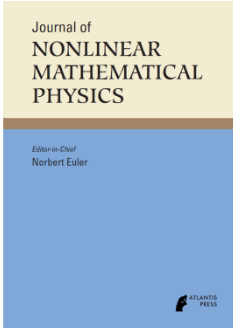

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Functional Representation of the Negative AKNS Hierarchy

V. E. Vekslerchik

To cite this article: V. E. Vekslerchik (2012) Functional Representation of the Negative AKNS Hierarchy, Journal of Nonlinear Mathematical Physics 19:3, 353–372, DOI: https://doi.org/10.1142/S1402925112500222

To link to this article: https://doi.org/10.1142/S1402925112500222

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 19, No. 3 (2012) 1250022 (20 pages)

DOI: 10.1142/S1402925112500222

FUNCTIONAL REPRESENTATION OF THE NEGATIVE AKNS HIERARCHY

V. E. VEKSLERCHIK

Usikov Institute for Radiophysics and Electronics
12, Proskura st., Kharkov 61085, Ukraine
vekslerchik@yahoo.com

Received 29 January 2012
Accepted 26 April 2012
Published 20 September 2012

This paper is devoted to the negative flows of the AKNS hierarchy. The main result of this work is the functional representation of the extended AKNS hierarchy, composed of both positive (classical) and negative flows. We derive a finite set of functional equations, constructed by means of the Miwa's shifts, which contains all equations of the hierarchy. Using the obtained functional representation we convert the nonlocal equations of the negative subhierarchy into local systems of higher order, derive the generating function of the conservation laws and the N-dark-soliton solutions for the extended AKNS hierarchy. As an additional result we obtain the functional representation of the Landau–Lifshitz hierarchy.

Keywords: AKNS hierarchy; negative flows; functional representation; Miwa's shifts; conservation laws; dark solitons; Landau–Lifshitz hierarchy.

Mathematics Subject Classification 2010: 37J35, 35Q51, 35Q55, 37K10

1. Introduction

This paper is devoted to the negative flows of the AKNS hierarchy. The most well-known physical models described by this class of equations are the sine-Gordon and self-induced transparency models (see, e.g., books [1, 9] and references therein). Mathematically, the simplest way to describe the subject of this work is to use the zero-curvature representation (ZCR), the approach which is the base of the inverse scattering transform (IST). The AKNS hierarchy is an infinite set of equations coming as compatibility conditions for the system of linear equations consisting of the Zakharov–Shabat scattering problem [18, 19],

$$\Psi_x = \mathsf{U}(\lambda)\Psi, \quad \mathsf{U}(\lambda) = i\begin{pmatrix} \lambda & R \\ Q & -\lambda \end{pmatrix} \tag{1.1}$$

and a linear problem describing the evolution,

$$\Psi_t = \mathsf{V}(\lambda)\Psi \tag{1.2}$$

where $V(\lambda)$ is a matrix polynomial in λ. The negative flows that form the negative AKNS hierarchy correspond to the case when $V(\lambda)$ is a polynomial in inverse powers of λ. Alternatively, negative (sub)hierarchies can be defined in terms of the recursion operator [5, 11, 12, 15]: if the hierarchy of higher-order symmetries (equations of the hierarchy) is generated by powers of the recursion operator, then the negative (sub)hierarchy can be constructed using its negative powers [15]. The recursion operator technique is widely exploited in the so-called "structural" approach [11] to integrable systems when one studies such questions as the Poisson structures, multi-Hamiltonian structures, mastersymmetries, R-matrices, etc.

A characteristic feature of the negative flows is their nonlocality. Among large number of known negative hierarchies there are only two, to our knowledge, examples of local ones: the Ablowitz–Ladik hierarchy, whose simplest positive and negative equations can be written as

$$i\frac{\partial q_n}{\partial t_{\pm 1}} = (1 + \kappa|q_n|^2)q_{n\pm 1}, \quad \kappa = \text{constant} \tag{1.3}$$

and the Manna–Neveu generalization [10] of the Hunter–Saxton equation [6] that was discussed in [3]. Looking for the simplest "negative" matrices $V(\lambda)$, $V(\lambda) \propto \lambda^{-1}$, one arrives at *nonlocal* equations. This nonlocality can be, in principle, eliminated by going to the equations that are not of the evolution type (the most well-known example is the sine-Gordon equation) or by introducing additional variables (as in equations that appear in the theory of the self-induced transparency). However, in any case the standard IST scheme is to be modified and probably because of this fact the negative AKNS hierarchy is not studied as comprehensively as the positive (classical) one.

The purpose of this paper, where we present some generalizations of the results obtained in [2, 7, 8], is twofold. First, we want to describe the extended AKNS hierarchy composed of both positive and negative flows. To do this we will use an approach that can be viewed as an alternative to the "traditional" ZCR or the method of [2]. The main result of this work is the functional representation of the positive and negative AKNS subhierarchies (Secs. 3–5). We derive a finite set of functional equations, constructed by means of the Miwa's shifts, which contains all equations of the extended hierarchy, that can be recovered by the power series expansion. This approach has some advantages over the standard IST because in its framework one can avoid introducing the "intermediate" objects of the inverse scattering method like Jost functions or scattering data and formulate results explicitly in terms of solutions (as, e.g., in the case of the generating function for the conservation laws presented in Sec. 3).

The second goal of this paper is to obtain some explicit solutions, the dark soliton ones, of the extended AKNS system. Here we use another advantage of the functional representation which in many situations facilitates the calculations, especially when one deals with the whole hierarchy (AKNS hierarchy in our case) instead of one of its equations (say, the nonlinear Schrödinger equation (NLSE) in our case). As is shown in Sec. 7, we can enhance the classical results of [19] using simple algebraic proceeding.

The obtained results have some interesting byproducts. In Sec. 6, we derive the functional representation of the Landau–Lifshitz hierarchy (LLH), that is known to be gauge equivalent to the AKNS hierarchy, and demonstrate that its negative part is symmetric to the positive one, which means that we have another example of local negative subhierarchy. Also we show that among the equations of the extended AKNS hierarchy one can find some models that were not associated with the AKNS system, such as, e.g., the Wadati–Konno–Ichikawa-like equations [16] and Zakharov $(1+2)$-dimensional NLSE [17].

2. Holonomy Representation of the Extended AKNS Hierarchy

Instead of the ZCR that is based on presenting the equations as the compatibility conditions for the linear systems

$$\frac{\partial}{\partial t_j}\Psi = \mathsf{U}_j\Psi, \quad \frac{\partial}{\partial \bar{t}_k}\Psi = \widetilde{\mathsf{U}}_k\Psi, \quad j,k = 1,2,\ldots \tag{2.1}$$

we will be dealing with the linear systems constructed by the Miwa's shift operators that are applied to functions of a doubly infinite number of arguments,

$$Q(\mathrm{t},\bar{\mathrm{t}}) = Q(t_1,t_2,\ldots,\bar{t}_1,\bar{t}_2,\ldots) = Q(t_j,\bar{t}_k)_{j,k=1,2,\ldots}, \tag{2.2}$$

and are defined by

$$\mathbb{E}_\xi Q(\mathrm{t},\bar{\mathrm{t}}) = Q(\mathrm{t}+i[\xi],\bar{\mathrm{t}}), \tag{2.3a}$$

$$\overline{\mathbb{E}}_\eta Q(\mathrm{t},\bar{\mathrm{t}}) = Q(\mathrm{t},\bar{\mathrm{t}}+i[\eta]) \tag{2.3b}$$

or

$$\mathbb{E}_\xi Q(t_j,\bar{t}_k)_{j,k=1,2,\ldots} = Q(t_j + i\xi^j/j,\bar{t}_k)_{j,k=1,2,\ldots}, \tag{2.4a}$$

$$\overline{\mathbb{E}}_\eta Q(t_j,\bar{t}_k)_{j,k=1,2,\ldots} = Q(t_j,\bar{t}_k + i\eta^k/k)_{j,k=1,2,\ldots}. \tag{2.4b}$$

Thus, our goal is to study the compatibility conditions of the systems of the following type:

$$\mathbb{E}_\xi\Psi = \mathsf{L}(\xi)\Psi \tag{2.5}$$

and

$$\overline{\mathbb{E}}_\eta\Psi = \bar{\mathsf{L}}(\eta)\Psi, \tag{2.6}$$

where L and $\bar{\mathsf{L}}$ are 2×2-matrices, that are given by

$$[\mathbb{E}_{\xi_1}\mathsf{L}(\xi_2)]\mathsf{L}(\xi_1) = [\mathbb{E}_{\xi_2}\mathsf{L}(\xi_1)]\mathsf{L}(\xi_2), \tag{2.7a}$$

$$[\mathbb{E}_\xi\bar{\mathsf{L}}(\eta)]\mathsf{L}(\xi) = [\overline{\mathbb{E}}_\eta\mathsf{L}(\xi)]\bar{\mathsf{L}}(\eta), \tag{2.7b}$$

$$[\overline{\mathbb{E}}_{\eta_1}\bar{\mathsf{L}}(\eta_2)]\bar{\mathsf{L}}(\eta_1) = [\overline{\mathbb{E}}_{\eta_2}\bar{\mathsf{L}}(\eta_1)]\bar{\mathsf{L}}(\eta_2). \tag{2.7c}$$

Using the standard strategy of the ZCR, that consists of introducing an auxiliary parameter, ζ, and looking for the matrices L and $\bar{\mathsf{L}}$ with simplest dependence on ζ, one can come to the matrices that lead to the AKNS hierarchy. It turns out that the matrices L and $\bar{\mathsf{L}}$ should be *linear* functions of ζ^{-1} and ζ. Indeed, expanding, for example, (2.5) in the power series in ξ one obtains an infinite set of equations of the following structure:

$$[\partial_j + \mathcal{D}(\partial_{j-1}, \ldots, \partial_1)]\Psi = \mathsf{L}_j \Psi, \quad j = 1, 2, \ldots, \tag{2.8}$$

where $\partial_j = \partial/\partial t_j$. Expressing recursively the derivatives $\partial_{j-1}, \ldots, \partial_2$ one can rewrite the last equation as

$$\partial_j \Psi = [\mathsf{L}_j - \tilde{\mathcal{D}}(\partial_1^j, \ldots, \partial_1)]\Psi, \quad j = 1, 2, \ldots. \tag{2.9}$$

If L_1 is linear in ζ^{-1}, then, after replacing the powers of ∂_1 by powers of L_1 one arrives at

$$\partial_n \Psi = \mathsf{U}_j \Psi, \quad j = 1, 2, \ldots, \tag{2.10}$$

where U_j is a jth order polynomial in ζ^{-1}. In such a way, the standard structure of the ZCR of the AKNS hierarchy is reproduced, with ζ^{-1} playing the role of the spectral parameter. Omitting the details of the calculations we present here the "minimal" solution of (2.7)

$$\mathsf{L}(\xi) = \begin{pmatrix} 1 - \xi/\zeta + \xi^2(\mathbb{E}_\xi Q)R & \xi R \\ \xi \mathbb{E}_\xi Q & 1 \end{pmatrix} \tag{2.11a}$$

and

$$\bar{\mathsf{L}}(\eta) = \mathbf{1} - \frac{\zeta\eta}{1 + (\overline{\mathbb{E}}_\eta q)r} \begin{pmatrix} 1 & r \\ \overline{\mathbb{E}}_\eta q & (\overline{\mathbb{E}}_\eta q)r \end{pmatrix}, \tag{2.11b}$$

where $\mathbf{1}$ is the unit matrix and the functions Q, R, q and r are subjected to some constraints that will be discussed below.

In the following sections we study the systems of equations (the subhierarchies of the extended AKNS hierarchy) that appear as the result of (2.7) combined with (2.11).

3. Positive AKNS Subhierarchy

The equations that follow from (2.7a) are

$$\begin{cases} (\xi_1 - \xi_2)\mathbb{E}_{\xi_1}\mathbb{E}_{\xi_2} Q = \Lambda(\xi_1, \xi_2)(\xi_1 \mathbb{E}_{\xi_1} Q - \xi_2 \mathbb{E}_{\xi_2} Q), \\ (\xi_1 - \xi_2)R = \Lambda(\xi_1, \xi_2)(\xi_1 \mathbb{E}_{\xi_2} R - \xi_2 \mathbb{E}_{\xi_1} R) \end{cases} \tag{3.1}$$

with

$$\Lambda(\xi_1, \xi_2) = 1 + \xi_1 \xi_2 (\mathbb{E}_{\xi_1}\mathbb{E}_{\xi_2} Q)R. \tag{3.2}$$

By simple algebra one can verify that (3.1) ensure vanishing of all components of the matrix equation (2.7a). The functional representation (3.1) of the positive AKNS hierarchy has been derived in [13, 14] (see also [4]). It can be simplified by various limiting procedures. For example, sending ξ_2 to zero one arrives at the system that was used in [13]:

$$\begin{cases} Q - \hat{Q} + i\xi\partial_1 \hat{Q} - \xi^2 \hat{Q}^2 R = 0, \\ \hat{R} - R - i\xi\partial_1 R - \xi^2 \hat{Q} R^2 = 0, \end{cases} \tag{3.3}$$

where the notation $\partial_j = \partial/\partial t_j$ and

$$\hat{Q} = \mathbb{E}_\xi Q \tag{3.4}$$

is used. Another reduction can be made by introducing the operator $\partial(\xi)$ by

$$\partial(\xi) = \sum_{j=1}^{\infty} \xi^j \partial_j. \tag{3.5}$$

Noting that

$$\lim_{\xi_1,\xi_2\to\xi} \frac{1}{\xi_1 - \xi_2} (\mathbb{E}_{\xi_1} \mathbb{E}_{\xi_2}^{-1} - 1) f = i\xi^{-1} \partial(\xi) f \tag{3.6}$$

one can rewrite (3.1) as

$$\begin{cases} -i\partial(\xi) Q = Q - \Omega(\xi)\hat{Q}, \\ i\partial(\xi) R = R - \Omega(\xi)\check{R}, \end{cases} \tag{3.7}$$

where

$$\Omega(\xi) = \frac{1}{1 + \xi^2 \hat{Q}\check{R}} \tag{3.8}$$

and

$$\check{R} = \mathbb{E}_\xi^{-1} R. \tag{3.9}$$

Expanding these functional equations in the power series in ξ one obtains an infinite set of differential equations, the first nontrivial of which are the NLSE,

$$\begin{cases} i\partial_2 Q + \partial_{11} Q + 2Q^2 R = 0, \\ -i\partial_2 R + \partial_{11} R + 2QR^2 = 0 \end{cases} \tag{3.10}$$

and the complex mKdV equation,

$$\begin{cases} \partial_3 Q + \partial_{111} Q + 6QR\partial_1 Q = 0, \\ \partial_3 R + \partial_{111} R + 6QR\partial_1 R = 0 \end{cases} \tag{3.11}$$

(here ∂_{jk} stand for $\partial^2/\partial t_j \partial t_k$, etc.) that are the first equations of the AKNS hierarchy.

The positive AKNS subhierarchy, to repeat, is the classical AKNS hierarchy, that has been introduced in the early 1970's and which is one of the best-studied integrable systems. That is why we do not discuss Eq. (3.1) here in details. The only thing that we need to illustrate some features of negative AKNS equations is the generating function for the conservation laws.

Constants of motion

The AKNS hierarchy, as an integrable system, possesses an infinite number of constants of motion that can be represented in the form

$$\mathcal{I}_m(t_2, t_3, \ldots) = \int_{\text{reg}} I_m(t_1, t_2, t_3, \ldots) dt_1, \quad m = 0, 1, \ldots \tag{3.12}$$

(the symbol "reg" indicates that one has to regularize, if necessary, the integrands by adding some constants depending on the boundary conditions that ensure the existence of the integrals). As has been shown in [13], the generating function for I_m,

$$I(\zeta) = \sum_{m=0}^{\infty} I_m \zeta^m \tag{3.13}$$

has a very simple form when rewritten in terms of Miwa's shifts:

$$I(\zeta) = (\mathbb{E}_\zeta Q) R. \tag{3.14}$$

Indeed, Eqs. (3.1) imply that

$$\partial(\xi) I(\zeta) = \partial_1 J(\xi, \zeta), \tag{3.15}$$

where

$$J(\xi, \zeta) = \xi \Omega(\xi) \frac{(\mathbb{E}_\xi \mathbb{E}_\zeta Q)(\mathbb{E}_\xi^{-1} R)}{1 + \xi\zeta (\mathbb{E}_\xi \mathbb{E}_\zeta Q) R} \tag{3.16}$$

which leads to $\partial_j \mathcal{I}_m = 0$ for any j and m. It should be noted that paper [13] is devoted to the NLSE and not to the whole AKNS hierarchy, so a reader can find there only the particular case of Eqs. (3.15) and (3.16). However, in order to not deviate from the main topic of this paper, we present them here "as is", leaving the proof for a separate publication.

4. Mixed AKNS Subhierarchy

The subhierarchy that is discussed in this section is closely related to the equations that usually appear in the works devoted to the negative flows of the AKNS hierarchy. The results presented below can be viewed as a generalization of the ones obtained in [7, 8]. Substituting matrices (2.11a) and (2.11b) into (2.7b) one arrives at

$$\begin{cases} \eta h(\eta) \bar{q} = \bar{Q} - Q, \\ \eta h(\eta) r = R - \bar{R} \end{cases} \tag{4.1}$$

and

$$\begin{cases} \hat{q} = (1 - \xi \hat{q} R)(q + \xi \hat{Q}), \\ r = (1 - \xi \hat{Q} r)(\hat{r} + \xi R), \end{cases} \tag{4.2}$$

where

$$h(\eta) = \frac{1}{1 + \bar{q} r} \tag{4.3}$$

with the shortcuts that are used throughout this section:

$$\hat{f} = \mathbb{E}_\xi f, \quad \bar{f} = \overline{\mathbb{E}}_\eta f. \tag{4.4}$$

Thus, we have four equations for four functions Q, R, q and r. Since we are discussing the AKNS hierarchy, our current task is to eliminate the last two and to obtain a closed system of equations for Q and R.

Equation (4.1) taken for two negative shifts, $\overline{\mathbb{E}}_{\eta_1}$ and $\overline{\mathbb{E}}_{\eta_2}$, can be rewritten in terms of the operator $\bar{\partial}(\eta)$,

$$\bar{\partial}(\eta) = \sum_{l=1}^{\infty} \eta^l \bar{\partial}_l = -i\eta \lim_{\eta_1,\eta_2 \to \eta} \frac{1}{\eta_1 - \eta_2} (\overline{\mathbb{E}}_{\eta_1} \overline{\mathbb{E}}_{\eta_2}^{-1} - 1) \tag{4.5}$$

(here $\bar{\partial}_l = \partial/\partial \bar{t}_l$) as

$$i\eta^{-1} \bar{\partial}(\eta) Q = \omega(\eta) \bar{q}, \tag{4.6a}$$

$$-i\eta^{-1} \bar{\partial}(\eta) R = \omega(\eta) \mathring{r}, \tag{4.6b}$$

where

$$\omega(\eta) = \frac{1}{1 + \bar{q}\mathring{r}} \tag{4.7}$$

and

$$\mathring{f} = \overline{\mathbb{E}}_{\eta}^{-1} f. \tag{4.8}$$

Equations (4.6a) and (4.6b) can be generalized by means of (4.1) and (4.2),

$$(1 - \xi\eta) i\eta^{-1} \bar{\partial}(\eta) \hat{Q} = \omega(\eta)(1 - \xi \hat{Q} \mathring{r})(\bar{q} + \xi \hat{Q}), \tag{4.9a}$$

$$-(1 - \xi\eta) i\eta^{-1} \bar{\partial}(\eta) \check{R} = \omega(\eta)(1 - \xi \bar{q} \check{R})(\mathring{r} + \xi \check{R}) \tag{4.9b}$$

with

$$\check{f} = \mathbb{E}_{\xi}^{-1} f. \tag{4.10}$$

Repeating this trick for two positive shifts, $\mathbb{E}_{\xi_1}$ and $\mathbb{E}_{\xi_2}$, one arrives at

$$i\xi^{-1} \partial(\xi) \frac{q}{1 + qr} = \frac{\Omega(\xi)}{1 + qr} (1 - qr - 2\xi q \check{R}) \hat{Q}, \tag{4.11a}$$

$$-i\xi^{-1} \partial(\xi) \frac{r}{1 + qr} = \frac{\Omega(\xi)}{1 + qr} (1 - qr - 2\xi \hat{Q} r) \check{R}. \tag{4.11b}$$

These equations suffice for achieving our goal of eliminating q and r. The resulting system can be written as

$$\begin{cases} 0 = (1 - \xi\eta) \partial(\xi) \bar{\partial}(\eta) Q + 2i\xi\beta(\eta) \partial(\xi) Q + i[\xi\eta - 2\xi\alpha(\xi)] \bar{\partial}(\eta) Q + 2\xi\beta(\eta) Q, \\ 0 = (1 - \xi\eta) \partial(\xi) \bar{\partial}(\eta) R - 2i\xi\beta(\eta) \partial(\xi) R - i[\xi\eta - 2\xi\alpha(\xi)] \bar{\partial}(\eta) R + 2\xi\beta(\eta) R, \end{cases} \tag{4.12}$$

where the functions α and β are defined by

$$\alpha(\xi) = \frac{\xi \hat{Q} \check{R}}{1 + \xi^2 \hat{Q} \check{R}}, \tag{4.13a}$$

$$\beta(\eta) = \frac{\eta}{2} \frac{1 - \bar{q}\mathring{r}}{1 + \bar{q}\mathring{r}} \tag{4.13b}$$

and are related by

$$\bar{\partial}(\eta) \alpha(\xi) = \partial(\xi) \beta(\eta). \tag{4.14}$$

The last identity, that can be verified directly, is very important from the viewpoint of the conservation laws of the hierarchy.

Expanding Eqs. (4.12) in the power series in ξ and η one arrives at

$$\begin{cases} 0 = (\partial_{k+1}\bar{\partial}_{l+1} - \partial_k\bar{\partial}_l + 2i\beta_{l+1}\partial_k - 2i\alpha_k\bar{\partial}_{l+1})Q, \\ 0 = (\partial_{k+1}\bar{\partial}_{l+1} - \partial_k\bar{\partial}_l - 2i\beta_{l+1}\partial_k + 2i\alpha_k\bar{\partial}_{l+1})R, \end{cases} \quad k, l = 1, 2, \dots, \tag{4.15}$$

where α_k and β_l are the coefficient of the Taylor's expansion of the functions $\alpha(\xi)$ and $\beta(\eta)$,

$$\alpha(\xi) = \sum_{k=1}^{\infty} \xi^k \alpha_k, \quad \beta(\eta) = \sum_{l=1}^{\infty} \eta^l \beta_l. \tag{4.16}$$

As in the case of positive subhierarchy, this two-parametric system can be simplified in several ways. First, Eqs. (4.12) in the $\xi \to 0$ limit can be represented in the form

$$\begin{cases} 0 = \partial_1\bar{\partial}(\eta)Q + i\eta\bar{\partial}(\eta)Q + 2\beta(\eta)Q, \\ 0 = \partial_1\bar{\partial}(\eta)R - i\eta\bar{\partial}(\eta)R + 2\beta(\eta)R, \end{cases} \tag{4.17}$$

where $\beta(\eta)$ can be viewed as an additional dependent variable related to Q and R by

$$\partial_1\beta(\eta) = \bar{\partial}(\eta)QR \tag{4.18}$$

which is the limiting form of (4.14). Here one can see the nonlocality (explicit expression for $\beta(\eta)$ invokes ∂_1^{-1} operator) that was observed in all works devoted to the negative AKNS flows. Equations (4.17) can be bilinearized by introducing the tau-functions τ, σ and ρ by

$$\beta(\eta) = \partial_1\bar{\partial}(\eta) \ln \tau \tag{4.19}$$

and

$$Q = \frac{\sigma}{\tau}, \quad R = \frac{\rho}{\tau}. \tag{4.20}$$

In new terms Eqs. (4.17) and (4.18) become

$$\begin{cases} 0 = (D_1 + i\eta)\bar{D}(\eta)\sigma \cdot \tau, \\ 0 = (D_1 - i\eta)\bar{D}(\eta)\rho \cdot \tau \end{cases} \tag{4.21}$$

and

$$D_{11}\tau \cdot \tau = 2\rho\sigma, \tag{4.22}$$

where D_1, D_{11} and $\bar{D}(\eta)$ are the Hirota's bilinear operators,

$$D_j u \cdot v = (\partial_j u)v - u(\partial_j v), \tag{4.23}$$

$D_{jk\dots} = D_j D_k \dots$ etc., and

$$\bar{D}(\eta) = \sum_{k=1}^{\infty} \eta^k \bar{D}_k \tag{4.24}$$

with

$$\bar{D}_k u \cdot v = (\bar{\partial}_k u)v - u(\bar{\partial}_k v). \tag{4.25}$$

Returning from the power series to "individual" flows one can rewrite (4.17) and (4.21) as

$$\begin{cases} 0 = (\partial_1 \bar\partial_{l+1} + i\bar\partial_l + 2\beta_{l+1})Q, \\ 0 = (\partial_1 \bar\partial_{l+1} - i\bar\partial_l + 2\beta_{l+1})R, \quad l = 1, 2, \ldots \\ \partial_1 \beta_l = \bar\partial_l QR, \end{cases} \tag{4.26}$$

and

$$\begin{cases} 0 = (D_1 \bar D_{l+1} + i\bar D_l)\sigma \cdot \tau, \\ 0 = (D_1 \bar D_{l+1} - i\bar D_l)\rho \cdot \tau, \end{cases} \quad l = 1, 2, \ldots \tag{4.27}$$

together with (4.22).

Equations (4.26) and (4.27) were discussed in [7, 8], thus Eqs. (4.17) and (4.21) can be viewed as their compact form while Eqs. (4.12) and (4.15) as their generalization.

The simplest (and hence most representative) equation of the mixed AKNS subhierarchy, discussed in this section, can be written as

$$\begin{cases} 0 = Q_{xy} + 2PQ, \\ 0 = R_{xy} + 2PR, \\ P_x = (QR)_y, \end{cases} \tag{4.28}$$

where $x = t_1$, $y = \bar t_1$ and $P = \beta_1$. The next one (Eq. (4.26) with $l = 1$),

$$\begin{cases} -iQ_t = Q_{xy} + 2PQ, \\ iR_t = R_{xy} + 2PR, \\ P_x = (QR)_y \end{cases} \tag{4.29}$$

($x = t_1$, $y = \bar t_2$, $t = \bar t_1$ and $P = \beta_2$) is nothing but the $(1+2)$-dimensional NLSE introduced by Zakharov [17].

Constants of motion

In Sec. 3, we have presented the generating function for the constants of motion, $I(\zeta) = (\mathbb{E}_\zeta Q)R$, for the positive (classical) AKNS hierarchy. It turns out that equation similar to (3.15) holds for the negative flows as well:

$$\bar\partial(\eta) I(\zeta) = \partial_1 \bar J(\eta, \zeta), \tag{4.30}$$

where

$$\bar J(\eta, \zeta) = \frac{\eta}{1 - \zeta\eta}\omega(\eta)[1 - \zeta(\mathbb{E}_\zeta Q)(\overline{\mathbb{E}}_\eta^{-1} r)]. \tag{4.31}$$

This means that the quantities I_m, as is expected, are the constants of both positive and negative AKNS subhierarchies. Again, the proof of (4.30) and (4.31) will be published elsewhere.

5. Negative AKNS Subhierarchy

Equations that follow from the commutativity condition (2.7c) can be written as

$$\begin{cases} \eta_1^{-1}(\overline{\mathbb{E}}_{\eta_1} - 1)h(\eta_2)\overline{\mathbb{E}}_{\eta_2} q = \eta_2^{-1}(\overline{\mathbb{E}}_{\eta_2} - 1)h(\eta_1)\overline{\mathbb{E}}_{\eta_1} q, \\ \eta_1^{-1}(\overline{\mathbb{E}}_{\eta_1} - 1)h(\eta_2)r = \eta_2^{-1}(\overline{\mathbb{E}}_{\eta_2} - 1)h(\eta_1)r, \end{cases} \tag{5.1}$$

where $h(\eta)$ is defined by (4.3). This is a closed system for the functions q and r which is closely related to the LLH that is discussed in Sec. 6. However, as in Sec. 4, we use the words "negative AKNS (sub)hierarchy" bearing in mind another system, the one for the functions Q and R that can be written in terms of Miwa's shifts and differential operators with respect to negative "times", $\overline{\mathbb{E}}_\eta$ and $\bar{\partial}_j = \partial/\partial \bar{t}_j$.

There are several ways to eliminate q and r together with $\mathbb{E}_\xi$ from Eqs. (2.7b) and (2.7c). The shortest one can be described as follows. Passing from $\overline{\mathbb{E}}_\eta$ to $\bar{\partial}(\eta)$ one can rewrite (5.1) as

$$\begin{cases} i\bar{\partial}(\eta)\omega_0 q = \kappa\omega(\eta)\bar{q} - \gamma(\eta)\omega_0 q, \\ -i\bar{\partial}(\eta)\omega_0 r = \kappa\omega(\eta)\mathring{r} - \gamma(\eta)\omega_0 r, \end{cases} \tag{5.2}$$

where $\omega(\eta)$ is defined in (4.7), $\bar{q} = \overline{\mathbb{E}}_\eta q$, $\mathring{r} = \overline{\mathbb{E}}_\eta^{-1} r$,

$$\gamma(\eta) = \frac{1 - \bar{q}\mathring{r}}{1 + \bar{q}\mathring{r}} = \frac{2}{\eta}\beta(\eta), \tag{5.3}$$

$\kappa = \gamma(0) = (1 - qr)/(1 + qr)$ and $\omega_0 = \omega(0) = 1/(1 + qr)$. Combining the above formulae with Eqs. (4.1) one can obtain the system

$$\begin{cases} 0 = \bar{\partial}_1\bar{\partial}(\eta)Q + i\eta^{-1}\kappa\bar{\partial}(\eta)Q - i\gamma(\eta)\bar{\partial}_1 Q, \\ 0 = \bar{\partial}_1\bar{\partial}(\eta)R - i\eta^{-1}\kappa\bar{\partial}(\eta)R + i\gamma(\eta)\bar{\partial}_1 R. \end{cases} \tag{5.4}$$

To finish the derivation of the negative AKNS equations one has to express $\gamma(\eta)$ in terms of Q and R. This can be easily achieved by means of Eqs. (4.1):

$$\gamma^2(\eta) = 1 - 4\eta^{-2}[\bar{\partial}(\eta)Q][\bar{\partial}(\eta)R]. \tag{5.5}$$

So we have a closed system of Eqs. (5.4) and (5.5) that can be viewed as the functional representation of the negative AKNS subhierarchy. Expanding (5.4) and (5.5) in the power series in η one arrives at an infinite number of partial differential equations describing the negative flows. The simplest one can be written as

$$\begin{cases} 0 = (i\bar{\partial}_2 + \kappa^2\bar{\partial}_1\kappa^{-1}\bar{\partial}_1)Q, \\ 0 = (i\bar{\partial}_2 - \kappa^2\bar{\partial}_1\kappa^{-1}\bar{\partial}_1)R \end{cases} \tag{5.6}$$

with

$$\kappa = \sqrt{1 - 4(\bar{\partial}_1 Q)(\bar{\partial}_1 R)}. \tag{5.7}$$

The second negative AKNS equation can be written as

$$\begin{cases} 0 = (2\bar{\partial}_3 - \bar{\partial}_{111} - 3i\kappa^2\bar{\partial}_2\kappa^{-1}\bar{\partial}_1)Q, \\ 0 = (2\bar{\partial}_3 - \bar{\partial}_{111} + 3i\kappa^2\bar{\partial}_2\kappa^{-1}\bar{\partial}_1)R. \end{cases} \tag{5.8}$$

One can see that the negative AKNS equations have the same nonlinearity structure as the Wadati–Konno–Ichikawa equations [16], that have been derived as a generalization of the *positive* AKNS hierarchy.

6. Landau–Lifshitz Hierarchy

In this section we discuss the "pure negative" equations, the ones stemming from (2.7c), from a standpoint different from that of Sec. 5. To this end it is convenient to introduce the matrix

$$\bar{\mathsf{S}}(\eta) = \omega(\eta)\begin{pmatrix} 1 - \bar{q}\mathring{r} & 2\mathring{r} \\ 2\bar{q} & -1 + \bar{q}\mathring{r} \end{pmatrix}, \tag{6.1}$$

where $\bar{q} = \overline{\mathbb{E}}_\eta q$ and $\mathring{r} = \overline{\mathbb{E}}_\eta^{-1} r$. In principle, one can express $\bar{\mathsf{S}}$ in terms of Q and R using (5.1),

$$\bar{\mathsf{S}}(\eta) = \begin{pmatrix} \gamma(\eta) & -2i\eta^{-1}\bar{\partial}(\eta)R \\ 2i\eta^{-1}\bar{\partial}(\eta)Q & -\gamma(\eta) \end{pmatrix}. \tag{6.2}$$

However, we will not use this relationship below, restricting ourselves to the consequences of (2.7c) that can be formulated in the terms of the matrix (6.1). By straightforward algebra one can show that Eqs. (5.1) lead to the following equations:

$$\eta_1^{-1}\bar{\partial}(\eta_1)\bar{\mathsf{S}}(\eta_2) = \eta_2^{-1}\bar{\partial}(\eta_2)\bar{\mathsf{S}}(\eta_1) \tag{6.3}$$

and

$$2i\eta\bar{\partial}_1\bar{\mathsf{S}}(\eta) = [\bar{\mathsf{S}}(\eta), \mathsf{S}], \tag{6.4}$$

where

$$\mathsf{S} = \bar{\mathsf{S}}(0). \tag{6.5}$$

Using (6.3) one can rewrite (6.4) as

$$2i\bar{\partial}(\eta)\mathsf{S} = [\bar{\mathsf{S}}(\eta), \mathsf{S}] \tag{6.6}$$

and calculate $\bar{\mathsf{S}}(\eta)$ in terms of S:

$$\bar{\mathsf{S}}(\eta) = -i\mathsf{S}\bar{\partial}(\eta)\mathsf{S} + \lambda(\eta)\mathsf{S}. \tag{6.7}$$

Here the function $\lambda(\eta)$ should be determined from the condition

$$\bar{\mathsf{S}}^2(\eta) = 1 \tag{6.8}$$

which leads to

$$\lambda^2(\eta) = -\frac{1}{2}\mathrm{tr}[\bar{\partial}(\eta)\mathsf{S}]^2. \tag{6.9}$$

Thus, we have a closed system of Eqs. (6.6), (6.7) and (6.9) for the matrix S.

Expanding these equations in the power series in η, one can obtain an infinite set of equations

$$0 = i\bar{\partial}_2 \mathsf{S} + \frac{1}{2}[\mathsf{S}'', \mathsf{S}], \tag{6.10}$$

$$0 = \bar{\partial}_3 \mathsf{S} + \mathsf{S}''' + \frac{3}{2}((\mathsf{S}')^2 \mathsf{S})', \tag{6.11}$$

$$\vdots$$

where the symbol $'$ is used to denote the derivative with respect to $\bar{t}_1$:

$$\mathsf{S}' = \bar{\partial}_1 \mathsf{S}. \tag{6.12}$$

The above equations are the simplest equations of the LLH. In other words, we have shown that Eq. (2.7c) lead to the LLH and derived the functional representation of the latter.

The fact that the LLH is closely related to the AKNS hierarchy is not new, it is known since the works of Zakharov and Takhtadzhyan [20]. However, we would like to note that the Landau–Lifshitz equation that was mentioned in [8], whose results are generalized in this section, and the Landau–Lifshitz equation that appear in [20] are not the same: the last one belongs to the positive subhierarchy, while the former describes the negative flows. This indicates that the symmetry between the positive and negative flows of the AKNS hierarchy, which is not visible in terms of Q and R, becomes apparent at the level of the LLH.

7. Dark Solitons of the Extended AKNS Hierarchy

In this section we would like to present the dark-soliton solutions of the extended (describing both positive and negative flows) AKNS hierarchy. We will not derive them from scratch but will use the classical results (for the positive subhierarchy) and extend them to cover the negative flows. The N-soliton solutions for the NLSE were obtained in the beginning of the seventies by Zakharov and Shabat who developed in [18, 19] the corresponding version of the IST. Since all equations of the AKNS hierarchy, considered from the viewpoint of the inverse scattering approach, are based on the same scattering problem, their solutions (dark solitons in our case) possess the same structure that the ones derived in [19]. Thus, to solve any equation of the hierarchy one can utilize a big part of the results of [19]. The only thing that one has to do is to establish some relations between parameters of the solutions (the so-called "dispersion laws") which are different for different equations of the hierarchy. These considerations suggest the following procedure: we look for the solutions whose structure is similar to the classical dark solitons and then find, using some simple algebraic calculations, the conditions that convert them into solutions of all equations of the hierarchy (both positive and negative).

The main building blocks for the dark-soliton solutions of the AKNS hierarchy are $N \times N$ matrices A that satisfy the "almost rank-one" condition

$$\mathsf{L}\mathsf{A} - \mathsf{A}\mathsf{L}^{-1} = |\ell\rangle\langle a|, \tag{7.1}$$

where L is a constant diagonal matrix, $|\ell\rangle$ is constant N-component column, $|\ell\rangle = (\ell_1, \ldots, \ell_N)^T$ and $\langle a|$ is N-component row depending on the coordinates describing the

AKNS flows, $\langle a(\mathrm{t},\bar{\mathrm{t}})| = (a_1(\mathrm{t},\bar{\mathrm{t}}),\ldots,a_N(\mathrm{t},\bar{\mathrm{t}}))$, and matrices H_ζ are defined by

$$\mathsf{H}_\zeta = (\zeta\mathsf{1} - \mathsf{L})(\zeta\mathsf{1} - \mathsf{L}^{-1})^{-1}, \tag{7.2}$$

where $\mathsf{1}$ is the $N \times N$ unit matrix.

The remarkable property of the above matrices, that will be repeatedly used below, is that the determinants

$$\omega(\mathsf{A}) = \det|\mathsf{1} + \mathsf{A}| \tag{7.3}$$

satisfy the Fay-like identity

$$(\xi - \eta)\omega_\zeta\omega_{\xi\eta} + (\eta - \zeta)\omega_\xi\omega_{\eta\zeta} + (\zeta - \xi)\omega_\eta\omega_{\zeta\xi} = 0, \tag{7.4}$$

where

$$\omega = \omega(\mathsf{A}), \quad \omega_\zeta = \omega(\mathsf{AH}_\zeta), \quad \omega_{\xi\eta} = \omega(\mathsf{AH}_\xi\mathsf{H}_\eta). \tag{7.5}$$

One can find an elementary proof of this identity in Appendix A. It is shown below that upon representing the Miwa's shifts as multiplication by combinations of matrices H_ζ it is possible to derive from (7.4) the identities similar to Eqs. (3.1), (4.1) and (5.1) that we want to solve.

7.1. *Solution of the equations of the positive subhierarchy*

First let us study the positive subhierarchy. The key feature is to assume that the dependence on the positive "times" $t_1, t_2, \ldots$ is governed by

$$\mathbb{E}_\xi\mathsf{A} = \mathsf{AH}_\alpha\mathsf{H}_0^{-1}, \tag{7.6}$$

where the function $\alpha = \alpha(\xi)$, $\alpha(0) = 0$, is specified below. Writing down Eq. (7.4) with $\xi = \alpha_1$, $\eta = \alpha_2$ and $\zeta = 0$ and the matrix A being replaced with AH_0^{-1} one arrives at

$$(\alpha_1 - \alpha_2)\omega(\mathsf{A})[\mathbb{E}_1\mathbb{E}_2\omega(\mathsf{B})] = \alpha_1[\mathbb{E}_1\omega(\mathsf{B})][\mathbb{E}_2\omega(\mathsf{A})] - \alpha_2[\mathbb{E}_1\omega(\mathsf{A})][\mathbb{E}_2\omega(\mathsf{B})], \tag{7.7}$$

where $\mathsf{B} = \mathsf{AH}_0$ and

$$\mathbb{E}_k = \mathbb{E}_{\xi_k}, \quad \alpha_k = \alpha(\xi_k) \quad (k = 1, 2). \tag{7.8}$$

In a similar way, Eq. (7.4) with $\xi = 1/\alpha_1$, $\eta = \alpha_2$, $\zeta = 0$ and $\mathsf{A} \to \mathsf{AH}_0^{-1}$ leads to

$$(1 - \alpha_1\alpha_2)\omega(\mathsf{A})[\mathbb{E}_1^{-1}\mathbb{E}_2\omega(\mathsf{A})] = [\mathbb{E}_1^{-1}\omega(\mathsf{A})][\mathbb{E}_2\omega(\mathsf{A})] - \alpha_1\alpha_2[\mathbb{E}_1^{-1}\omega(\mathsf{C})][\mathbb{E}_2\omega(\mathsf{B})], \tag{7.9}$$

where $\mathsf{C} = \mathsf{AH}_0^{-1}$. Rewriting (7.7) and (7.9) in terms of functions Q and R defined by

$$Q = U\frac{\omega(\mathsf{B})}{\omega(\mathsf{A})}, \quad R = V\frac{\omega(\mathsf{C})}{\omega(\mathsf{A})}, \tag{7.10}$$

where U and V are two auxiliary functions one can obtain

$$\mathbb{E}_{12}Q = \frac{(\mathbb{E}_{12}U)}{\alpha_1 - \alpha_2}\,\frac{[\mathbb{E}_1\omega(\mathsf{A})][\mathbb{E}_2\omega(\mathsf{A})]}{\omega(\mathsf{A})[\mathbb{E}_{12}\omega(\mathsf{A})]}\left[\frac{\alpha_1}{\mathbb{E}_1 U}(\mathbb{E}_1 Q) - \frac{\alpha_2}{\mathbb{E}_2 U}(\mathbb{E}_2 Q)\right] \tag{7.11}$$

and

$$\frac{[\mathbb{E}_1\omega(\mathsf{A})][\mathbb{E}_2\omega(\mathsf{A})]}{\omega(\mathsf{A})[\mathbb{E}_{12}\omega(\mathsf{A})]} = \frac{1}{1-\alpha_1\alpha_2}\left[1 - \frac{\alpha_1\alpha_2}{(\mathbb{E}_{12}U)V}(\mathbb{E}_{12}Q)R\right]. \tag{7.12}$$

It is easy to see that these equations become (3.1) if the following conditions hold:

$$\begin{cases} \dfrac{(\xi_1-\xi_2)\alpha_k}{(\alpha_1-\alpha_2)(1-\alpha_1\alpha_2)}\dfrac{\mathbb{E}_{12}U}{\mathbb{E}_k U} = \xi_k \quad (k=1,2), \\ \alpha_1\alpha_2 = -\xi_1\xi_2(\mathbb{E}_{12}U)V. \end{cases} \tag{7.13}$$

The simplest way to satisfy these equations is to take

$$\frac{\mathbb{E}_\xi U}{U} = f(\xi) \tag{7.14}$$

and

$$\alpha(\xi) = \alpha_* \xi f(\xi) \tag{7.15}$$

which reduces (7.13) to

$$\alpha_*^2 = -UV = \text{constant} \tag{7.16}$$

and

$$(\xi_1-\xi_2)f(\xi_1)f(\xi_2) = [1-\alpha_*^2\xi_1\xi_2 f(\xi_1)f(\xi_2)][\xi_1 f(\xi_1) - \xi_2 f(\xi_2)]. \tag{7.17}$$

The last equation can be transformed, in $\xi_2 \to 0$ limit, into "ordinary" one,

$$\alpha_*^2\xi^2 f^2(\xi) + (f_*\xi - 1)f(\xi) + 1 = 0 \tag{7.18}$$

with an arbitrary constant f_*. Solution of this quadratic equation that satisfies $f(0) = 1$ determines the dependence of α on ξ.

In a similar way one can show that R defined in (7.10) satisfies the second of equations (3.1). Thus, definitions (7.10) together with (7.14), (7.15) and (7.18) provide N-dark-soliton solutions for the functional equations (3.1) describing the classical AKNS hierarchy. Below one can find a more detailed version of these formulae written down for the physically relevant case $R = -\overline{Q}$.

7.2. *Solution of the equations of the negative subhierarchy*

Assuming

$$\overline{\mathbb{E}}_\eta \mathsf{A} = \mathsf{A}\mathsf{H}_{\beta_0}^{-1}\mathsf{H}_\beta, \tag{7.19}$$

where $\beta = \beta(\eta)$ and $\beta_0 = \beta(0)$ one can obtain from (7.4) with $\xi = \beta$, $\eta = \beta_0$, $\zeta = 0$ and the shift $\mathsf{A} \to \mathsf{A}\mathsf{H}_{\beta_0}^{-1}$

$$(\beta-\beta_0)\omega(\mathsf{A}^r)[\overline{\mathbb{E}}_\eta\omega(\mathsf{B}^q)] = \beta\omega(\mathsf{A})[\overline{\mathbb{E}}_\eta\,\omega(\mathsf{B})] - \beta_0\omega(\mathsf{B})[\overline{\mathbb{E}}_\eta\,\omega(\mathsf{A})]. \tag{7.20}$$

In a similar way, Eq. (7.4) with $\xi = \beta$, $\eta = 1/\beta_0$, $\zeta = 0$ and $\mathsf{A} \to \mathsf{A}\mathsf{H}_0^{-1}$ leads to

$$(1-\beta_0\beta)\omega(\mathsf{A})[\overline{\mathbb{E}}_\eta\omega(\mathsf{A})] = \omega(\mathsf{A}^r)[\overline{\mathbb{E}}_\eta\omega(\mathsf{A}^q)] - \beta_0\beta\omega(\mathsf{C}^r)[\overline{\mathbb{E}}_\eta\omega(\mathsf{B}^q)]. \tag{7.21}$$

Here the matrices A, B and C are the ones defined above while

$$\begin{aligned} &\mathsf{B}^q = \mathsf{A}\mathsf{H}_{\beta_0}, \quad \mathsf{C}^r = \mathsf{A}\mathsf{H}_{\beta_0}^{-1}, \\ &\mathsf{A}^q = \mathsf{A}\mathsf{H}_{1/\beta_0}^{-1}, \quad \mathsf{A}^r = \mathsf{A}\mathsf{H}_{1/\beta_0}. \end{aligned} \tag{7.22}$$

In terms of the functions

$$q = u\frac{\omega(\mathsf{B}^q)}{\omega(\mathsf{A}^q)}, \quad r = v\frac{\omega(\mathsf{C}^r)}{\omega(\mathsf{A}^r)} \tag{7.23}$$

these equations can be represented in the form

$$\omega(\mathsf{A})[\overline{\mathbb{E}}_\eta\omega(\mathsf{A})]\left[\frac{\beta}{\overline{\mathbb{E}}_\eta U}(\overline{\mathbb{E}}_\eta Q) - \frac{\beta_0}{U}Q\right] = \frac{\beta-\beta_0}{\overline{\mathbb{E}}_\eta u}\omega(\mathsf{A}^r)[\overline{\mathbb{E}}_\eta\omega(\mathsf{A}^q)]\overline{\mathbb{E}}_\eta q \tag{7.24}$$

and

$$(1-\beta_0\beta)\omega(\mathsf{A})[\overline{\mathbb{E}}_\eta\omega(\mathsf{A})] = \omega(\mathsf{A}^r)[\overline{\mathbb{E}}_\eta\,\omega(\mathsf{A}^q)]\left[1 - \frac{\beta_0\beta}{(\overline{\mathbb{E}}_\eta u)v}(\overline{\mathbb{E}}_\eta q)r\right] \tag{7.25}$$

which leads to (4.1) after imposing the conditions

$$\begin{cases} \beta_0\beta = -(\overline{\mathbb{E}}_\eta u)v, \\ \beta U = \beta_0(\overline{\mathbb{E}}_\eta U), \\ U(\beta-\beta_0)(1-\beta_0\beta) = \eta\beta_0\overline{\mathbb{E}}_\eta u. \end{cases} \tag{7.26}$$

These restrictions can be resolved as follows:

$$\frac{\overline{\mathbb{E}}_\eta U}{U} = \frac{\overline{\mathbb{E}}_\eta u}{u} = g(\eta) \tag{7.27}$$

and

$$\beta = \beta_0 g(\eta), \quad \beta_0^2 = -uv, \quad \frac{u}{U} = \frac{v}{V} = g_*, \tag{7.28}$$

where g_* is an arbitrary constant and $g(\eta)$ is the solution of the quadratic equation

$$[g(\eta)-1][1-\beta_0^2 g(\eta)] = g_*\eta g(\eta) \tag{7.29}$$

satisfying $g(0) = 1$.

This completes settling the problem of finding N-dark-soliton solutions of the extended AKNS hierarchy because one can show by straightforward algebra that the functions q and r presented in this section satisfy Eqs. (5.1) as well.

7.3. $R = -\overline{Q}$ case

This section is devoted to the situation that appears in the physical applications of the NLSE (and hence, of the whole AKNS hierarchy):

$$R(\mathrm{t},\bar{\mathrm{t}}) = -\overline{Q(\mathrm{t},\bar{\mathrm{t}})}, \tag{7.30}$$

where overbar stands for the complex conjugation. In this case the background solutions U and V can be represented in the form

$$U = Q_* \exp[i\varphi(\mathrm{t}, \bar{\mathrm{t}})], \tag{7.31a}$$

$$V = R_* \exp[-i\varphi(\mathrm{t}, \bar{\mathrm{t}})] \tag{7.31b}$$

with the constants Q_* and R_* being related by

$$R_* = -\overline{Q_*}. \tag{7.32}$$

Similar formulae can be written for q and r,

$$u = q_* \exp[i\varphi(\mathrm{t}, \bar{\mathrm{t}})], \tag{7.33a}$$

$$v = r_* \exp[-i\varphi(\mathrm{t}, \bar{\mathrm{t}})] \tag{7.33b}$$

with

$$r_* = -\overline{q_*}. \tag{7.34}$$

The phase φ is determined by the equations

$$\begin{cases} \exp[i(\mathbb{E}_\xi - 1)\varphi] = f(\xi), \\ \exp[i(\overline{\mathbb{E}}_\eta - 1)\varphi] = g(\eta). \end{cases} \tag{7.35}$$

Presenting φ as

$$\varphi(t, \bar{t}) = \varphi_0 + \sum_{k=1}^{\infty} (\varphi_k t_k + \tilde{\varphi}_k \bar{t}_k) \tag{7.36}$$

one can obtain for the generating function for the coefficients φ_k and $\tilde{\varphi}_k$ the following expressions:

$$\sum_{k=1}^{\infty} \varphi_k \xi^k / k = -\ln f(\xi), \tag{7.37a}$$

$$\sum_{k=1}^{\infty} \tilde{\varphi}_k \eta^k / k = -\ln g(\eta) \tag{7.37b}$$

or, after applying the $\xi d/d\xi$ and $\eta d/d\eta$ operators and using Eqs. (7.18) and (7.29) for $f(\xi)$ and $g(\eta)$,

$$\sum_{k=1}^{\infty} \varphi_k \xi^k = 1 - \frac{f(\xi)}{1 - |Q_*|^2 \xi^2 f^2(\xi)}, \tag{7.38a}$$

$$\sum_{k=1}^{\infty} \tilde{\varphi}_k \eta^k = \frac{[1 - g(\eta)][1 - |q_*|^2 g(\eta)]}{1 - |q_*|^2 g^2(\eta)} \tag{7.38b}$$

with arbitrary real φ_0. Then, it is easy to check that to ensure the necessary properties of the matrices A one has to choose

$$\mathsf{L} = \mathrm{diag}(e^{i\theta_n})_{n=1,\dots,N}. \tag{7.39}$$

In this case the matrices describing the t- and $\bar{t}$-evolution are unitary,

$$\mathsf{H}_0^{-1}\mathsf{H}_{\alpha(\xi)} = \operatorname{diag}(e^{i\phi_n(\xi)})_{n=1,\dots,N} \tag{7.40}$$

and

$$\mathsf{H}_{\beta_0}^{-1}\mathsf{H}_{\beta(\eta)} = \operatorname{diag}(e^{i[\psi_n(\eta)-\psi_n(0)]})_{n=1,\dots,N} \tag{7.41}$$

with

$$\phi_n(\xi) = 2\arg(1 - |Q_*|\xi f(\xi)e^{-i\theta_n}), \tag{7.42a}$$

$$\psi_n(\xi) = 2\arg(1 - |q_*|g(\eta)e^{-i\theta_n}). \tag{7.42b}$$

Now one can establish the dependence of the matrix A on the variables t_k and $\bar{t}_k$:

$$\mathsf{A}(\mathrm{t},\bar{\mathrm{t}}) = \mathsf{A}_*\operatorname{diag}(e^{\nu_n(\mathrm{t},\bar{\mathrm{t}})})_{n=1,\dots,N}, \tag{7.43}$$

where A_* is a constant matrix and

$$\nu_n(\mathrm{t},\bar{\mathrm{t}}) = \nu_{n0} + \sum_{k=1}^{\infty}(\nu_{nk}t_k + \tilde{\nu}_{nk}\bar{t}_k) \tag{7.44}$$

with arbitrary real ν_{n0} and

$$\sum_{k=1}^{\infty}\nu_{nk}\xi^k/k = \phi_n(\xi), \tag{7.45a}$$

$$\sum_{k=1}^{\infty}\tilde{\nu}_{nk}\eta^k/k = \psi_n(\eta) - \psi_n(0) \tag{7.45b}$$

or

$$\sum_{k=1}^{\infty}\nu_{nk}\xi^k = \xi\phi_n'(\xi), \tag{7.46a}$$

$$\sum_{k=1}^{\infty}\tilde{\nu}_{nk}\eta^k = \eta\psi_n'(\eta). \tag{7.46b}$$

Upon noting that the determinants (7.3) are invariant under transformations $\mathsf{A} \to \mathsf{M}^{-1}\mathsf{A}\mathsf{M}$ one can eliminate, without loss of generality, the constants ℓ_m ($\ell_m \to 1$) by redefining the functions $a_n(\mathrm{t},\bar{\mathrm{t}})$ ($a_n(\mathrm{t},\bar{\mathrm{t}}) \to a_n(\mathrm{t},\bar{\mathrm{t}})\ell_n$) thus arriving at the final expressions for the N-dark-soliton solutions of the extended AKNS hierarchy:

$$Q(\mathrm{t},\bar{\mathrm{t}}) = |Q_*|e^{i\varphi(\mathrm{t},\bar{\mathrm{t}})}\frac{\Delta_1(\mathrm{t},\bar{\mathrm{t}})}{\Delta_0(\mathrm{t},\bar{\mathrm{t}})}, \tag{7.47}$$

$$R(\mathrm{t},\bar{\mathrm{t}}) = -|Q_*|e^{-i\varphi(\mathrm{t},\bar{\mathrm{t}})}\frac{\Delta_{-1}(\mathrm{t},\bar{\mathrm{t}})}{\Delta_0(\mathrm{t},\bar{\mathrm{t}})}. \tag{7.48}$$

Here, the determinants Δ_ϵ ($\epsilon = 0, \pm 1$) are given by

$$\Delta_\epsilon(\mathrm{t}, \bar{\mathrm{t}}) = \det \left| \delta_{mn} + \frac{\exp(\nu_n(\mathrm{t}, \bar{\mathrm{t}}) + 2i\epsilon\theta_n)}{\sin(\frac{\theta_m + \theta_n}{2})} \right|_{m,n=1,\ldots,N} \tag{7.49}$$

while the phase of Q_* and the real constants $a_n(0, \bar{0})$ are respectively absorbed into φ_0 and ν_{n0}.

Appendix A. Proof of (7.4)

In this appendix we present some identities for the matrices defined in Sec. 7 which provide a proof of the Fay's identity (7.4). Consider the matrix A satisfying

$$\mathsf{LA} - \mathsf{AM} = |\ell\rangle\langle a| \tag{A.1}$$

with arbitrary diagonal matrices L and M together with the matrices H_ζ defined by

$$\mathsf{H}_\zeta = (\zeta - \mathsf{L})(\zeta - \mathsf{M})^{-1}, \tag{A.2}$$

where $(\zeta - \mathsf{L})$ stands for $(\zeta\mathsf{1} - \mathsf{L})$ etc. It follows from (A.1) that

$$(\zeta - \mathsf{M})(\mathsf{1} + \mathsf{H}_\zeta\mathsf{A})(\zeta - \mathsf{M})^{-1} = \mathsf{1} + \mathsf{A} - |\ell\rangle\langle b_\zeta| \tag{A.3}$$

with

$$\langle b_\zeta| = \langle a|(\zeta - \mathsf{M})^{-1} \tag{A.4}$$

which leads to

$$\det|\mathsf{1} + \mathsf{AH}_\zeta| = \det|\mathsf{1} + \mathsf{A}| \cdot \det|\mathsf{1} - |e\rangle\langle b_\zeta|| \tag{A.5}$$

with

$$|e\rangle = (\mathsf{1} + \mathsf{A})^{-1}\,|\,\ell\rangle \tag{A.6}$$

and hence to

$$\frac{\omega_\zeta}{\omega} = 1 - \langle b_\zeta\,|\,e\rangle. \tag{A.7}$$

A little bit more cumbersome calculations lead to the following "two-point" analogue of (A.3):

$$\begin{aligned}(\xi - \mathsf{M})(\eta - \mathsf{M})(\mathsf{1} + \mathsf{H}_\xi\mathsf{H}_\eta\mathsf{A})(\xi - \mathsf{M})^{-1}(\eta - \mathsf{M})^{-1} \\ = \mathsf{1} + \mathsf{A} + \frac{\xi - \mathsf{L}}{\eta - \xi}|\ell\rangle\langle b_\eta| + \frac{\eta - \mathsf{L}}{\xi - \eta}|\ell\rangle\langle b_\xi|\end{aligned} \tag{A.8}$$

and

$$\frac{\omega_{\xi\eta}}{\omega} = \det|\mathsf{1} + |u_1\rangle\langle v_1| + |u_2\rangle\langle v_2||. \tag{A.9}$$

Here, the rows $\langle v_{1,2}|$ and the columns $|u_{1,2}\rangle$ are defined by

$$\langle v_1| = \langle b_\eta|, \quad \langle v_2| = \langle b_\xi| \tag{A.10}$$

and

$$|u_1\rangle = \frac{1}{\eta - \xi}|c_\xi\rangle, \quad |u_2\rangle = \frac{1}{\xi - \eta}|c_\eta\rangle \tag{A.11}$$

with

$$|c_\zeta\rangle = (1 + \mathsf{A})^{-1}(\zeta - \mathsf{L})|\ell\rangle. \tag{A.12}$$

Rewriting the determinant in the right-hand side of (A.9) as

$$\frac{\omega_{\xi\eta}}{\omega} = \begin{vmatrix} 1 + \langle v_1|u_1\rangle & \langle v_1|u_2\rangle \\ \langle v_2|u_1\rangle & 1 + \langle v_2|u_2\rangle \end{vmatrix} \tag{A.13}$$

and calculating the scalar products,

$$\langle v_2|u_1\rangle = \frac{\varphi_\xi}{\eta - \xi}, \tag{A.14}$$

$$\langle v_1|u_2\rangle = \frac{\varphi_\eta}{\xi - \eta} \tag{A.15}$$

with

$$\varphi_\zeta = \langle b_\zeta \,|\, c_\zeta\rangle \tag{A.16}$$

and

$$1 + \langle v_1|u_1\rangle = \frac{\omega_\eta}{\omega} + \frac{\varphi_\eta}{\eta - \xi}, \tag{A.17}$$

$$1 + \langle v_2|u_2\rangle = \frac{\omega_\xi}{\omega} + \frac{\varphi_\xi}{\xi - \eta} \tag{A.18}$$

(here Eq. (A.7) was used) one arrives at

$$\frac{\omega\omega_{\xi\eta}}{\omega_\xi\omega_\eta} = 1 + \frac{\omega}{\xi - \eta}\left[\frac{\varphi_\xi}{\omega_\xi} - \frac{\varphi_\eta}{\omega_\eta}\right] \tag{A.19}$$

which leads to "separation of variables",

$$(\xi - \eta)\frac{\omega\omega_{\xi\eta}}{\omega_\xi\omega_\eta} = \Omega_\xi - \Omega_\eta \tag{A.20}$$

where

$$\Omega_\zeta = \zeta + \frac{\omega\varphi_\zeta}{\omega_\zeta}. \tag{A.21}$$

Upon adding three copies of (A.20) for (ξ, η), (η, ζ) and (ζ, ξ) one can obtain the identity (7.4) that we want to prove.

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).

[2] H. Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, The complex Sine–Gordon equation as a symmetry flow of the AKNS hierarchy, *J. Phys. A* **33** (2000) L331–L337.

[3] H. Baran, Can we always distinguish between positive and negative hierarchies? *J. Phys. A* **38** (2005) L301–L306.

[4] A. Dimakis and F. Müller-Hoissen, Functional representations of integrable hierarchies, *J. Phys. A* **39** (2006) 9169–9186.

[5] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Backlund transformations and hereditary symmetries, *Physica D* **64** (1981) 47–66.

[6] J. K. Hunter and R. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991) 1498–1521.

[7] J. Ji, J.-B. Zhang and D.-J. Zhang, Soliton solutions for a negative order AKNS equation hierarchy, *Commun. Theor. Phys.* **52** (2009) 395–397.

[8] A. M. Kamchatnov and M. V. Pavlov, On generating functions in the AKNS hierarchy, *Phys. Lett. A* **301** (2002) 269–274.

[9] A. I. Maimistov and A. M. Basharov, *Nonlinear Optical Waves*, Fundamental Theories of Physics, Vol. 104 (Kluwer Academic Publishers, Dordrecht, 1999).

[10] M. A. Manna and A. Neveu, A singular integrable equation from short capillary-gravity waves, preprint (2003), arXiv:physics/0303085.

[11] W. Oevel and O. Ragnisco, *R*-matrices and higher poisson brackets for integrable systems, *Physica A* **161** (1989) 181–220.

[12] P. J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.* **18** (1977) 1212–1215.

[13] G. M. Pritula and V. E. Vekslerchik, Conservation laws for the nonlinear Schrödinger equation in Miwa variables, *Inverse Problems* **18** (2002) 1355–1360.

[14] V. E. Vekslerchik, Functional representation of the Ablowitz–Ladik hierarchy. II, *J. Nonlinear Math. Phys.* **9** (2002) 157–180.

[15] J. M. Verosky, Negative powers of Olver recursion operators, *J. Math. Phys.* **32** (1991) 1733–1736.

[16] M. Wadati, K. Konno and Y.-H. Ichikawa, A generalization of inverse scattering method, *J. Phys. Soc. Japan* **46** (1979) 1965–1966.

[17] V. E. Zakharov, The inverse scattering method, in *Solitons*, eds. R. K. Bullough and J. P. Caudrey, Topics in Current Physics, Vol. 17 (Springer-Verlag, Berlin, Heidelberg, 1980), pp. 243–285.

[18] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Zh. Eksp. Teor. Fiz.* **61** (1971) 118–134 (in Russian); *Sov. Phys. JETP* **34** (1972) 62–69 (in English).

[19] V. E. Zakharov and A. B. Shabat, Interaction between solitons in a stable medium, *Zh. Eksp. Teor. Fiz.* **64** (1973) 1627–1639 (in Russian); *Sov. Phys. JETP* **37** (1973) 823–828 (in English).

[20] V. E. Zakharov and L. A. Takhtadzhyan, Equivalence of the nonlinear Schrödinger equation and the equation of a Heisenberg ferromagnet, *Teor. Mat. Fiz.* **38** (1979) 26–35 (in Russian); *Theor. Math. Phys.* **38** (1979) 17–23 (in English).