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ON THE NONLOCAL SYMMETRIES OF THE μ -CAMASSA–HOLM EQUATION

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The μ -Camassa–Holm (μ CH) equation is a nonlinear integrable partial differential equation closely related to the Camassa–Holm and the Hunter–Saxton equations. This equation admits quadratic pseudo-potentials which allow us to compute some first-order nonlocal symmetries. The found symmetries preserve the mean of solutions. Finally, we discuss also the associated μ CH equation.

Keywords: μ CH equation; pseudo-potential; nonlocal symmetry.

Mathematics Subject Classification 2000: 35Q53, 37K05, 37K25

1. Introduction

In this paper, we study the μ CH equation, which was derived recently in [12, 16] as

$$\mu(u_t) - u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx}, \quad (1.1)$$

where $u(x, t)$ is a spatially periodic real-valued function of time variable t and space variable $x \in S^1 = [0, 1)$, $\mu(u) = \int_0^1 u dx$ denotes its mean. The μ CH equation describes the propagation of weakly nonlinear orientation waves in a massive liquid crystal with external magnetic field and self-interaction. In this form the μ CH equation appears as the geodesic equation on the diffeomorphism group of the circle corresponding to a natural right invariant Sobolev metric.

By introducing $m = \mathcal{A}u = \mu(u) - u_{xx}$, where $\mathcal{A} := \mu - \partial^2$ is the inertia operator (∂ stands for $\frac{\partial}{\partial x}$), Eq. (1.1) becomes

$$m_t = -um_x - 2mu_x, \quad m = \mu(u) - u_{xx}. \quad (1.2)$$

The μ CH equation is closely related to the Camassa–Holm (CH) equation [1, 6] with $\mathcal{A} = 1 - \partial^2$

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx} \quad (1.3)$$

and to the Hunter–Saxton (HS) equation [11] with $\mathcal{A} = -\partial^2$

$$-u_{txx} = 2u_xu_{xx} + uu_{xxx}. \quad (1.4)$$

Similar to its relatives (1.3), (1.4) the μ CH equation is a model for wave breaking, that is, it admits smooth solutions which break in finite time in such a way that the wave remains bounded while its slope becomes unbounded [12]. The μ CH equation also admits peaked solutions (peakons): for any $c \in \mathbb{R}$, the peakon $u(x, t) = c\varphi(x - ct)$, where

$$\varphi(x) = \frac{1}{26}(12x^2 + 23), \quad \text{for } x \in [-1/2, 1/2] \quad (1.5)$$

and φ is extended periodically to the real line, is a solution to (1.1). It is proven in [2] that the periodic peakons of the μ CH equation are orbitally stable in $H^1(S^1)$.

The μ CH equation is also well-posed (see [12]). This equation enjoys other geometric descriptions [7], for example, it is geometrically integrable. Moreover, its Kuperschmidt deformation is also geometrically integrable [4].

The μ CH equation is formally integrable (see Sec. 2) and bi-Hamiltonian. Let us define the Hamiltonians

$$H_1 = \frac{1}{2} \int umdx, \quad H_2 = \int \left(\mu(u)u^2 + \frac{1}{2}uu_x^2 \right) dx. \quad (1.6)$$

Then, Eq. (1.2) can be presented as

$$m_t = -\mathcal{B}^1 \frac{\delta H_2}{\delta m} = -\mathcal{B}^2 \frac{\delta H_1}{\delta m}, \quad (1.7)$$

where $\mathcal{B}^1 = \partial\mathcal{A} = -\partial^3$, $\mathcal{B}^2 = m\partial + \partial m$ are the two compatible Hamiltonian operators.

In fact, there exists an infinite sequence of conservation laws $H_n[m]$, $n = 0, \pm 1, \pm 2, \dots$, such that

$$\mathcal{B}^1 \frac{\delta H_n}{\delta m} = \mathcal{B}^2 \frac{\delta H_{n-1}}{\delta m},$$

the first few of them in the hierarchy are H_2, H_1 given above and

$$H_0 = \int m dx, \quad H_{-1} = \int \sqrt{m} dx, \quad H_{-2} = -\frac{1}{16} \int \frac{m_x^2}{m^{5/2}} dx. \quad (1.8)$$

Note that

$$H_0 = \int m dx = \int (\mu(u) - u_{xx}) dx = \mu(u). \quad (1.9)$$

Then $\mu(u_t) = 0$ on solutions to the μ CH equation — this fact can be seen also if we integrate both sides of Eq. (1.1) over the circle and use the periodicity. This implies that the mean of any solution u is a constant in time and hence is completely determined by the initial

condition [12]. This fact is crucial for the calculations in Secs. 3 and 4. Then, Eq. (1.1) can be written in the form:

$$-u_{txx} = -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx} \quad (1.10)$$

just as it is introduced in [12] under the name μ HS equation.

There is a lot of activity in extending the CH, HS and μ CH equations to multi-component ones recently. In [22] the authors consider the geometric integrability of two-component CH and HS systems. They also obtain a class of nonlocal symmetries for these systems. In [26] the author proposes a two-component generalization of the μ CH equation. Then he shows that this generalization is a bi-hamiltonian Euler equation and can be viewed as a bi-variational equation. In [13] the author studies the periodic $\mu - b$ equation which contains the μ CH equation for $b = 2$ and μ Degasperis–Procesi equation for $b = 3$, respectively. Then the author shows that the $\mu - b$ equation can be realized as a metric Euler equation on $\text{Diff}^\infty(S^1)$ if and only if $b = 2$ i.e. for the μ CH equation.

Let us mention also the paper [24] where the authors introduce the multi-component Hunter–Saxton and μ CH systems. They show that these multi-component systems are geometrically integrable. For the three-component CH and HS systems they find nonlocal symmetries depending on the pseudo-potentials.

The paper is organized as follows. Section 2 contains some facts around the notion of a scalar partial differential equation describing pseudo-spherical surface. Then the pseudo-spherical character of the μ CH equation is recalled. A quadratic pseudo-differential is presented. In Sec. 3, we first review the theory of nonlocal symmetries of partial differential equations. We follow mainly [10]. The more detailed description can be found in Krasil'shchik and Vinogradov [14, 15] (see also [19]). Another point of view can be seen in [5] where higher degree potential symmetries are introduced which lead to nonlocal conservation laws and nonlocal transformations for the equations. Then we give nonlocal symmetries for the μ CH equation. We consider only symmetries which preserve the mean of solutions, because they are found in a simple way. In the general case, one has to solve an integro-differential equation for the characteristic of the symmetry. The general approach for finding symmetries of nonlocal equations is given by Zawistowski [25]. In his approach there is no need in introducing nonlocal variables, so it is different from the approach taken in this paper. The Zawistowski's approach naturally leads to the solving of a system of integro-differential equations for the coefficient determining the generator of the symmetry.

In Sec. 4, we first discuss the existence of other symmetries, different from those found in Sec. 3. Then the associated μ CH ($A\mu$ CH) equation is introduced by analogy. The Lie algebra of nonlocal symmetries for $A\mu$ CH is presented and an one-parameter family of solutions is given.

2. The μ CH Equation and Pseudo-Spherical Surfaces

In this section we recall some definitions and facts about the equations of pseudo-spherical type. They are introduced by Chern and Tenenblat [3]. One can consult, for example [18, 20] for more details.

Definition 2.1. A scalar differential equation $\Xi(x, t, u, u_x, \dots, u_{x^n t^m}) = 0$ in two independent variables x, t , where $u_{x^n t^m} = \partial^{n+m} u / (\partial x^n \partial t^m)$, is of pseudo-spherical type (or, it

describes pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \dots, u_{x^r t^p})dx + f_{\alpha 2}(x, t, u, \dots, u_{x^s t^q})dt, \quad \alpha = 1, 2, 3, \quad (2.1)$$

whose coefficients $f_{\alpha\beta}$ are smooth functions which depend on x, t and finite number of derivatives of u , such that the 1-forms $\bar{\omega}^\alpha = \omega^\alpha(u(x, t))$ satisfy the structure equations

$$d\bar{\omega}^1 = \bar{\omega}^3 \wedge \bar{\omega}^2, \quad d\bar{\omega}^2 = \bar{\omega}^1 \wedge \bar{\omega}^3, \quad d\bar{\omega}^3 = \bar{\omega}^1 \wedge \bar{\omega}^2, \quad (2.2)$$

whenever $u = u(x, t)$ is a solution of $\Xi = 0$.

Equations (2.2) can be interpreted as follows. The graphs of local solutions of equations of pseudo-spherical type can be equipped with structure of pseudo-spherical surface (see [3, 18, 20]): if $\bar{\omega}^1 \wedge \bar{\omega}^2 \neq 0$ the tensor $\bar{\omega}^1 \otimes \bar{\omega}^1 + \bar{\omega}^2 \otimes \bar{\omega}^2$ defines a Riemannian metric of constant Gaussian curvature -1 on the graph of solution $u(x, t)$ and $\bar{\omega}^3$ is the corresponding metric connection one-form.

An equation of pseudo-spherical type is the integrability condition for a $sl(2, \mathbb{R})$ -valued problem

$$d\psi = \Omega\psi,$$

where Ω is the matrix-valued one-form

$$\Omega = Xdx + Tdt = \frac{1}{2} \begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}. \quad (2.3)$$

Definition 2.2. An equation $\Xi = 0$ is geometrically integrable if it describes a nontrivial one-parameter family of pseudo-spherical surfaces.

Here, by a nontrivial one-parameter family of pseudo-spherical surfaces we mean that it is not a constant and further, the parameter cannot be removed via transformations which preserve the Riemannian structure of the pseudo-spherical surface (see [8] for a discussion).

Hence, if $\Xi = 0$ is geometrically integrable, it is the integrability condition of one-parameter family of linear problems $\psi_x = X\psi, \psi_t = T\psi$. In fact, this is equivalent to the zero curvature equation

$$X_t - T_x + [X, T] = 0, \quad (2.4)$$

which is an essential ingredient of integrable equations.

Another important property of equations of pseudo-spherical type is that they admit quadratic pseudo-potentials. Pseudo-potentials are a generalization of conservation laws.

Proposition 2.3 [18]. *Let $\Xi = 0$ be a differential equation describing pseudo-spherical surfaces with associated one-forms ω^α . The following two Pfaffian systems are completely integrable whenever $u(x, t)$ is a solution of $\Xi = 0$:*

$$-2d\Gamma = \omega^3 + \omega^2 - 2\Gamma\omega^1 + \Gamma^2(\omega^3 - \omega^2) \quad (2.5)$$

and

$$2d\gamma = \omega^3 - \omega^2 - 2\gamma\omega^1 + \gamma^2(\omega^3 + \omega^2). \quad (2.6)$$

Moreover, the one-forms

$$\Theta = \omega^1 - \Gamma(\omega^3 - \omega^2) \quad \text{and} \quad \hat{\Theta} = -\omega^1 + \gamma(\omega^3 + \omega^2) \quad (2.7)$$

are closed whenever $u(x, t)$ is a solution of $\Xi = 0$ and Γ (respectively γ) is a solution of (2.5) (respectively (2.6)). \square

Geometrically, the Pfaffian systems (2.5) and (2.6) determine geodesic coordinates on the pseudo-spherical surfaces associated with the equation $\Xi = 0$ [3, 18].

Now consider the μ CH equation (1.2).

Proposition 2.4. *The μ CH equation (1.2) describes pseudo-spherical surfaces, and hence, is geometrically integrable.*

For validation of the Proposition 2.4 we give the associated with (1.2) 1-forms (see for example [4, 7]). Note that $\mu(u_x) = \mu(u_t) = 0$ is used since the structure equations are valid on the solutions to the μ CH equation

$$\begin{aligned} \omega^1 &= \frac{1}{2} \left(\eta m - \frac{\eta^2}{2} + 2 \right) dx + \frac{1}{2} \left[\frac{\eta^2}{2} u - \eta \left(u_x + um + \frac{1}{2} \right) + \mu(u) - 2u + \frac{2}{\eta} \right] dt, \\ \omega^2 &= \eta dx + (1 - \eta u + u_x) dt, \\ \omega^3 &= \frac{1}{2} \left(\eta m - \frac{\eta^2}{2} - 2 \right) dx + \frac{1}{2} \left[\frac{\eta^2}{2} u - \eta \left(u_x + um + \frac{1}{2} \right) + \mu(u) + 2u - \frac{2}{\eta} \right] dt. \end{aligned} \quad (2.8)$$

For the matrices X and T in (2.4) we get

$$X = \frac{1}{2} \begin{pmatrix} \eta & 2 \\ \eta m - \frac{\eta^2}{2} & -\eta \end{pmatrix}, \quad T = \frac{1}{2} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & -T_{11} \end{pmatrix}, \quad (2.9)$$

where

$$T_{11} = 1 - \eta u + u_x, \quad T_{12} = 2 \left(-u + \frac{1}{\eta} \right), \quad T_{21} = \frac{\eta^2}{2} u - \eta \left(u_x + um + \frac{1}{2} \right) + \mu(u). \quad (2.10)$$

Hence, we have a zero curvature representation $X_t - T_x + [X, T] = 0$ for the system (1.2). From (2.9) it is straightforward to obtain the corresponding scalar linear problem

$$\begin{aligned} \psi_{xx} &= \left(\frac{\eta}{2} m \right) \psi, \\ \psi_t &= \left(-u - \eta w + \frac{1}{\eta} \right) \psi_x + \frac{u_x + \eta w_x}{2} \psi, \end{aligned} \quad (2.11)$$

which coincides with those in [12] upon setting $\lambda = \eta/2$.

In order to find pseudo-potentials for the μ CH equation we denote

$$\omega_{\text{new}}^1 = \omega^2, \quad \omega_{\text{new}}^2 = -\omega^1, \quad \omega_{\text{new}}^3 = \omega^3.$$

With these forms the Pfaffian system (2.6) becomes

$$2\gamma_x = -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \quad (2.12)$$

$$2\gamma_t = -\frac{2\gamma^2}{\eta} + 2\gamma^2 u - 2\gamma(1 - \eta u + u_x) + \left[\frac{\eta^2}{2} u - \eta \left(u_x + m + \frac{1}{2} \right) + \mu(u) \right]. \quad (2.13)$$

After some manipulations the above system obtains the form:

$$\begin{aligned} 2\gamma_x &= -2\gamma^2 - 2\eta\gamma + \eta m - \frac{\eta^2}{2}, \\ 2\gamma_t &= -\frac{2}{\eta}\gamma^2 - [(2\gamma + \eta)u]_x + \mu(u) - 2\gamma - \frac{\eta}{2}. \end{aligned}$$

Applying the transform $\gamma \mapsto \gamma - \eta/2$ we get

$$\gamma_x = -\gamma^2 + \frac{\eta}{2}m, \quad (2.14)$$

$$\gamma_t = -\frac{\gamma^2}{\eta} - (\gamma u)_x + \frac{\mu(u)}{2}. \quad (2.15)$$

Multiplying the first Eq. (2.14) by $-1/\eta$ and then adding the result to the second Eq. (2.15) we get the following result denoting $\lambda = \eta/2$.

Proposition 2.5 [4]. *The μ CH equation (1.2) admits a quadratic pseudo-potential γ , defined by the equations*

$$m = \frac{\gamma^2}{\lambda} + \frac{\gamma_x}{\lambda}, \quad (2.16)$$

$$\gamma_t = -\frac{2\gamma^2}{\lambda} - (\gamma u)_x + \frac{\mu(u)}{2}, \quad (2.17)$$

where $\lambda \neq 0, m = \mu(u) - u_{xx}$. Moreover, Eq. (1.2) possesses the parameter dependent conservation law

$$\gamma_t = \frac{1}{2\lambda}(\gamma + \lambda u_x - 2\lambda u\gamma)_x. \quad (2.18)$$

As an application we use pseudo-potential γ to obtain some conserved densities for the μ CH equation. One possible expansion of γ is

$$\gamma = \lambda^{1/2}\gamma_1 + \gamma_0 + \sum_{n=1}^{\infty} \lambda^{-n/2}\gamma_{-n}. \quad (2.19)$$

Substituting this into (2.14) gives

$$\gamma_1 = \sqrt{m}, \quad \gamma_0 = -\frac{m_x}{4m}, \quad \gamma_{-1} = \frac{1}{32} \frac{m_x^2}{m^{5/2}} + \frac{1}{8} \left(\frac{m_x}{m^{3/2}} \right)_x$$

and the other γ_{-n} are obtained recurrently by

$$\gamma_{-(n+1)} = -\frac{1}{2\gamma_1} \left[(\gamma_{-n})_x + \sum_{j=0}^n \gamma_{-j} \gamma_{j-n} \right], \quad n \geq 2.$$

In this way, we can obtain local functionals H_{-1}, H_{-2} (1.8) and so forth, see also [12].

3. Nonlocal Symmetries for the μ CH Equation

Nonlocal symmetries have been studied rigorously by Krasil'schik and Vinogradov [14, 15]. Here we give a brief description of the accompanying notions and facts. Note that there is a substantial geometry which we do not even present here. We follow mainly [9, 10]. Before starting we recall some usual conventions. The independent variables are denoted by $x^i, i = 1, \dots, n$ and dependent variables by $u^\alpha, \alpha = 1, \dots, m$. Partial derivatives with respect to x^i are indicated with sub-indices. The total derivative with respect x^i is denoted by

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^m \sum_{\#J \geq 0} u_{Ji}^\alpha \frac{\partial}{\partial u_J^\alpha}, \quad (3.1)$$

where the unordered k -tuple $J = (j_1, \dots, j_k), 0 \leq j_1, j_2, \dots, j_k \leq n$ indicates a multi-index of order $\#J = k, u_{Ji}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i}$ and $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$.

Definition 3.1. Let N be a nonzero integer or $N = \infty$. An N -dimensional covering π of a (system of) partial differential equation(s) $\Xi_a = 0, a = 1, \dots, k$, is a triplet

$$(\{\gamma^b, b = 1, \dots, N\}; \{X_{ib}, b = 1, \dots, N, i = 1, \dots, n\}; \{\tilde{D}_i, i = 1, \dots, n\}) \quad (3.2)$$

of variables γ^b -“nonlocal variables”, smooth functions X_{ib} depending on x^i, u^α, γ^b and finite number of partial derivatives of u^α , and linear operators

$$\tilde{D}_i = D_i + \sum_{b=1}^N X_{ib} \frac{\partial}{\partial \gamma^b} \quad (3.3)$$

such that the equations

$$\tilde{D}_i(X_{jb}) = \tilde{D}_j(X_{ib}), \quad i, j = 1, \dots, n, \quad b = 1, \dots, N \quad (3.4)$$

hold whenever $u^\alpha(x^i)$ is a solution to $\Xi_a = 0$.

The operators \tilde{D}_i satisfy $\tilde{D}_i(\gamma^b) = X_{ib}$ and these equations are compatible due to (3.4). Since we expect that on solutions to the system $\Xi_a = 0$ the total derivatives \tilde{D}_i become usual partial derivatives, the equations

$$\frac{\partial \gamma^b}{\partial x^i} = X_{ib} \quad (3.5)$$

have to be satisfied on the solutions $u^\alpha(x^i)$ of $\Xi_a = 0$. These compatible equations give the relations between u^α and new dependent variables γ^b . Conversely, a set of equations of the

form (3.5) which are compatible on solutions to the system $\Xi_a = 0$, determines a covering $\pi = (\gamma^b, X_{ib}, \tilde{D}_i)$ where the differential operators are defined as in (3.3).

We define the nonlocal symmetries as follows

Definition 3.2. Let $\Xi_a = 0, a = 1, \dots, k$ be a system of partial differential equations, $\pi = (\gamma^b, X_{ib}, \tilde{D}_i)$ be a covering of $\Xi_a = 0$. A nonlocal π -symmetry of $\Xi_a = 0$ is a generalized symmetry

$$X = \sum_i \xi^i \frac{\partial}{\partial x^i} + \sum_\alpha \phi^\alpha \frac{\partial}{\partial u^\alpha} + \sum_b \varphi^b \frac{\partial}{\partial \gamma^b}$$

of the augmented system

$$\Xi_a = 0, \quad \frac{\partial \gamma^b}{\partial x^i} = X_{ib}. \quad (3.6)$$

Hence, in order to find nonlocal symmetries, we can proceed as in the local case considered, for example, in Olver [17]. We need to check the conditions [15, 17]

$$prX(\Xi_a) = 0, \quad \text{and} \quad prX\left(\frac{\partial \gamma^b}{\partial x^i} - X_{ib}\right) = 0 \quad (3.7)$$

in which

$$prX = X + \sum_{\alpha, J} \phi_J^\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{b, J} \varphi_J^b \frac{\partial}{\partial \gamma_J^b}$$

and

$$\phi_J^\alpha = D_J \left(\phi^\alpha - \sum_i \xi^i u_i^\alpha \right) + \sum_i \xi^i u_{Ji}^\alpha, \quad \varphi_J^b = D_J \left(\varphi^b - \sum_i \xi^i \gamma_i^b \right) + \sum_i \xi^i \gamma_{Ji}^b.$$

As elaborated in [17, Chap. 5], it is enough to consider “evolutionary” vector fields X of the form:

$$X = \sum_{\alpha=1}^m G^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{b=1}^N H^b \frac{\partial}{\partial \gamma^b}. \quad (3.8)$$

Using this form of the vector field X the symmetry conditions (3.7) can be transformed further (see [9, 10]). Note that the nonlocal symmetries send solutions to the system $\Xi_a = 0$ into the solutions of the same system.

We now examine the nonlocal symmetries of the μ CH equation (1.10) considered as a system of equations for the variables m and u , namely (1.2). We search the nonlocal symmetries that preserve the mean of solutions, that is, the integral $\mu(u) = \int u dx$ remains the same constant after the action of any symmetry on a solution.

We have already found a pseudo-potential γ in (2.16) and (2.17), given by

$$\gamma_x = \lambda m - \gamma^2, \quad \gamma_t = \left(\frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma \right)_x. \quad (3.9)$$

Let δ be the potential defined via compatible system of equations

$$\delta_x = \gamma, \quad \delta_t = \frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma. \quad (3.10)$$

Proposition 3.3. *The evolutionary vector field*

$$V = G^1 \frac{\partial}{\partial u} + G^2 \frac{\partial}{\partial m} = \gamma e^{2\delta} \frac{\partial}{\partial u} - \lambda(m_x + 4m\gamma) e^{2\delta} \frac{\partial}{\partial m}. \quad (3.11)$$

is a nonlocal symmetry for the μ CH equation (1.2).

For the proof of this proposition we need to explore only the first part in the symmetry conditions (3.7). Then a long, but straightforward computations give the result.

Following [9, 10, 18], we note that Proposition 3.3 simply says that the infinitesimal variations of u and m along the flow of the vector field V are given by

$$u_\tau = \gamma e^{2\delta}, \quad m_\tau = -\lambda(m_x + 4m\gamma) e^{2\delta}, \quad (3.12)$$

where τ is the parameter along the flow and for each solution $u(x, t), m(x, t)$ of the μ CH equation (1.2), the deformed $u(x, t) + \tau u_\tau(x, t)$ and $m(x, t) + \tau m_\tau(x, t)$ satisfy (1.2) to first-order in τ . Note that as u and m move along the flow of V , so do γ and δ . Hence, in order to find the flow of V we need to find their variations with respect to V .

Let us consider the potential β determined by the compatible system of equations

$$\beta_x = m e^{2\delta}, \quad \beta_t = \left(\frac{\gamma^2}{2\lambda^2} - um \right) e^{2\delta}. \quad (3.13)$$

The system of Eqs. (3.9), (3.10) and (3.13) allow us to define a three-dimensional covering π of the μ CH equation (1.2) with the nonlocal variables γ, δ and β .

Theorem 3.4. *The following vector fields are the first-order generalized symmetries for the augmented μ CH system (1.2), (3.9), (3.10) and (3.13), which preserve the mean of the solutions to the μ CH equation (1.2)*

$$\begin{aligned} W_1 = & -u_t \frac{\partial}{\partial u} + (m_x u + 2m u_x) \frac{\partial}{\partial m} - \left[\frac{\mu(u)}{2} + \gamma^2 \left(u - \frac{1}{2\lambda} \right) - \gamma u_x - \lambda u m \right] \frac{\partial}{\partial \gamma} \\ & - \left(\frac{u_x}{2} + \frac{\gamma}{2\lambda} - u\gamma \right) \frac{\partial}{\partial \delta} - \left(\frac{\gamma^2}{2\lambda^2} - um \right) e^{2\delta} \frac{\partial}{\partial \beta}, \end{aligned} \quad (3.14)$$

$$W_2 = u_x \frac{\partial}{\partial u} + m_x \frac{\partial}{\partial m} + (\lambda m - \gamma^2) \frac{\partial}{\partial \gamma} + \gamma \frac{\partial}{\partial \delta} + m e^{2\delta} \frac{\partial}{\partial \beta}, \quad (3.15)$$

$$W_3 = \frac{\partial}{\partial \delta} + 2\beta \frac{\partial}{\partial \beta}, \quad (3.16)$$

$$W_4 = \frac{\partial}{\partial \beta}, \quad (3.17)$$

$$\begin{aligned}
 W_5 = & \gamma e^{2\delta} \frac{\partial}{\partial u} - \lambda(m_x + 4m\gamma) e^{2\delta} \frac{\partial}{\partial m} - \lambda^2 m e^{2\delta} \frac{\partial}{\partial \gamma} \\
 & - \lambda^2 \beta \frac{\partial}{\partial \delta} - (\lambda m e^{4\delta} + \lambda^2 \beta^2) \frac{\partial}{\partial \beta}.
 \end{aligned} \tag{3.18}$$

Consequently, these vector fields are nonlocal symmetries of the μ CH equation (1.2).

Again, the proof of Theorem 3.4 is a straightforward computation (see a comment on the availability of other symmetries at the beginning of the next section).

Corollary 3.5. *The five nonlocal symmetries (3.14)–(3.18) generate a Lie algebra \mathcal{L} and their commutators are presented in Table 1.*

Remark 3.6. If we introduce the vector fields $h := -W_3$, $e := \frac{1}{\lambda}W_4$, $f := -\frac{1}{\lambda}W_5$, we find that the commutators $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, i.e. e, f, h generate a copy of $sl(2, \mathbb{R})$. Therefore, \mathcal{L} is isomorphic to the direct sum of $sl(2, \mathbb{R})$ and the Abelian Lie algebra, generated by W_1 and W_2 .

Remark 3.7. Note that W_1 and W_2 are merely the generators of the shifts with respect to the independent variables — they are $\frac{\partial}{\partial t}$ and $-\frac{\partial}{\partial x}$, respectively.

Next we study the flow of the vector field (3.18). We take it because the others are simpler — W_1 and W_2 correspond to translations with respect to t and x , respectively, and W_3 and W_4 do not involve the main variables u and m . Let again τ be a parameter along the flow, so the governing equations are

$$\frac{\partial u}{\partial \tau} = \gamma e^{2\delta}, \tag{3.19}$$

$$\frac{\partial m}{\partial \tau} = -\lambda \left(\frac{\partial m}{\partial x} + 4m\gamma \right) e^{2\delta}, \tag{3.20}$$

$$\frac{\partial \gamma}{\partial \tau} = -\lambda^2 m e^{2\delta}, \tag{3.21}$$

$$\frac{\partial \delta}{\partial \tau} = -\lambda^2 \beta, \tag{3.22}$$

$$\frac{\partial \beta}{\partial \tau} = -\lambda m e^{4\delta} - \lambda^2 \beta^2, \tag{3.23}$$

Table 1. The commutation table of μ CH nonlocal symmetry algebra.

	W_1	W_2	W_3	W_4	W_5
W_1	0	0	0	0	0
W_2	0	0	0	0	0
W_3	0	0	0	$-2W_4$	$2W_5$
W_4	0	0	$2W_4$	0	$-\lambda^2 W_3$
W_5	0	0	$-2W_5$	$\lambda^2 W_3$	0

with initial conditions

$$u(x, t, 0) = u_0, \quad m(x, t, 0) = m_0, \quad \gamma(x, t, 0) = \gamma_0, \quad \delta(x, t, 0) = \delta_0, \quad \beta(x, t, 0) = \beta_0 \quad (3.24)$$

in which $u_0(x, t), m_0(x, t), \gamma_0(x, t), \delta_0(x, t)$ and $\beta_0(x, t)$ are particular solutions to (1.2), (3.9), (3.10) and (3.13).

We can obtain from the general theorems on existence, uniqueness, and regularity of solutions to symmetric hyperbolic quasi-linear systems such as (3.19)–(3.23) (see Taylor [23, Chap. 16]) that if we start with initial data

$$u_0(x, t), \quad m_0(x, t), \quad \gamma_0(x, t), \quad \delta_0(x, t), \quad \beta_0(x, t) \quad (3.25)$$

belonging to the Sobolev space $H^k(S^1)$, with $k > 3/2$, then the system (3.19)–(3.23) with the initial conditions (3.24) has solutions $u(x, t, \tau), m(x, t, \tau), \gamma(x, t, \tau), \delta(x, t, \tau)$ and $\beta(x, t, \tau)$ on an interval $I, \tau = 0 \in I$, belonging to $L^\infty(I, H^k(S^1)) \cap \text{Lip}(I, H^{k-1}(S^1))$. The local solutions $u(x, t, \tau), m(x, t, \tau), \gamma(x, t, \tau), \delta(x, t, \tau)$ and $\beta(x, t, \tau)$ are smooth provided that the initial data are smooth. Moreover, if we start with smooth initial conditions (3.25), globally defined for $x \in S^1$, we can find (at least for small values of τ) families of solutions to the μ CH equation also globally defined for $x \in S^1$.

Our aim is to obtain explicit formulas for the functions $u(x, t, \tau), m(x, t, \tau), \gamma(x, t, \tau), \delta(x, t, \tau)$ and $\beta(x, t, \tau)$. Similarly to the case of the CH equation [9, 10, 18], we have the following proposition.

Proposition 3.8. *If the variables m and u are related by (1.2), the functions γ, δ, β are defined by the Eqs. (3.9), (3.10), (3.13) and m, γ, δ, β satisfy Eqs. (3.20)–(3.23), then u satisfies (3.19).*

Proof. We compute the derivative $\beta_{t,\tau}$ using (3.13), $\beta_{\tau,t}$ using (3.23) and simplify the obtained expressions using (3.9), (3.10), (3.13) and (3.20)–(3.23). The result is intuitively clear since the operator $\mathcal{A} = \mu - \partial^2$ is invertible [12]. \square

Therefore, we can restrict ourselves to the projection of (3.18) on the space of m, γ, δ and β .

$$W_{pr} = -\lambda(m_x + 4m\gamma)e^{2\delta} \frac{\partial}{\partial m} - \lambda^2 m e^{2\delta} \frac{\partial}{\partial \gamma} - \lambda^2 \beta \frac{\partial}{\partial \delta} - (\lambda m e^{4\delta} + \lambda^2 \beta^2) \frac{\partial}{\partial \beta} \quad (3.26)$$

or we study the Eqs. (3.20)–(3.23) with initial conditions

$$m(x, t, 0) = m_0, \quad \gamma(x, t, 0) = \gamma_0, \quad \delta(x, t, 0) = \delta_0, \quad \beta(x, t, 0) = \beta_0. \quad (3.27)$$

As in [10] we change the independent variables τ, x with the variables $\xi = \tau, \eta = \eta(\tau, x)$, subjected to the conditions

$$\eta(\tau = 0, x) = x, \quad \eta_\tau = -\lambda \eta_x e^{2\delta}. \quad (3.28)$$

Then after simplifying the resulting equations with the expressions for $\gamma_x, \delta_x, \beta_x$ from (3.9) (3.10) and (3.13), we get

$$\frac{\partial m}{\partial \tau} = -4\lambda m(\tau, \eta) \gamma(\tau, \eta) e^{2\delta(\tau, \eta)}, \quad (3.29)$$

$$\frac{\partial \gamma}{\partial \tau} = -\lambda \gamma(\tau, \eta)^2 e^{2\delta(\tau, \eta)}, \quad (3.30)$$

$$\frac{\partial \delta}{\partial \tau} = \lambda \gamma(\tau, \eta) e^{2\delta(\tau, \eta)} - \lambda^2 \beta(\tau, \eta), \quad (3.31)$$

$$\frac{\partial \beta}{\partial \tau} = -\lambda^2 \beta(\tau, \eta)^2, \quad (3.32)$$

together with (3.28), which is equivalent to

$$\frac{\partial x}{\partial \tau} = \lambda e^{2\delta(\tau, \eta)}. \quad (3.33)$$

The following Proposition provides the explicit solution of the above system (note that η appears here as a parameter).

Proposition 3.9. *The initial value problem (3.29)–(3.33), with initial conditions $m_0 = m(0, \eta)$, $\gamma_0 = \gamma(0, \eta)$, $\delta_0 = \delta(0, \eta)$, $\beta_0 = \beta(0, \eta)$ and $x_0 = x(0, \eta) = \eta$, has the solution*

$$m = \left[\frac{1 + \tau(\lambda^2 \beta_0 - \lambda \gamma_0 e^{2\delta_0})}{1 + \lambda^2 \beta_0 \tau} \right]^4 m_0, \quad (3.34)$$

$$\gamma = \gamma_0 - \lambda \gamma_0^2 e^{2\delta_0} \frac{\tau}{1 + \lambda^2 \beta_0 \tau}, \quad (3.35)$$

$$\delta = \delta_0 - \ln(1 + \tau \lambda^2 \beta_0 - \tau \lambda \gamma_0 e^{2\delta_0}), \quad (3.36)$$

$$\beta = \frac{\beta_0}{1 + \lambda^2 \beta_0 \tau}, \quad (3.37)$$

$$x = \eta + \frac{\tau \lambda e^{2\delta_0}}{1 + \tau \lambda^2 \beta_0 - \tau \lambda \gamma_0 e^{2\delta_0}}. \quad (3.38)$$

Corollary 3.10. *Let $u_0(x, t)$ be a solution of the μ CH equation. Then the solution $u(x, t, \tau)$ to the initial problem*

$$\frac{\partial u}{\partial \tau} = \gamma(x, \tau) e^{2\delta(x, \tau)}, \quad u(x, t, 0) = u_0(x, t), \quad (3.39)$$

in which $\gamma(x, t, \tau)$ and $\delta(x, t, \tau)$ are determined by (3.35), (3.36) and (3.38), is a one-parameter family of solutions to the μ CH equation (1.2).

Remark 3.11. The formulas of the above type are used in [10] and [19] for computation of explicit particular solutions to the CH equation and the Kaup–Kupershmidt equation, respectively. Let us recall that the solutions of the μ CH equation are periodic by its definition. The trivial solution and the constant solution do not give much. The next in order of complexity smooth periodic solutions are the traveling waves [12]. They are expressed via elliptic functions. This makes the calculations in applying the above formulas and Corollary 3.10 very difficult. Anyway, the derivation of the formulas (3.33)–(3.37) is not in vain.

They can be used to construct a Darboux-like transformation for the μ CH equation. The construction is completely similar to the one appearing in [9, 10], so we omit it.

4. Discussion

In this paper, we use the approach from [10] to compute some first-order nonlocal symmetries for the μ CH equation. Only symmetries that preserve the mean of solutions are considered. It is needed to point out that we do not find all nonlocal symmetries, since they depend essentially on the possibility to construct nonlocal variables and the corresponding equations. There exist other symmetries for certain. Here is an example of a symmetry which do not preserve $\mu(u)$ (kindly provided by the referee)

$$x \mapsto x, \quad t \mapsto \frac{t}{\tau}, \quad u \mapsto \tau u,$$

where τ is a parameter. Then the μ CH equation is invariant. However, this symmetry cannot be extended to a symmetry for the augmented μ CH system (1.2), (3.9), (3.10) and (3.13).

It may be interesting to see another object connected with the μ CH equation. Recall that the so called associated Camassa–Holm (ACH) equation is introduced by Schiff [21]. Let us give by analogy the associated μ -Camassa–Holm ($A\mu$ CH) equation. Define

$$p = \sqrt{m}, \quad dy = p dx - p u dt, \quad dT = dt \quad (4.1)$$

and replace in Eq. (1.2). Note that this change of variables is justified since if $m(0)$ is positive, then $m(x) > 0$ as long as $u(x, t)$ exists (see [12] for the proof). One finds

$$p_T = -p^2 u_y, \quad -p \left(\frac{p_T}{p} \right)_y + \frac{p^2}{2} = \mu(u). \quad (4.2)$$

This is the analogue of the ACH equation — the associated μ -Camassa–Holm ($A\mu$ CH) equation. It is not clear yet whether this equation is of use. Nevertheless, our aim is to study nonlocal symmetries of the $A\mu$ CH equation. First of all, we transform the equations for γ, δ and β (3.9), (3.10) and (3.13) using (4.1).

Proposition 4.1. *The $A\mu$ CH equation (4.2) admits a pseudo-potential γ and potentials δ, β determined by the compatible equations, respectively*

$$\gamma_y = \frac{\lambda p}{2} - \frac{\gamma^2}{p}, \quad \gamma_T = \frac{\mu(u)}{2} - \frac{\gamma^2}{2\lambda} - p\gamma u_y, \quad (4.3)$$

$$\delta_y = \frac{\gamma}{p}, \quad \delta_T = \frac{p u_y}{2} + \frac{\gamma}{2\lambda}, \quad (4.4)$$

$$\beta_y = \frac{p}{2} e^{2\delta}, \quad \beta_T = \frac{\gamma^2}{2\lambda^2} e^{2\delta}. \quad (4.5)$$

As in the previous section, we consider symmetry vector fields which preserve $\mu(u)$.

$$W = G^1 \frac{\partial}{\partial u} + G^2 \frac{\partial}{\partial p} + H^1 \frac{\partial}{\partial \gamma} + H^2 \frac{\partial}{\partial \delta} + H^3 \frac{\partial}{\partial \beta}, \quad (4.6)$$

where G^a, H^b are functions of the variables $y, T, u, p, \gamma, \delta, \beta$ and p_y, u_y, u_T .

Theorem 4.2. *The following vector fields are first-order generalized symmetries for the augmented $A\mu CH$ system (4.2)–(4.5)*

$$W_1 = u_T \frac{\partial}{\partial u} - p^2 u_y \frac{\partial}{\partial p} + \left(\frac{\mu(u)}{2} - \frac{\gamma^2}{2\lambda} - p\gamma u_y \right) \frac{\partial}{\partial \gamma} + \left(\frac{pu_y}{2} + \frac{\gamma}{2\lambda} \right) \frac{\partial}{\partial \delta} + \frac{\gamma^2}{2\lambda^2} e^{2\delta} \frac{\partial}{\partial \beta}, \quad (4.7)$$

$$W_2 = u_y \frac{\partial}{\partial u} + p_y \frac{\partial}{\partial p} + \left(\frac{\lambda p}{2} - \frac{\gamma^2}{p} \right) \frac{\partial}{\partial \gamma} + \frac{\gamma}{p} \frac{\partial}{\partial \delta} + \frac{p}{2} e^{2\delta} \frac{\partial}{\partial \beta}, \quad (4.8)$$

$$W_3 = \frac{1}{2} \frac{\partial}{\partial \delta} + \beta \frac{\partial}{\partial \beta}, \quad (4.9)$$

$$W_4 = \frac{\partial}{\partial \beta}, \quad (4.10)$$

$$W_5 = -(\gamma + \lambda p u_y) e^{2\delta} \frac{\partial}{\partial u} + 2\lambda p \gamma e^{2\delta} \frac{\partial}{\partial p} + \lambda \gamma^2 e^{2\delta} \frac{\partial}{\partial \gamma} + (\lambda^2 \beta - \lambda \gamma e^{2\delta}) \frac{\partial}{\partial \delta} + \lambda^2 \beta^2 \frac{\partial}{\partial \beta}. \quad (4.11)$$

Therefore, these vector fields are nonlocal symmetries for the $A\mu CH$ equation (4.2).

Corollary 4.3. *The five nonlocal symmetries (4.7)–(4.11) generate a Lie algebra \mathcal{L} and their commutators are presented in the Table 2.*

Note that \mathcal{L} is again isomorphic to a direct sum of $sl(2, \mathbb{R})$ and Abelian algebra generated by W_1 and W_2 , which are equivalent to $-\frac{\partial}{\partial T}$, $-\frac{\partial}{\partial y}$, respectively.

One can find a Darboux transform for the $A\mu CH$ equation just in a way described in [10, 21]). Instead taking this direction, we conclude the section by obtaining solutions to the $A\mu CH$ equation using the nonlocal symmetries. We consider solutions generated by the flow of the vector field (4.11). The flow of (4.11) is governed by the system of equations

$$\frac{\partial u}{\partial \tau} = -(\gamma + \lambda p u_y) e^{2\delta}, \quad (4.12)$$

$$\frac{\partial p}{\partial \tau} = 2\lambda p \gamma e^{2\delta}, \quad (4.13)$$

Table 2. The commutation table of $A\mu CH$ nonlocal symmetry algebra.

	W_1	W_2	W_3	W_4	W_5
W_1	0	0	0	0	0
W_2	0	0	0	0	0
W_3	0	0	0	$-W_4$	W_5
W_4	0	0	W_4	0	$-2\lambda^2 W_3$
W_5	0	0	$-W_5$	$2\lambda^2 W_3$	0

$$\frac{\partial \gamma}{\partial \tau} = \lambda \gamma^2 e^{2\delta}, \quad (4.14)$$

$$\frac{\partial \delta}{\partial \tau} = -\lambda \gamma e^{2\delta} + \lambda^2 \beta, \quad (4.15)$$

$$\frac{\partial \beta}{\partial \tau} = \lambda^2 \beta^2, \quad (4.16)$$

with initial conditions $u(y, T, 0) = u_0, p(y, T, 0) = p_0, \gamma(y, T, 0) = \gamma_0, \delta(y, T, 0) = \delta_0, \beta(y, T, 0) = \beta_0$. Easy calculations produce

$$\gamma(\tau) = \gamma_0 \left(1 + \frac{\tau \lambda \gamma_0 e^{2\delta_0}}{1 - \tau \lambda^2 \beta_0} \right), \quad \delta(\tau) = \delta_0 - \ln(1 - \tau \lambda^2 \beta_0 + \tau \lambda \gamma_0 e^{2\delta_0}), \quad (4.17)$$

$$p(\tau) = p_0 \left(1 + \frac{\tau \lambda \gamma_0 e^{2\delta_0}}{1 - \tau \lambda^2 \beta_0} \right)^2, \quad \beta(\tau) = \frac{\beta_0}{1 - \tau \lambda^2 \beta_0}. \quad (4.18)$$

It remains to obtain $u(\tau)$ from (4.12). Note that in contrast to the ACH equation, here it is not possible to get $u(\tau)$ directly from (4.2). To find $u(\tau)$ we need to solve the initial value problem

$$\frac{\partial u}{\partial \tau} = - \left(\gamma(y, T, \tau) + \lambda p(y, T, \tau) \frac{\partial u}{\partial y} \right) e^{2\delta(y, T, \tau)}, \quad u(y, T, 0) = u_0(y, T), \quad (4.19)$$

where $u_0(y, T)$ is a particular solution to the $A\mu$ CH equation. Solving this initial value problem, we will find one-parameter family of solutions to the $A\mu$ CH equation.

Example 4.4. As we mentioned above $\mu(u)$ is a constant on solutions. Let us suppose that $\mu(u) = \lambda > 0$. It is not difficult to be verified that the system (4.2)–(4.5) has a particular solution

$$\beta_0 = \frac{z}{\lambda^2(2-z)}, \quad \gamma_0 = \lambda \left(\frac{2+z}{2-z} \right), \quad \delta_0 = \frac{1}{2}(\sqrt{2\lambda}y + T) - \ln \frac{2+z}{2}, \quad (4.20)$$

$$p_0 = \sqrt{2\lambda} \left(\frac{2+z}{2-z} \right)^2, \quad u_0 = -\frac{4}{\lambda} \frac{z}{(2+z)^2}, \quad (4.21)$$

where $z := \lambda^2 \exp(\sqrt{2\lambda}y + T)$. Then using (4.17) and (4.18) we find

$$\beta(y, T, \tau) = \frac{z}{\lambda^2[2 - z(\tau + 1)]}, \quad (4.22)$$

$$\gamma(y, T, \tau) = \lambda \left(\frac{2 + z(\tau + 1)}{2 - z(\tau + 1)} \right), \quad (4.23)$$

$$\delta(y, T, \tau) = \frac{1}{2} \ln \frac{4z}{\lambda^2[2 + z(\tau + 1)]}, \quad (4.24)$$

$$p(y, T, \tau) = \sqrt{2\lambda} \left(\frac{2 + z(\tau + 1)}{2 - z(\tau + 1)} \right)^2. \quad (4.25)$$

Now with these $\gamma(y, T, \tau), \delta(y, T, \tau), p(y, T, \tau)$ we solve the initial value problem (4.19), where $u_0(y, T) = u_0 = -\frac{4}{\lambda} \frac{z}{(2+z)^2}$. Some computations produce

$$u(y, T, \tau) = -\frac{4}{\lambda} \frac{z(\tau + 1)}{[2 + z(\tau + 1)]^2}. \quad (4.26)$$

Therefore, the functions $p(y, T, \tau), u(y, T, \tau)$ from (4.25) and (4.26) provide one parametric family of solution to the $A\mu$ CH equation (4.2).

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