + \mathcal{G}(-\alpha,-\lambda)) \qquad (2.22a)$$

and

$$\mathcal{G}(\alpha,\lambda) - \mathcal{G}(-\alpha,\lambda)$$

$$= \frac{\cos(\pi\sigma(1-\lambda))\Gamma(1+2\sigma\lambda)\Gamma(2\sigma(1-\lambda))}{\cos(\pi\sigma(1+\lambda))\Gamma(1-2\sigma\lambda)\Gamma(2\sigma(1+\lambda))}(\mathcal{G}(\alpha,-\lambda) - \mathcal{G}(-\alpha,-\lambda)). \qquad (2.22b)$$

This pair of equations allows for the derivation of three-term equations featuring $G(\alpha,\lambda)$. One such example is given by

$$\mathcal{G}(\alpha,\lambda) = \frac{\Gamma(1+2\sigma\lambda)\Gamma(2\sigma(1-\lambda))\Gamma(1-2\sigma(1+\lambda))}{\pi\Gamma(1-2\sigma\lambda)}$$

$$\times (\sin(2\pi\sigma)\mathcal{G}(\alpha,-\lambda) - \sin(2\pi\sigma\lambda)\mathcal{G}(-\alpha,-\lambda)). \qquad (2.23)$$

Now $\mathcal{G}(\alpha,\lambda)$ has been shown to satisfy a particular three term functional relation. From (2.23), specific parameter choices allow for identities which only feature $\mathcal{G}(\alpha,\lambda)$ twice. One example is given by $\lambda = 1 - 1/2\sigma$. When this is the case $\mathcal{G}(\alpha,\lambda)$ simplifies to a $_2F_1$ hypergeometric series, which is in-turn expressed in terms of gamma functions by Gauss'

theorem. The subsequent relation is that

$$\frac{\Gamma^2(1-2\sigma)\Gamma(2-2\sigma)}{\Gamma(2\sigma)\Gamma(1-\sigma(1+\delta))\Gamma(1-\sigma(1-\delta))\Gamma(2-4\sigma)}$$

$$= \mathcal{G}\left(2\delta, 1 - \frac{1}{2\sigma}\right) + \mathcal{G}\left(-2\delta, 1 - \frac{1}{2\sigma}\right). \tag{2.24}$$

Similar such reductions can be found by any simplification of the $_3F_2$ hypergeometric series in (2.21) being substituted into the three-term relation in (2.23).

3. $\mathcal{PT}$-Symmetric Schrödinger Problems

Now we consider a class of non-Hermitian eigenvalue problems related to the radial problems of Sec. 2, given for $K \in \mathbb{N}$ by the Schrödinger equation

$$-\phi'' + \left((-1)^K(ix)^{2M} - \alpha(ix)^{M-1} + \frac{\lambda^2 - \frac{1}{4}}{x^2}\right)\phi = E\phi. \tag{3.1}$$

The boundary conditions are that $\phi \in L^2(\mathcal{C}(x))$ where $\mathcal{C}(x)$ is a quantization contour that asymptotes towards the anti-Stokes rays with complex arguments $-\pi/2 \pm \pi(K+1)/(2M+2)$. These anti-Stokes rays are shown in Fig. 1. Such a contour must also avoid the branch cut along the positive imaginary axis, employed to ensure that the potential is single-valued for general M.

Surprisingly these Schrödinger problems, with complex potentials and boundary conditions defined in the complex plane, can have an entirely real spectrum [12, 13, 18, 28]. The suggested reason for the possibility of spectral reality lies in the $\mathcal{PT}$-symmetry of the problem, meaning that (3.1) and the boundary conditions are invariant under a reflection in the imaginary axis. Specifically the spectrum is known to be real if the $\mathcal{PT}$-symmetry is unbroken, meaning that the eigenfunctions are invariant under the combined $\mathcal{PT}$ operation [13]. The revelation of a non-Hermitian problem exhibiting real eigenvalues has since spawned a huge research effort with hundreds of papers. Recent review papers of the subject are also available [10, 19].
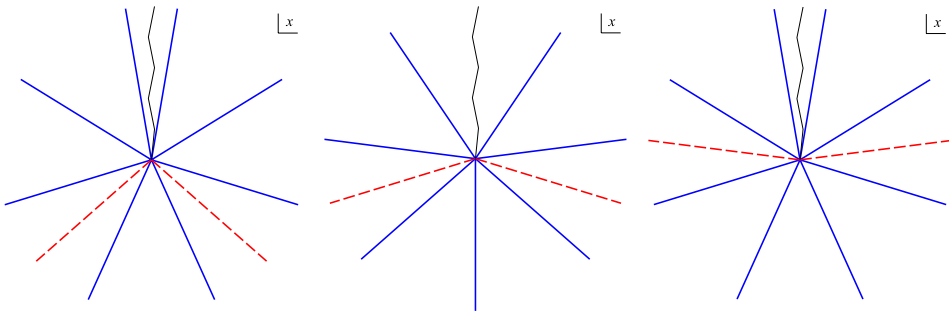

Fig. 1. (Color online) Stokes rays for $M = 2.7$ and $K = 1, 2, 3$ are given by the solid (blue) lines. The dashed (red) lines are the relevant anti-Stokes rays and we choose the quantization contour to asymptotically tend towards these, while avoiding the branch cut along the imaginary axis. The different boundary conditions thus define separate spectra.

Rather than investigating spectral reality, our interest will be in the relationships between the radial problems in Sec. 2 and the $\mathcal{PT}$-symmetric of this section. For fixed M, different choices of K will result in quantization contours which asymptotically lie within different pairs of Stokes sectors. Generally these different problems are then expected to have unique spectra for fixed M, α, λ, as was demonstrated in [13]. To these separate spectra we associate the eigenvalues $\{E_k^K\}$ and the spectral functions

$$C_K(E) \equiv C_K(0) \prod_{k=0}^{\infty} \left(1 - \frac{E}{E_k^K}\right) \quad \text{and} \quad \mathcal{Z}_K(s) \equiv \sum_{k=0}^{\infty} \frac{1}{(E_k^K)^s}. \tag{3.2}$$

Despite the different spectra appearing to have no obvious analytic relationship, in [21, 29] a number of results are given which link together different $C_K(E)$ and hence the associated eigenvalues. Following work in [21], using the normalizations in [17], one such result is the "fused" quantum Wronskian equation

$$2\lambda \omega^{\frac{\alpha}{2}(1+(-1)^K K)} C_K(-E, \alpha)$$
$$= \overline{\omega}^{(K+1)\lambda} D_-(\overline{\omega}^{K+1} E, (-i)^{K+1}\alpha) D_+(\omega^{K+1} E, i^{K+1}\alpha)$$
$$- \omega^{(K+1)\lambda} D_-(\omega^{K+1} E, i^{K+1}\alpha) D_+(\overline{\omega}^{K+1} E, (-i)^{K+1}\alpha), \tag{3.3}$$

where $M > 1$ and λ is again restricted to exclude the values in (1.7).

The fused quantum Wronskian is essentially a simple extension of (2.7), which can be verified by setting $K = 0$ and $C_0(E) \equiv 1$ in (3.3). The function $C_0(E)$ is understood to be a constant as the choice $K = 0$ dictates that the quantization contour of the eigenvalue problem asymptotically lies within two adjacent Stokes sectors, meaning an empty spectrum [21]. Therefore any sum rules derived from (3.3) must reproduce those derived from (2.7) by setting $K = 0$ and defining $Z_0(s) \equiv 0$.

In contrast to (2.7), the fused quantum Wronskian will feature only two spectral determinants on the right-hand side when $\alpha \neq 0$ and K is odd. At such values of K the sum rules must be of similar form to those in (2.10).

3.1. *Fused sum rules and simplifications*

Calculating $\mathcal{Z}_K(n)$ directly can be handled by the Green's function method used previously. This was first implemented in [15, 26] for the anharmonic ($\alpha = 0$, $\lambda = 1/2$) oscillators, where the results were used to verify conjectures on the spectral reality of the $\mathcal{PT}$-symmetric problems. However there is an easier method than direct calculation, using sum rules and the exact forms for $Z_{\mp}(n)$ calculated in (2.4) and (2.5). To determine sum rules between $\mathcal{Z}_K(n)$ and $Z_{\mp}(n)$, $C_K(-E)$ is expanded near the origin as

$$C_K(-E) = C_K(0) \exp\left(-\sum_{n=1}^{\infty} \frac{\mathcal{Z}_K(n)}{n}(-E)^n\right). \tag{3.4}$$

The fused sum rules are calculated by substituting (2.9) and (3.4) into (3.3) and comparing coefficients of the powers of E. Temporarily suppressing the dependence on K when

$\alpha \neq 0$, the first few fused sum rules are

$$\mathcal{Z}_K(1) = -L_1 Z_-(1) - L_{-1} Z_+(1), \tag{3.5a}$$

$$\mathcal{Z}_K(2) = L_2 Z_-(2) + L_{-2} Z_+(2) + (L_1^2 - L_2)\tilde{Z}(1)^2, \tag{3.5b}$$

$$\mathcal{Z}_K(3) = -L_3 Z_-(3) - L_{-3} Z_+(3) + \frac{3}{2}(L_3 - L_2 L_1)\tilde{Z}(2)\tilde{Z}(1)$$

$$+ \frac{1}{2}\left(L_3 - 3L_2 L_1 + 2L_1^3\right)\tilde{Z}(1)^3, \tag{3.5c}$$

where

$$L_a \equiv \sin(\pi\sigma(K+1)(\lambda+a))\csc(\pi\sigma(K+1)\lambda).$$

For $\alpha = 0$ the fused sum rules apply for all K, whereas for $\alpha \neq 0$ this is only true when K is odd. In the latter situation $\mathcal{Z}_K(n, \alpha)$ is given in terms of $Z_\mp(n, -\alpha)$ for $K = 4n - 3$ and in terms of $Z_\mp(n, \alpha)$ for $K = 4n - 1$.

To calculate $\mathcal{Z}_K(n)$ it is now only necessary to calculate $Z_-(n), \ldots, Z_-(1)$, determine $Z_+(n, \lambda)$ from (1.6) and then substitute these into the appropriate fused sum rule. This method implies that $\mathcal{Z}_K(n)$ is always real-valued, despite no guarantee that the eigenvalues will be real. The explanation is that all $\mathcal{PT}$-symmetric systems have a real characteristic equation, implying that the eigenvalues are either real or appear in complex conjugate pairs [11].

Given the fundamentally differences between the radial problems and the $\mathcal{PT}$-symmetric problems, there is *a priori* no reason to expect such simple connections between the differing sets of spectra. Specific examples of the fused sum rules can be written in many ways, depending on the relationships between $Z_-(n)$ and $Z_+(n)$ as determined by the radial sum rules (2.10).

Example 3.1. The anharmonic quartic oscillator has the sum rules

$$\mathcal{Z}_1(1) = 2Z_-(1),$$

$$\mathcal{Z}_1(2) = Z_-(1)^2 - Z_-(2),$$

$$2\mathcal{Z}_1(3) = 3Z_-(2)Z_-(1) + Z_-(1)^3.$$

As with the radial sum rules, the fused sum rules are also useful for deriving simplifications of $\mathcal{Z}_K(n)$. For example, comparing (2.10b) and (3.5b) shows for $m \in \mathbb{N}$ that

$$\mathcal{Z}_K\left(2, \sigma = \frac{m}{(\lambda+2)}, \alpha = 0\right)$$

$$= \frac{\pi^4 \sigma^{4-4\sigma} \Gamma^2(1-2\sigma)\csc^2(\pi\sigma)}{16^{1-\sigma}\Gamma^2(1+\sigma-m)\Gamma^2(1-3\sigma+m)\Gamma^4(1-\sigma)}$$

$$\times \left(\frac{(\csc(3\pi\sigma) + \csc(\pi\sigma))^2}{\csc^2(\pi\sigma(K+1))\sin^2(2\pi\sigma(K+1))} - \frac{\sin(4\pi\sigma(K+1))(1+\sec(2\pi\sigma))}{(1+2\cos(2\pi\sigma))^2\sin^2(\pi\sigma)\sin(2\pi\sigma(K+1))}\right). \tag{3.6}$$

Example 3.2. The anharmonic cubic oscillator has the zeta value

$$\mathcal{Z}_1(2) = \left(1 - \frac{2}{\sqrt{5}}\right) Z_+(1)^2 = \frac{16(\sqrt{5}-2)\pi^4}{5^{\frac{29}{10}}\Gamma^4(\frac{4}{5})\Gamma^2(\frac{3}{5})}.$$

Certain choices of M and K reduce the quantum Wronskian to the form $C_K(-E) \propto D_+(\pm E)D_-(\pm E)$ and hence give the zeta function identity $\mathcal{Z}_K(n) = (\mp 1)^n Z(n)$. Additionally (3.5) will allow for the form $\mathcal{Z}_K(n) = \tilde{Z}(n) + f(Z_\mp(n-1), \ldots, Z_\mp(1))$ under appropriate parameter choices. Therefore the simplifications for $Z(2)$ and $\tilde{Z}(2)$, as in (2.14) and (2.15), apply to $\mathcal{Z}_K(2)$. Two examples of these are:

Example 3.3. The quartic oscillator has the zeta value

$$\mathcal{Z}_2\left(2, \alpha = 0, \lambda = \frac{3}{2}\right) = \left(\frac{3}{2}\right)^{\frac{1}{3}} \Gamma^2\left(\frac{2}{3}\right).$$

Example 3.4. The anharmonic sextic oscillator has the zeta value

$$\mathcal{Z}_2(2) = \frac{(3 - 2\sqrt{2})\pi^5}{16\Gamma^4(\frac{3}{4})\Gamma^2(\frac{7}{8})\Gamma^2(\frac{5}{8})}.$$

3.2. *Nonlocal integrals of motion*

An application for the fused sum rules in (3.5) is within the "ODE/IM Correspondence" [8, 9, 20, 21] which links the eigenvalue problems in this paper to a class of integrable models. The correspondence, which was established by functional relations such as the quantum Wronskian, is summarized in-depth in the review paper [19].

Two essential components for realizing the correspondence are the continuum analogues of the $\mathbb{T}$- and $\mathbb{Q}$-operators which appear in the study of integrable systems such as the six-vertex model. These analogues were introduced in [4, 5] to study the integrable structure of conformal field theory. When $M > 1$ and $\alpha = 0$ the vacuum eigenvalues of these analogues, denoted $T(s)$ and $Q(s)$, are related simply to the spectral determinants of the radial and $\mathcal{PT}$-symmetric problems. This remarkable connection was shown in [8, 21] and is summarized by the relations

$$T(s) = C_1(-\nu s^2) \tag{3.7}$$

and

$$Q(s) = \frac{1}{D_-(0)} D_-(\nu s^2) \tag{3.8}$$

under the parameter identifications

$$\beta^2 = \sigma, \quad p = \frac{\sigma\lambda}{2} \quad \text{and} \quad \nu \equiv \left(\frac{\sigma}{2}\right)^{2\sigma-2} \Gamma^2(1-\sigma). \tag{3.9}$$

Considering (3.8), direct relations are given in [5] between the nonlocal integrals of motion (IMs) H_n, used in a power series expansion for $Q(s)$, and the zeta functions $Z_-(n)$.

Additionally conjectured in [5] were analytic continuations for $Z_-((2n-1)/(2\sigma-2))$ in terms of local IMs I_{2n-1} and for $Z_-(-n/\sigma)$ in terms of nonlocal IMs $\tilde{H}_n$. Therefore there appear to be good reasons to consider the spectral zeta functions in relation to the nonlocal IMs. Specifically we will work with G_n, which were given in [4] to be nonlocal IMs appearing in the power series expansion

$$T(s) = T(0) + \sum_{n=1}^{\infty} G_n s^{2n}. \tag{3.10}$$

These functions are found directly from the integral

$$G_n \equiv 2 \int_0^{2\pi} du_1 \int_0^{u_1} dv_1 \int_0^{v_1} du_2 \int_0^{u_2} dv_2 \cdots$$

$$\times \int_0^{v_{n-1}} du_n \int_0^{u_n} dv_n \cos\left(2p\left(\pi + \sum_{i=1}^{n} v_i - u_i\right)\right)$$

$$\times \prod_{j>i}^{n} \left[4 \sin\left(\frac{u_i - u_j}{2}\right) \sin\left(\frac{v_i - v_j}{2}\right)\right]^{2\beta^2}$$

$$\times \prod_{j>i}^{n} \left[2 \sin\left(\frac{v_i - u_j}{2}\right)\right]^{-2\beta^2} \prod_{j\geq i}^{n} \left[2 \sin\left(\frac{u_i - v_j}{2}\right)\right]^{-2\beta^2}, \tag{3.11}$$

with the first of these being given simply as

$$G_1 = \frac{4\pi^2 \Gamma(1 - 2\beta^2)}{\Gamma(1 - \beta^2 - 2p)\Gamma(1 - \beta^2 + 2p)}. \tag{3.12}$$

The nested integration required to calculate G_n is reminiscent of the nested integration required to compute $Z_-(n)$ as in Sec. 2. The difference is that while $Z_-(2)$ can be written exactly, no general form for G_2 is known, although alternative approaches have been used in [6, 7, 24]. To calculate G_n, (3.7) is expanded on both sides as a power series by using (3.4) and (3.10). Comparing powers of s and using $T(0) = C_1(0) = 2\cos(\pi\sigma\lambda)$ [4, 21], the first few G_n are found:

$$G_1 = 2\left(\frac{\sigma}{2}\right)^{2\sigma-2} \Gamma^2(1-\sigma)\cos(\pi\sigma\lambda)\mathcal{Z}_1(1), \tag{3.13a}$$

$$G_2 = \left(\frac{\sigma}{2}\right)^{4\sigma-4} \Gamma^4(1-\sigma)\cos(\pi\sigma\lambda)(\mathcal{Z}_1(1)^2 - \mathcal{Z}_1(2)), \tag{3.13b}$$

$$G_3 = \frac{1}{6}\left(\frac{\sigma}{2}\right)^{6\sigma-6} \Gamma^4(1-\sigma)\cos(\pi\sigma\lambda)(\mathcal{Z}_1(1)^3 + 2\mathcal{Z}_1(3) - 3\mathcal{Z}_1(2)\mathcal{Z}_1(1)). \tag{3.13c}$$

Now that G_n are expressed in terms of the zeta functions $\mathcal{Z}_1(n)$, they can be written in terms of $Z_{\mp}(n)$ by (2.10). This circumvents the need to calculate G_n directly, expressing them instead in terms of zeta functions which can be calculated from work in Sec. 2. Given that $Z_-(n)$ becomes more complicated as n increases, identities for G_n are expected to reflect this property. However the integral expression for $Z_-(n)$ in (2.2) can always be given

in terms of some infinite power series. This provides a computational advantage over (3.11), which has not been directly evaluated for $n \geq 2$.

The validity of (3.13) can be checked by first substituting (2.5a) into (3.5a). The general identity for $\mathcal{Z}_1(1)$ is then substituted into (3.13a) and, after the variable changes in (3.9), found to agree with (3.12) exactly. The process is simply extended to G_n and the higher nonlocal IMs giving, as an example, that

$$
G_2 = \frac{4\pi^4 \Gamma^2(1-2\sigma)\sec(\pi\sigma\lambda)}{\Gamma^2(1-\sigma(1-\lambda))\Gamma^2(1-\sigma(1+\lambda))} \left(1 - \frac{\cos^4(\pi\sigma)}{\sin^2(\pi\sigma(1-\lambda))\sin^2(\pi\sigma(1+\lambda))}\right)
$$
$$
+ \left(\frac{\sigma}{2}\right)^{4\sigma-4} \Gamma^4(1-\sigma)\cos(\pi\sigma\lambda)\left(\frac{\sin(2\pi\sigma(2-\lambda))}{\sin(2\pi\sigma\lambda)}Z_+(2) - \frac{\sin(2\pi\sigma(2+\lambda))}{\sin(2\pi\sigma\lambda)}Z_-(2)\right),
$$
$$(3.14)$$

where the identifications in (3.9) must be taken into account.

This expression for G_2 can be checked to agree with some special cases given in [6]. Practically (3.14) provides little extra information over the methods given in [6, 7, 24] due to the lack of known analytic properties of the $_5F_4$ hypergeometric terms appearing in (2.5b). However the simplifications for $\mathcal{Z}_K(2)$, for example (3.6), allow for concise special values of G_2 to be found, in some cases being as simple as the general formula for G_1.

Example 3.5. For $\beta^2 = 2/5$ and $p = 11/10$ the second nonlocal IM is given by

$$
G_2 = \frac{32(5+\sqrt{5})\pi^4\Gamma^2(\frac{3}{5})}{45\Gamma^4(\frac{4}{5})}.
$$

The above methods can be extended to compute G_3 and higher nonlocal IMs. Simplifications available by the sum rules would mean that, for specific parameter choices, G_3 can be given without having to calculate a general form for $Z_-(3)$ directly as in Sec. 2.

4. Summary and Future Work

We have calculated the spectral zeta functions $Z_\mp(1)$ and $Z_\mp(2)$ pertaining to the radial problems in (1.5). Using the quantum Wronskian, sum rules were established which give relationships between the different $Z_\mp(n)$. As well as illustrating the deep connections between the two problems, these sum rules allow for the simplification of $Z_\mp(2)$ with many examples found in the anharmonic oscillators. The physical reasons for such simplifications, if any, is not yet understood.

Our methods were then applied to determine special properties of the gamma function and specific hypergeometric series. We have not found another way to obtain the functional relations between these hypergeometric series and a separate derivation would be of interest. This technique could be applied in general to any Schrödinger problems where sum rules can be established between the zeta functions. Furthermore the functional relations established encodes many known, closed-form identities for specific hypergeometric series. A full study of this topic may therefore prove interesting in the future.

In Sec. 3 the zeta functions $\mathcal{Z}_K(n)$ of a related, $\mathcal{PT}$-symmetric problem were calculated not directly but using the fused sum rules relating them to $Z_\mp(n)$. This requires fewer

calculations than the direct method and serves to highlight the deep connections between the real-line and monodromy problems. After calculating $\mathcal{Z}_K(n)$, we showed how the ODE/IM correspondence can be used to calculate the vacuum nonlocal IMs G_n. This was restricted to $\alpha = 0$, although we expect our work will be useful in testing the correspondence postulated in [1, 2, 9], where this restriction is not imposed.

We deliberately excluded the possibility of a zero-energy eigenvalue. However our initial investigations show that this need not be the case. In particular we have found that spectral zeta functions can be used to detect the presence (and classify the order) of zero-energy exceptional points in non-Hermitian eigenvalue problems. A natural application of this method is to the $\mathcal{PT}$-symmetric problems specified in (3.1), where the fused sum rules are used. We hope to work more on this in the future.

Acknowledgments

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