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GEOMETRY OF THE RECURSION OPERATORS FOR THE GMV SYSTEM

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We consider the Recursion Operator approach to the soliton equations related to a auxiliary linear system introduced recently by Gerdjikov, Mikhailov and Valchev (GMV system) and their interpretation as dual of Nijenhuis tensors on the manifold of potentials.

Keywords: Lax representation; recursion operators; Nijenhuis tensors.

2010 Mathematics Subject Classification: 35Q51, 37K05, 37K10

1. Introduction

The so-called soliton equations or completely integrable equations have been the object of intense study right from their discovery. Their most essential property is that they admit a Lax representation $[L, A] = 0$, see for example [6]. Here L, A are linear operators on ∂_x, ∂_t depending also on some functions $q_\alpha(x,t)$, $1 \leq \alpha \leq p$ ("potentials") and a spectral parameter λ. The equation $[L, A] = 0$ should be satisfied identically in λ, which in case A depends linearly on ∂_t and thereby makes the Lax equation $[L, A] = 0$ equivalent to a system of differential equations of the type

$$(q_\alpha)_t = F_\alpha(q, q_x, \dots), \quad \text{where } q = (q_\alpha)_{1\leq\alpha\leq p}. \tag{1.1}$$

Usually one fixes the linear problem $L\psi = 0$ (it is then called auxiliary linear problem) and considers all the evolution equations that can be obtained by changing the operator A. These equations are called the nonlinear evolution equations (NLEEs) associated (related) with L (or with the linear system $L\psi = 0$). There are several schemes through which the above equations could be resolved but the essential is that the Lax representation permits to pass from the original evolution defined by the Eq. (1.1) to the evolution of some spectral data related to the problem $L\psi = 0$ (it is then called auxiliary linear problem). The evolution of the spectral data is then easy to find and the solution to the problem (1.1) is obtained by recovering the potentials from the spectral data — the process is called Inverse Scattering Transform Method. This issues are not in the scope of the present paper, one can see details of the Inverse Scattering Transform Method in a lot of sources, for example in the monograph books [6, 12].

The Generalized Zakharov–Shabat (GZS) system we write below is a paradigm of auxiliary linear problem. It can be written as follows

$$L\psi = (\mathrm{i}\partial_x + q(x) - \lambda J)\psi = 0. \tag{1.2}$$

Here $q(x)$ and J belong to the fixed simple Lie algebra $\mathfrak{g}$ in some finite-dimensional irreducible representation. The element J is real and regular, that is the kernel of $\mathrm{ad}_J(\mathrm{ad}_J(X) \equiv [J, X],\ X \in \mathfrak{g})$ is the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The potential $q(x)$ belongs to the orthogonal complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$ with respect to the Killing form:

$$\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y), \quad X, Y \in \mathfrak{g} \tag{1.3}$$

and therefore $q(x) = \sum_{\alpha\in\Delta} q_\alpha E_\alpha$ where E_α are the root vectors, Δ is the root system of $\mathfrak{g}$. The scalar functions $q_\alpha(x)$ are defined on $\mathbb{R}$, are complex valued, smooth, and tend to zero as $x \to \pm\infty$. The functions q_α are called "potentials", we consider $q(x)$ as a point in infinite-dimensional manifold. Usually the case when $q_\alpha(x)$ are of Schwartz type is considered, for example, for the geometric picture this is enough. The classical Zakharov–Shabat system is obtained for $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C})$, $J = \mathrm{diag}(1, -1)$.

Remark 1.1. We shall assume that the basic properties of the semisimple Lie algebras (real and complex) are known. All definitions and normalizations we use coincide with those made in [17].

Remark 1.2. When GZS system on different algebras are involved we say that we have GZS $\mathfrak{g}$-system (or GZS system on $\mathfrak{g}$) to underline the relation with $\mathfrak{g}$, but when we work on a fixed algebra its symbol is usually omitted.

Let us mention also, that some authors refer to the problem (1.2) as Caudrey–Beals–Coifman system, see [3], when the element J is complex and only in the case when J is real they call it GZS system. The point is that the spectral theory of L is of a primary importance for the development of the inverse scattering techniques for L and for the spectral theory the cases when J is real or complex are very different. We however shall call it GZS system in all cases since from the geometric point of view there are no differences.

The GZS system (which in fact should be called the GZS in canonical gauge) has another form which is obtained from (1.2) by a gauge transformation of the type $\psi \mapsto \psi_0^{-1}\psi = \tilde{\psi}$ where ψ_0 is a fundamental solution to GZS system corresponding to $\lambda = 0$. Then if we denote $S = \psi_0^{-1} J \psi_0$ and the orbit $\mathcal{O}_J$ passing through J of the adjoint representation of the group G corresponding to $\mathfrak{g}$ by $\mathcal{O}_J$ we shall obtain that $\tilde{\psi}$ is a solution of the following linear problem:

$$\tilde{L}\tilde{\psi} = \mathrm{i}\partial_x\tilde{\psi} - \lambda S\tilde{\psi} = 0, \quad S(x) \in \mathcal{O}_J \tag{1.4}$$

(with appropriate conditions on $S(x)$ when $x \mapsto \pm\infty$) called GZS system in pole gauge. One can choose different fundamental solutions ψ_0 and one will obtain different limit values for $S(x)$ when $x \mapsto \pm\infty$ but usually for ψ_0 is taken the Jost solution that satisfies $\lim_{x\to-\infty}\psi_0 = \mathbf{1}$. The system (1.4) is called GZS system in pole gauge in contrast to the system (1.2) which is called GZS system in canonical gauge.

The theory of the NLEEs for the GZS auxiliary problem in canonical gauge is closely related to the theory of the NLEEs for $\tilde{L}\tilde{\psi} = 0$ — the GZS auxiliary problem in pole gauge. The NLEEs for both systems are in one-to-one correspondence and are called gauge-equivalent equations. This beautiful construction has been introduced for the first time in the famous work of Zakharov and Takhtadjan, [37]. In that paper there has been proved the gauge-equivalence of two famous equations — the Heisenberg Ferromagnet equation and the Nonlinear Schrödinger equation.

The Zakharov–Shabat system in canonical gauge has been object of many studies, we mention [16] where it is proved the completeness of the so-called adjoint solutions of L when L is considered in arbitrary faithful representation of the algebra $\mathfrak{g}$. Referring for the details to this work we remind that the adjoint solutions of L are functions of the type $w = mXm^{-1}$ where X is a constant element from $\mathfrak{g}$ and m is fundamental solution of $Lm = 0$. Let us denote by w^{a} and w^{d} the orthogonal projection (with respect to the Killing form) of w on $\mathfrak{h}^{\perp}$ and $\mathfrak{h}$, respectively. If one designates the orthogonal projector on $\mathfrak{h}^{\perp}$ by π_0 then of course $w^{\mathrm{a}} = \pi_0 w$ and $w^{\mathrm{d}} = (\mathbf{1} - \pi_0)w$. One of the most important facts from the theory of GZS system is that if a suitable family of adjoint solutions $(w_i(x,\lambda))$ is taken then roughly speaking for λ on the spectrum of L the functions $(w_i^{\mathrm{a}}(x,\lambda))$ form a complete set in the space of potentials. If one expands a potential over this complete family as coefficients one gets the minimal scattering data for L, that is the minimal data from which one can recover the potential. Thus passing from the potentials to the scattering data can be considered as a sort of Fourier Transform, called Generalized Fourier Transform. For this transform $(w_i^{\mathrm{a}}(x,\lambda))$ play the role the exponents play in the Fourier Transform. This interpretation was given for the first time in [1] and subsequently has been developed in a number of works, see for example the monograph books [12, 19] for comprehensive bibliography and detailed study of the $\mathrm{sl}(2,\mathbb{C})$-case. For more general situations see [3, 16].

I. One of the ways the Recursion Operators (Generating Operators, Λ-operators) for the system L could be introduced is to look for operators for which the functions $w_i^{\mathrm{a}}(x,\lambda)$ are eigenfunctions. This means that for the Generalized Fourier Transform they play the same role as the differentiation operator plays in the usual Fourier Transform method and naturally such objects should be important in the theory of soliton equations — in fact it is a theoretical tool which apart from explicit solutions can give most of the information about the NLEEs [12, 36]. Through them can be obtained:

- The hierarchies of the NLEEs solvable through L
- The conservation laws for these NLEEs
- The hierarchies of Hamiltonian structures for these NLEEs

So the study of these operators is an area of continuing research. It is not hard to get that the Recursion Operators related to L have the form

$$\Lambda_{\pm}(X(x)) = \mathrm{ad}_J^{-1}\left(\mathrm{i}\frac{\partial X}{\partial x} + \pi_0[q,X] + \mathrm{i}\,\mathrm{ad}_q \int_{\pm\infty}^{x} (\mathbf{1}-\pi_0)[q(y),X(y)]\mathrm{d}y\right), \tag{1.5}$$

where of course $\mathrm{ad}_q(X) = [q,X]$ and X is a smooth, fast decreasing function with values in $\mathfrak{h}^{\perp}$.

The constructions for the system L and its gauge equivalent $\tilde{L}$ are in complete analogy. We have instead of the fixed Cartan subalgebra $\mathfrak{h} = \ker \mathrm{ad}_J$ a "moving" Cartan subalgebra $\mathfrak{h}_S(x) = \ker \mathrm{ad}_{S(x)}$, "moving" orthogonal (with respect to the Killing form) complementary space $\mathfrak{h}_S^{\perp}(x)$ to $\mathfrak{h}_S(x)$ etc. We have the corresponding adjoint solutions $\tilde{w} = \tilde{\psi} Y \tilde{\psi}^{-1}$ where $\tilde{\psi}$ is a solution of $\tilde{L}\tilde{\psi} = 0$ and Y is a constant element in $\mathfrak{g}$. If we denote by $\tilde{w}^{\mathrm{a}}$ and $\tilde{w}^{\mathrm{d}}$ the projections of $\tilde{w}(x)$ on $\mathfrak{h}_S^{\perp}(x)$ and $\mathfrak{h}_S(x)$, respectively then the corresponding Recursion Operators are constructed using the fact that the functions $\tilde{w}^{\mathrm{a}}$ must be eigenfunctions for them.

Let us make a remark about the notation we use. Though the Cartan subalgebra $\mathfrak{h}_S(x) = \ker \mathrm{ad}_{S(x)}$, its orthogonal space $\mathfrak{h}_{S(x)}^{\perp}$ and the orthogonal projector $\pi_S(x) = \mathrm{Ad}(\psi_0^{-1}) \circ \pi_0 \circ \mathrm{Ad}(\psi_0)$ on $\mathfrak{h}_{S(x)}^{\perp}$, depend on x we shall not write this dependence explicitly unless we fear that there will be confusion. For example, we write $\mathfrak{h}_S^{\perp}$ and we understand that what is written is a field of subspaces. If $X(x)$ is a function with values in $\mathrm{sl}(3,\mathbb{C})$ but $X(x) \in \mathfrak{h}^{\perp}(x)$ for all x we shall write $X \in \mathfrak{h}^{\perp}$ and so on.

Continuing our brief explanations about the gauge-equivalent system the evolution equations associated with the system (1.4) and gauge-equivalent to the equations related with GZS system can be calculated through $\tilde{\Lambda}_{\pm}$ — the Recursion Operator for $\tilde{L}$ [12]. One can see that

$$\tilde{\Lambda}_{\pm} = \mathrm{Ad}(\psi_0^{-1}) \circ \Lambda_{\pm} \circ \mathrm{Ad}(\psi_0),$$

where Ad denotes the adjoint action of the simply connected Lie group G having algebra $\mathfrak{g}$. So for the GZS system in pole gauge all the facts related with GZS in canonical gauge can be reformulated and the only difficulty is to calculate all the quantities that are expressed through q and its derivative, through S and its derivatives. There is a clear procedure on how to achieve that goal. The procedure has been developed in detail in the Ph.D. thesis [32], outlined in [13, 15] (for the sl(2ℂ) case) and in more general cases in [14]. In the case $\mathrm{sl}(3,\mathbb{C})$ the procedure has been carried out in detail in [33] — for all these references see also [12]. The interest in the Recursion Operators for the system (1.4) is motivated from the fact that it is an auxiliary linear problem for a number of interesting equations related to continuous versions of magnetic models, in $\mathrm{sl}(2,\mathbb{C})$ case to the Heisenberg Ferromagnet model equation, in $\mathrm{sl}(3,\mathbb{C})$ to some other models, [4]. The Recursion Operators for the LandauLifshiftz equation (closely related to the Heisenberg Ferromagnet model equation) have been also studied, see [2].

However, the technical difficulty we mentioned — to express everything through the new "potential" $S(x)$ is sometimes big enough in order to make it preferable that the pole-gauge GZS is considered as auxiliary system on its own right. This is especially the case when there are reductions, see below.

II. The Recursion Operators for GZS have also beautiful geometric meaning. In fact their adjoint operators can be interpreted as Nijenhuis tensors on the manifolds of "potentials" on which the evolution defined by $[L, A] = 0$ occurs. The point is that in general the soliton equations are Hamiltonian with respect to two different but compatible Poisson structures. The property is called bi-Hamiltonian property of the NLEEs solvable through the corresponding linear problem. We recall that a Poisson structure (Poisson tensor) on a manifold $\mathcal{M}$ is a field of linear maps $m \mapsto P_m : T^*_m(\mathcal{M}) \mapsto T_m(\mathcal{M})$ such that for any two smooth functions f, g the expression $\{f, g\}(m) = \langle dg_m, P_m(df)_m\rangle$ is a Poisson bracket. (Here $\langle\, ,\, \rangle$ denotes the canonical pairing between $T_m(\mathcal{M})$ and $T^*_m(\mathcal{M})$ — the tangent and cotangent spaces at $m \in \mathcal{M}$). Compatible Poisson structures are called such Poisson structures P, Q that each their linear combination $aP + bQ$ (where a, b are constants) is also a Poisson tensor. It turns out that compatible Poisson structures give rise to Nijenhuis tensors in case one of them is invertible. Indeed, if Q is invertible, then one can define $N = P \circ Q^{-1}$ and N is a field of linear maps $m \mapsto N_m : T_m(\mathcal{M}) \mapsto T_m(\mathcal{M})$ such that the so-called Nijenhuis bracket $[N, N]$ of N is zero. Then the manifold of potentials is endowed with a very special geometric structure — Poisson–Nijenhuis structure (P–N structure) consisting of coupled Poisson tensor and Nijenhuis tensor. The properties of the P–N structure are responsible to the fact that the symmetries of the soliton equations have "hereditary" properties and that there are infinitely many Hamiltonian structures for the corresponding NLEEs. The above interpretation was found by Magri in his pioneer works [21, 22] and one can see all the details of the theory in [5] or in [12]. Here we shall assume that the theory of the P–N manifolds is known and shall not describe it in detail. Let us also mention that the theory of the Nijenhuis operators (endomorphisms associated with an invariant mixed tensors with a vanishing Nijenhuis torsion) has its independent value for the discussion of integrability issues. If we assume some spectral properties, as doubly degenerate eigenspaces this theory allows to extend to field theories the classical Liouville theorem for the complete integrability of Hamiltonian dynamics, see [8, 9, 28, 31]. Relaxing the conditions on the eigenspaces, the ideas behind Liouville integrability can be extended even to dissipative dynamics, [7]. There is a nice picture of the relation of the P–N structures on the manifold of potentials for the GZS system in canonical gauge, the manifold of potentials for the same system in pole gauge and the manifold of the corresponding Jost solutions, see [12, Chap. 15].

With this we end our quick introduction to the theory of $\mathrm{sl}(3, \mathbb{C})$ GZS Generating Operators in general position and we pass to the objectives of the present work. In some recent papers, [10, 11], there has been studied the auxiliary linear problem

$$\tilde{L}^0\psi = (\mathrm{i}\partial_x + \lambda L_1)\psi = 0, \quad L_1 = \begin{pmatrix} 0 & u & v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}. \tag{1.6}$$

In the above u, v (the potentials) are smooth complex valued functions on x belonging to the real line and by * is denoted the complex conjugation. In addition, the functions u and v satisfy the relations:

$$|u|^2 + |v|^2 = 1, \tag{1.7}$$

$$\lim_{x\to\pm\infty} u(x) = 0, \quad \lim_{x\to\pm\infty} v(x) = v_\pm = e^{\pm\eta}, \tag{1.8}$$

where $\pm\eta$ are real constants. We shall call this system GMV system.

As described in [10, 11] the GMV system arises naturally when one looks for integrable system having a Lax representation $[L, A] = 0$ with L of the form $\mathrm{i}\partial_x + \lambda L_1$ and L, A subject to Mikhailov-type reduction requirements, see [20, 26, 27]. In this particular case the Mikhailov reduction group G_0 is generated by the two elements g_1 and g_2 acting in the following way on the fundamental solutions of the system (1.6):

$$g_1(\psi)(x,\lambda) = [\psi(x,\lambda^*)^\dagger]^{-1}, \tag{1.9}$$

$$g_2(\psi)(x,\lambda) = H\psi(x,-\lambda)H, \quad H = \mathrm{diag}(-1,1,1), \tag{1.10}$$

where $\dagger$ denotes Hermitian conjugation. Since $g_1g_2 = g_2g_1$ and $g_1^2 = g_2^2 = \mathrm{id}$ we see that $G_0 = \mathbb{Z}_2 \times \mathbb{Z}_2$. It should be mentioned that $h : X \mapsto HXH = HXH^{-1}$ is involutive automorphism of $\mathrm{sl}(3,\mathbb{C})$ which commutes with the complex conjugation σ that defines the real form $\mathrm{su}(3)$ of $\mathrm{sl}(3,\mathbb{C})$, $(\sigma(X) = -X^\dagger)$. Then introducing the spaces

$$\mathfrak{g}_j = \{X : h(X) = (-1)^j X\}, \quad j = 0, 1 \tag{1.11}$$

we get the splittings

$$\mathrm{sl}(3,\mathbb{C}) = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \tag{1.12}$$

$$\mathrm{su}(3) = (\mathfrak{g}_0 \cap \mathrm{su}(3)) \oplus (\mathfrak{g}_1 \cap \mathrm{su}(3)). \tag{1.13}$$

The spaces $\mathfrak{g}_0, \mathfrak{g}_1$ as easily seen consist: $\mathfrak{g}_1$ of all off-diagonal matrices $X = (x_{ij})$ for which $x_{23} = x_{32} = 0$ and $\mathfrak{g}_0$ of all traceless matrices $X = (x_{ij})$ for which $x_{12} = x_{21} = x_{13} = x_{31} = 0$. Of course, $L_1 \in \mathfrak{g}_1$. Also, since h is automorphism, the spaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are orthogonal with respect to the Killing form.

The invariance under the reduction group G_0 means that if ψ is the common G_0-invariant fundamental solution of (1.6) and a linear problem of the type:

$$A\psi = \mathrm{i}\partial_t\psi + \left(\sum_{i=0}^{n} \lambda^k A_k\right)\psi = 0, \quad A_k \in \mathrm{sl}(3,\mathbb{C}) \tag{1.14}$$

we must have

$$A_{2k+1} \in \mathfrak{g}_1 \cap \mathrm{isu}(3), \quad A_{2k} \in \mathfrak{g}_0 \cap \mathrm{isu}(3), \quad k = 0, 1, 2, \ldots. \tag{1.15}$$

In the same way $S \in \mathfrak{g}_1 \cap \mathrm{isu}(3))$ which forces S to be of the form L_1.

The simplest nontrivial G-invariant A operator (the one that is polynomial of degree 2 in λ) has then the form

$$A\psi = (\mathrm{i}\partial_t + \lambda A_1 + \lambda^2 A_2)\psi = 0, \tag{1.16}$$

$$\mathrm{i}A_1(x) \in \mathfrak{g}_0 \cap \mathrm{su}(3), \quad \mathrm{i}A_2(x) \in \mathfrak{g}_1 \cap \mathrm{su}(3),$$

$$A_2 = \frac{1}{3}(\mathrm{tr}L_1^2)\mathbf{1} - L_1^2, \quad A_1 = -\mathrm{i}\,\mathrm{ad}_{L_1}^{-1} A_{2x} + \alpha L_1, \tag{1.17}$$

where α is a real constant. One must prove here of course that in the above expression the inverse of the operator $\mathrm{ad}_{L_1}(X) = [L_1, X]$ is well-defined but this will be discussed later.

In the papers [10, 11] the study of the spectral properties of (1.6) has been started and the generating operators for the system have been calculated using two different techniques — assuming they are operators for which the adjoint solutions of (1.6) should be eigenfunctions, and a symmetry approach based on technique developed in the work [18]. The Recursion Operators obtained raise several interesting questions.

It can be checked that the matrix L_1 has constant eigenvalues. Indeed, as pointed in [10] we have $g^{-1}L_1g = J_0$, where g is unitary matrix ($g^\dagger = g^{-1}$) of the form:

$$g = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ u^* & \sqrt{2}v & u^* \\ v^* & -\sqrt{2}u & v^* \end{pmatrix}, \quad J_0 = \mathrm{diag}(1, 0-1) \in \mathfrak{g}_0 \cap \mathrm{su}(3). \tag{1.18}$$

Also,

$$q_0 = g_x g^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u_x^* u + v_x v^* & u_x^* v - v_x u \\ 0 & v_x^* u - u_x v^* & v_x^* v + u_x u^* \end{pmatrix} \in \mathfrak{g}_0 \cap \mathrm{su}(3). \tag{1.19}$$

Then one can show that GMV system is gauge-equivalent to a GZS system on $\mathrm{sl}(3,\mathbb{C})$ and the corresponding Recursion Operators can be found as restrictions of the corresponding Recursion Operators for the general GZS system in pole gauge, [35]. For this reason we shall use the variable $S(x) = -L_1(x)$ and shall consider GMV system as GZS system in pole gauge subject to some restrictions.

As already mentioned, in [10, 11] have been considered the spectral theory aspects of the generating operators related to (1.6). The geometric aspects of the theory has not been developed yet, an issue that we shall address in this paper.

2. P–N Structure for GZS Pole Gauge Hierarchy

The $\mathbf{sl(3,\mathbb{C})}$ case:

Having fixed the element J for the GZS $\mathfrak{g}$-system in pole gauge the smooth function $S(x)$ with domain $\mathbb{R}$, see (1.4), is subject only to the restrictions that $S(x) \in \mathcal{O}_J$ and $S(x)$ tends fast enough to some constant values when $x \mapsto \pm\infty$. Let us consider first the more general case when $S(x)$ is smooth, takes values in $\mathfrak{g}$ and when $x \to \pm\infty$ tends fast enough to constant values. The functions of this type form an infinite-dimensional manifold which we shall denote by $\mathcal{M}$. Then it is reasonable to assume that the tangent space $T_S(\mathcal{M})$ at S consists of all the smooth functions $X : \mathbb{R} \mapsto \mathfrak{g}$ that tend to zero fast enough when $x \mapsto \pm\infty$. We denote that space by $\mathfrak{F}(\mathfrak{g})$. We shall also assume that the "dual space" $T_S^*(\mathcal{M})$ is equal to $\mathfrak{F}(\mathfrak{g})$ and if $\alpha \in T_S^*(\mathcal{M})$, $X \in T_S(\mathcal{M})$ then

$$\alpha(X) = \langle\langle \alpha, X \rangle\rangle \equiv \int_{-\infty}^{+\infty} \langle \alpha(x), X(x)\rangle \mathrm{d}x, \tag{2.1}$$

where $\langle\, ,\, \rangle$ is the Killing form of $\mathfrak{g}$.

Remark 2.1. In other words, we identify $T_S^*(\mathcal{M})$ and $T_S(\mathcal{M})$ using the bilinear form $\langle\langle\, ,\, \rangle\rangle$. We do not want to make the definitions more precise, since we speak rather about a geometric picture than about precise results. Such results can be obtained only after profound study of the spectral theory of L and $\tilde{L}$. In particular, we put dual space in quotation marks because it is clearly not equal to the dual of $\mathfrak{F}(\mathfrak{g})$. We mention also that the term "allowed" functional H means that $\frac{\delta H}{\delta S} \in T^*_S(\mathcal{M}) \sim T_S(\mathcal{M})$.

First we note that the operators

$$\alpha \mapsto P(X) = \mathrm{i}\partial_x \alpha, \tag{2.2}$$

$$\alpha \mapsto Q(\alpha) = \mathrm{ad}_S(\alpha), \quad S \in \mathcal{M}, \quad \alpha \in T^*_S(\mathcal{M}), \tag{2.3}$$

can be interpreted as Poisson tensors on the manifold $\mathcal{M}$. These facts are well-known, see for example [12], where they have been discussed in detail and the relevant references are given. Of course, it can be verified also directly, checking that if H_1, H_2 are two functions (allowed functionals) on the manifold of potentials $\mathcal{M}$ then

$$\{H_1, H_2\}_P = \left\langle\!\left\langle \frac{\delta H_1}{\delta S}, \partial_x \frac{\delta H_2}{\delta S} \right\rangle\!\right\rangle, \tag{2.4}$$

$$\{H_1, H_2\}_Q = \left\langle\!\left\langle \frac{\delta H_1}{\delta S}, \left[S, \frac{\delta H_2}{\delta S}\right] \right\rangle\!\right\rangle, \tag{2.5}$$

are Poisson brackets. It is also a fact from the general theory that these Poisson tensors are compatible, see [12, Chap. 15]. In other words $P + Q$ is also a Poisson tensor. Let us also mention that the tensor Q is the canonical Kirillov tensor which acquires the above form because the algebra is simple and coadjoint and adjoint representations are equivalent.

Now let $\mathcal{O}_J$ be the orbit of the coadjoint representation of G passing through J. Let us consider the set of smooth functions $f : \mathbb{R} \mapsto \mathcal{O}_J$ such that when $x \to \pm\infty$ they tend fast enough to constant values. The set of this functions will be denoted by $\mathcal{N}$ and clearly can be considered as submanifold of $\mathcal{M}$. If $S \in \mathcal{N}$ the tangent space $T_S(\mathcal{N})$ consists of all smooth functions X, that vanish fast enough when $x \mapsto \pm\infty$ and such that $X(x) \in T_{S(x)}(\mathcal{O}_J)$ (Recall that $\mathcal{O}_J$ is a smooth manifold in a classical sense.) We again assume that $T^*_S(\mathcal{N}) \sim T_S(\mathcal{N})$ and that these spaces are identified via $\langle\langle , \rangle\rangle$.

The Poisson tensors P and Q can be restricted from $\mathcal{M}$ to $\mathcal{N}$. The question about the conditions under which a restriction of a Poisson tensor on submanifold can be done has been solved in [25], see also [29, 30]. We shall use a simplified version of the results obtained in these papers which has been introduced in [23, 24] and we shall call it First Restriction Theorem:

Theorem 2.2. *Let $\mathcal{M}$ be Poisson manifold with Poisson tensor P and $\bar{\mathcal{M}} \subset \mathcal{M}$ be a submanifold. Let us denote by* j *the inclusion map of $\bar{\mathcal{M}}$ into $\mathcal{M}$, by $\mathcal{X}^*_P(\bar{\mathcal{M}})_m$ the subspace of covectors $\alpha \in T^*_m(\mathcal{M})$ such that*

$$P_m(\alpha) \in \mathrm{dj}_m(T_m(\bar{\mathcal{M}})) = \mathrm{Im}(\mathrm{dj}_m), \quad m \in \bar{\mathcal{M}} \tag{2.6}$$

(*where* Im *denotes the image*) *and by $T^{\perp}(\bar{\mathcal{M}})_m$ — the set of all covectors at $m \in \mathcal{M}$ vanishing on the subspace* $\mathrm{Im}(\mathrm{dj}_m)$, $m \in \bar{\mathcal{M}}$ (*also called the annihilator of* $\mathrm{Im}(\mathrm{dj}_m)$ *in*

$T_m^*(\mathcal{M})$). *Let the following relations hold*:

$$\mathcal{X}_P^*(\bar{\mathcal{M}})_m + T^\perp(\bar{\mathcal{M}})_m = T_m^*(\mathcal{M}), \quad m \in \bar{\mathcal{M}}, \tag{2.7}$$

$$\mathcal{X}_P^*(\bar{\mathcal{M}})_m \cap T^\perp(\bar{\mathcal{M}})_m \subset \ker(P_m), \quad m \in \bar{\mathcal{M}}. \tag{2.8}$$

Then there exists unique Poisson tensor $\bar{P}$ *on* $\bar{\mathcal{M}}$, j-*related with* P, *that is*

$$P_m = \mathrm{dj}_m \circ \bar{P}_m \circ (\mathrm{dj}_m)^*, \quad m \in \bar{\mathcal{M}}. \tag{2.9}$$

The proof of the theorem is constructive, one takes $\beta \in T_m^*(\bar{\mathcal{M}})$, then represents $(\mathrm{j}^*\beta)_m$ as $\alpha_1 + \alpha_2$ where $\alpha_1 \in \mathcal{X}_P^*(\bar{\mathcal{M}})_m$, $\alpha_2 \in T^\perp(\bar{\mathcal{M}})_m$ and puts $\bar{P}_m(\beta) = P_m(\alpha_1)$ (we identify m and $\mathrm{j}(m)$ here).

In the simplest case $\mathfrak{g} = \mathrm{sl}(2,\mathbb{C})$ the above construction has been preformed in various works, see for example [24]. We do it now in the case $\mathfrak{g} = \mathrm{sl}(3,\mathbb{C})$. Restricting the Poisson tensor Q is easy, one readily gets that the restriction $\bar{Q}$ is given by the same formula as before:

$$\alpha \mapsto \bar{Q}(\alpha) = \mathrm{ad}_S(\alpha), \quad S \in \mathcal{N}, \quad \alpha \in T_S^*(\mathcal{N}). \tag{2.10}$$

The tensor P is a little harder to restrict. Let us introduce some notation and some ingredients that are needed for that. First, since J is a regular element from the Cartan subalgebra $\mathfrak{h}$ then each element S from the orbit $\mathcal{O}_J$ also is a regular element. Therefore $\mathfrak{h}_S \equiv \ker \mathrm{ad}_S$ is a Cartan subalgebra of $\mathrm{sl}(3,\mathbb{C})$ and we have

$$\mathrm{sl}(3,\mathbb{C}) = \mathfrak{h}_S(x) \oplus \mathfrak{h}_S^\perp(x). \tag{2.11}$$

If $X \in T_S(\mathcal{N})$, then $X(x) \in \mathfrak{h}_S^\perp(x)$ and X vanishes rapidly when $x \mapsto \pm\infty$. We shall denote the set of these functions by $\mathfrak{F}(\mathfrak{h}_S^\perp)$ so $X \in \mathfrak{F}(\mathfrak{h}_S^\perp)$ (this means a little more than simply $X \in \mathfrak{h}_S^\perp$ where by $\mathfrak{h}_S^\perp$ we understand the space of smooth functions $Z(x)$ defined on $\mathbb{R}$ and such that $Z(x) \in \mathfrak{h}_{S(x)}^\perp$). Using the same logic, for $X \in \mathfrak{F}(\mathfrak{h}_S^\perp)$ we write $\mathrm{ad}_S(X) \in \mathfrak{F}(\mathfrak{h}_S^\perp)$ which means that the function $\mathrm{ad}_{S(x)}X(x)$ belongs to $\mathfrak{F}(\mathfrak{h}_S^\perp)$.

The following facts about $\mathrm{sl}(3,\mathbb{C})$ are relevant here.

(1) The matrices J and $J_1 = J^2 - \frac{2}{3}\mathbf{1}$ span the Cartan subalgebra $\mathfrak{h}$ of $\mathrm{sl}(3,\mathbb{C})$ (it consists of all 3×3 traceless diagonal matrices). As a consequence, from $S \in \mathcal{O}_J$ follows that the matrices S and $S_1 = S^2 - \frac{2}{3}\mathbf{1}$ span the Cartan subalgebra $\mathfrak{h}_S(x)$ of $\mathrm{sl}(3,\mathbb{C})$.
(2) On $\mathfrak{h}_S^\perp(x)$ the operator $\mathrm{ad}_S(x)$ is invertible and on $\mathfrak{h}_S^\perp$ the operator ad_S is invertible.
(3) We have

$$\langle J, J\rangle = 12, \quad \langle J_1, J_1\rangle = 4, \quad \langle J, J_1\rangle = 0, \tag{2.12}$$

$$\langle S, S\rangle = 12, \quad \langle S_1, S_1\rangle = 4, \quad \langle S, S_1\rangle = 0. \tag{2.13}$$

Consequently, if one denotes by π_S and $\mathbf{1} - \pi_S$ the orthogonal the projectors on $\mathfrak{h}_S^\perp$ and $\mathfrak{h}_S$ relative to the splitting (2.11) we shall have

$$(\mathbf{1} - \pi_S)X = \frac{S}{12}\langle S, X\rangle + \frac{S_1}{4}\langle S_1, X\rangle. \tag{2.14}$$

(4) From (2.13) and from the fact that $S, S_x \in \mathfrak{g}_1$ while $S_1, (S_1)_x \in \mathfrak{g}_0$ immediately follows that

$$\partial_x S = S_x, \quad \partial_x S_1 = \partial_x S^2, \quad S_x, (S_1)_x \in \mathfrak{F}(\mathfrak{h}_S^\perp). \tag{2.15}$$

Taking into account the above, one can use the First Restriction Theorem to find the Generating Operators for the GZS system on sl(3, ℂ), [33, 34]. They are also useful in the study of GMV system, [35]. The details of the restriction of P on $\mathcal{N}$ can be found in [34]. Since we shall need the facts of the P–N structure for sl(3, ℂ) GZS in pole gauge we cite them:

For $S \in \mathcal{N}$ and for $\beta \in T_S^*(\mathcal{N})$ the restriction $\bar{P}$ of P on $\mathcal{N}$ has the form:

$$\bar{P}(\beta) = \mathrm{i}\pi_S \partial_x \beta - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x \beta(x), S(x)\rangle - \mathrm{i}\frac{(S_1)_x}{6}\partial_x^{-1}\langle \partial_x \beta(x), S_1(x)\rangle. \tag{2.16}$$

Remark 2.3. In the theory of the Recursion Operators and their geometric interpretation the expressions on which the operator ∂_x^{-1} acts are total derivatives. Thus the same results will be obtained choosing for ∂_x^{-1} any of the following operators:

$$\int_{-\infty}^{x} .\ \mathrm{d}y, \quad \int_{+\infty}^{x} .\ \mathrm{d}y, \quad \frac{1}{2}\left(\int_{-\infty}^{x} .\ \mathrm{d}y + \int_{+\infty}^{x} .\ \mathrm{d}y\right). \tag{2.17}$$

However, one uses more frequently the third expression when one writes the corresponding Poisson tensors in order to make them explicitly skew-symmetric.

The Poisson tensor $\bar{Q}$ is invertible on $\mathcal{N}$, so one can construct a Nijenhuis tensor:

$$\begin{aligned} N(X) = \bar{P} \circ \mathrm{ad}_S^{-1}(X)\mathrm{i}\pi_S \partial_x(\mathrm{ad}_S^{-1}X) - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x(\mathrm{ad}_S^{-1}X), S(x)\rangle \\ - \mathrm{i}\frac{(S_1)_x}{4}\partial_x^{-1}\langle \partial_x(\mathrm{ad}_S^{-1}X), S_1(x)\rangle, \quad X \in \mathfrak{F}(\mathfrak{h}_S^\perp). \end{aligned} \tag{2.18}$$

Taking into account that $\langle \mathrm{ad}_S^{-1}(X), S\rangle = \langle \mathrm{ad}_S^{-1}(X), S_1\rangle = 0$ the above can be cast into equivalent form:

$$\begin{aligned} N(X) = \mathrm{i}\pi_S \partial_x(\mathrm{ad}_S^{-1}X) + \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \mathrm{ad}_S^{-1}X, S_x\rangle \\ - \mathrm{i}\frac{(S_1)_x}{4}\partial_x^{-1}\langle \mathrm{ad}_S^{-1}X, (S_1)_x\rangle, \quad X \in \mathfrak{F}(\mathfrak{h}_S^\perp). \end{aligned} \tag{2.19}$$

From the general theory of the compatible Poisson tensors now follows the following theorem.

Theorem 2.4. *The Poisson tensor field $\bar{Q}$ and the Nijenhuis tensor field N endow the manifold $\mathcal{N}$ with P–N structure.*

The final step is to calculate the dual of the tensor N with respect to the pairing $\langle\langle\, ,\, \rangle\rangle$. A quick calculation, taking into account that ad_S is skew-symmetric with respect to the

Killing form, gives:

$$N^*(\alpha) = \mathrm{iad}_S^{-1}\left[\pi_S\partial_x\alpha + \frac{S_x}{12}\partial_x^{-1}\langle\alpha, S_x\rangle + \frac{(S_1)_x}{6}\partial_x^{-1}\langle\alpha, (S_1)_x\rangle\right], \quad \alpha \in \mathfrak{F}(\mathfrak{h}_S^{\perp}) \quad (2.20)$$

or equivalently

$$N^*(\alpha) = \mathrm{iad}_S^{-1}\left[\pi_S\partial_x\alpha - \frac{S_x}{12}\partial_x^{-1}\langle\alpha_x, S\rangle - \frac{(S_1)_x}{6}\partial_x^{-1}\langle\alpha_x, S_1\rangle\right], \quad \alpha \in \mathfrak{F}(\mathfrak{h}_S^{\perp}). \quad (2.21)$$

But these are the Recursion Operators $\tilde{\Lambda}_{\pm}$ for the sl$(3,\mathbb{C})$-GZS system in pole gauge, see [33]. This of course confirms the general fact that the Recursion Operators and the Nijenhuis tensors are dual objects.

3. The Manifold of Potentials for the GMV System

The GMV hierarchy has some specifics, because we must restrict on a submanifold of $\mathcal{N}$ of matrices having the form:

$$S = \begin{pmatrix} 0 & a & b \\ a^* & 0 & 0 \\ b^* & 0 & 0 \end{pmatrix}, \quad (3.1)$$

where $a = a(x)$, $b = b(x)$ are some smooth functions and $|a|^2 + |b|^2 = 1$. One sees that the functions $S(x)$ we had in the general situation must now obey the following restrictions:

(1) $S(x) \in \mathrm{isu}(3)$ where su(3) is the Lie algebra of 3×3 anti-Hermitian matrices. In other words S must be Hermitian.
(2) Let $h : X \mapsto HXH$ where $H = \mathrm{diag}(1, -1, -1)$ be the involutive automorphism introduced earlier. The algebra sl$(3,\mathbb{C})$ splits into a direct sum of its eigenspaces $\mathfrak{g}_0, \mathfrak{g}_1$, see (1.11), and $S(x) \in \mathfrak{g}_1$.
(3) In addition, $|a|^2 + |b|^2 = 1$.

The first two conditions lead to the fact that S has the form (3.1). The first condition means that all our matrices must belong to isu(3) and as discussed earlier this will not change any of our formulae. Now, if we rewrite the GZS canonical gauge problem into the form $(\partial_x - \mathrm{i}q + \mathrm{i}\lambda J)\psi = 0$ then all our matrices will belong to su(3), the fundamental solutions will be in SU(3) and hence $S(x)$ will be in the orbit $\mathcal{O}_J(\mathrm{SU}(3))$ of J with respect to SU(3) (it is a submanifold of isu(3) of course). But if we assume that S of the type (3.1) and it is in $\mathcal{O}_J(\mathrm{SU}(3))$ then the condition $|a|^2 + |b|^2 = 1$ holds automatically. Indeed, the eigenvalues of the matrix (3.1) are $\mu_1 = 0, \mu_2 = -\mu_3 = \sqrt{|a|^2 + |b|^2}$ and since S is in the orbit of J they must coincide with $0, \pm 1$ so we must have $|a|^2 + |b|^2 = 1$. Thus the first and the third requirement are related and hold automatically when we restrict on the orbit of $\mathcal{O}_J(\mathrm{SU}(3))$. Summarizing, to get from the sl$(3,\mathbb{C})$-GZS problem $(\mathrm{i}\partial_x - \lambda S)\tilde{\psi} = 0$ the GMV problem we must require the condition: $S(x) \in \mathcal{O}_J(\mathrm{SU}(3)) \cap \mathfrak{g}_1$.

Below we shall need some facts related to the automorphism h. Let us forget for a while about the other conditions and let us assume that for the involutive automorphism h (the one that induces the splitting (1.11)) we have $h(S) = -S$. Then we obtain

$$h \circ \mathrm{ad}_S = -\mathrm{ad}_S \circ h \tag{3.2}$$

and consequently for any x the spaces $\ker \mathrm{ad}_{S(x)} = \mathfrak{h}_S(x)$ and $\mathfrak{h}_S^{\perp}(x)$ are invariant under h. Each of them splits into two eigenspaces (for eigenvalues $+1$ and -1 respectively) for h:

$$\mathfrak{h}_S^{\perp}(x) = \mathfrak{f}_S^0(x) \oplus \mathfrak{f}_S^1(x), \quad \mathfrak{h}_S(x) = \mathfrak{h}_S^0(x) \oplus \mathfrak{h}_S^1(x). \tag{3.3}$$

Using the same type of notation as before, we write the above as

$$\mathfrak{h}_S^{\perp} = \mathfrak{f}_S^0 \oplus \mathfrak{f}_S^1, \quad \mathfrak{h}_S = \mathfrak{h}_S^0 \oplus \mathfrak{h}_S^1 \tag{3.4}$$

and denote the sets of smooth, rapidly decreasing functions $X(x)$ with $X(x)$ belonging to the above spaces (for the same x) by $\mathfrak{F}(\mathfrak{f}_S^0), \mathfrak{F}(\mathfrak{f}_S^1)$ and $\mathfrak{F}(\mathfrak{h}_S^0), \mathfrak{F}(\mathfrak{h}_S^1)$. We have of course

$$\mathfrak{F}(\mathfrak{h}_S^{\perp}) = \mathfrak{F}(\mathfrak{f}_S^0) \oplus \mathfrak{F}(\mathfrak{f}_S^1), \quad \mathfrak{F}(\mathfrak{h}_S) = \mathfrak{F}(\mathfrak{h}_S^0) \oplus \mathfrak{F}(\mathfrak{h}_S^1). \tag{3.5}$$

From the above considerations there are several consequences:

- $\mathrm{ad}_{S(x)}$ and $\mathrm{ad}_{S(x)}^{-1}$ interchange the spaces $\mathfrak{f}_S^0(x)$ and $\mathfrak{f}_S^1(x)$. Therefore, they have (complex) dimension 3.
- Since $S \in \mathfrak{g}_1$, $S_1 = S^2 - \frac{2}{3}\mathbf{1} \in \mathfrak{g}_0$ the spaces $\mathfrak{h}_S^0$, $\mathfrak{h}_S^1$ are one-dimensional (complex) and are spanned by S and $S_1 = S^2 - \frac{2}{3}\mathbf{1}$ respectively.
- From the previous item follows that

$$S_x \in \mathfrak{g}_1, \quad (S_1)_x = (S^2)_x \in \mathfrak{g}_0 \tag{3.6}$$

and since S_x and $(S_1)_x$ are orthogonal to $\mathfrak{h}_S$ we have

$$S_x \in \mathfrak{F}(\mathfrak{f}_S^1), \quad (S_1)_x = (S^2)_x \in \mathfrak{F}(\mathfrak{f}_S^0), \tag{3.7}$$

where by $\mathfrak{F}(\mathfrak{f}_S^0)$ and $\mathfrak{F}(\mathfrak{f}_S^0)$ we denote the spaces of smooth, rapidly decreasing function $\tilde{Z}(x)$ with values in $\mathfrak{f}_S^0(x)$ and $\mathfrak{f}_S^1(x)$ respectively. Since $\mathfrak{f}_S^0$ and $\mathfrak{f}_S^1$ are orthogonal to the space $\mathfrak{h}_S$, we have

$$N(X) = \mathrm{i}\pi_S\partial_x(\mathrm{ad}_S^{-1}X) - \mathrm{i}\frac{(S_1)_x}{4}\partial_x^{-1}\langle\partial_x(\mathrm{ad}_S^{-1}X), S_1(x)\rangle, \quad X \in \mathfrak{F}(\mathfrak{f}_S^0),$$
$$N(X) \in \mathfrak{F}(\mathfrak{f}_S^1), \tag{3.8}$$

$$N(X) = \mathrm{i}\pi_S\partial_x(\mathrm{ad}_S^{-1}X) - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle\partial_x(\mathrm{ad}_S^{-1}X), S(x)\rangle, \quad X \in \mathfrak{F}(\mathfrak{f}_S^1),$$
$$N(X) \in X \in \mathfrak{F}(\mathfrak{f}_S^0), \tag{3.9}$$

$$N^*(\alpha) = \mathrm{i}\mathrm{ad}_S^{-1}\left[\pi_S\partial_x\alpha + \frac{S_x}{12}\partial_x^{-1}\langle\alpha, S_x\rangle\right], \quad \alpha \in \mathfrak{F}(\mathfrak{f}_S^0),\ N^*(\alpha) \in \mathfrak{F}(\mathfrak{f}_S^1), \tag{3.10}$$

$$N^*(\alpha) = \mathrm{i}\mathrm{ad}_S^{-1}\left[\pi_S\partial_x\alpha + \frac{(S_1)_x}{6}\partial_x^{-1}\langle\alpha, (S_1)_x\rangle\right], \ \alpha \in \mathfrak{F}(\mathfrak{f}_S^1),\ N^*(\alpha) \in \mathfrak{F}(\mathfrak{f}_S^0). \tag{3.11}$$

The last expressions are exactly the Recursion Operators $\Lambda_1^\pm$ and $\Lambda_2^\pm$ (taking into account that $S = -L_1$) obtained in [11]. We have

$$(N|_{\mathfrak{F}(\mathfrak{f}_S^0)})^* = N^*|_{\mathfrak{F}(\mathfrak{f}_S^1)} = \tilde{\Lambda}^\pm|_{\mathfrak{F}(\mathfrak{f}_S^1)} = \Lambda_2^\pm, \tag{3.12}$$

$$(N|_{\mathfrak{F}(\mathfrak{f}_S^1)})^* = N^*|_{\mathfrak{F}(\mathfrak{f}_S^0)} = \tilde{\Lambda}^\pm|_{\mathfrak{F}(\mathfrak{f}_S^0)} = \Lambda_1^\pm. \tag{3.13}$$

Let us return now to the GMV system case. As we have seen, the condition that $S(x) \in \mathfrak{g}_1 \cap \mathcal{O}_J(\mathrm{SU}(3))$ means that the three conditions that $S(x)$ must satisfy are equivalent to the requirement that $S(x)$ has the form (3.1) with $|a|^2 + |b|^2 = 1$, that is $\mathfrak{g}^1 \cap \mathcal{O}_J(\mathrm{SU}(3))$ is a real manifold $\mathcal{P}$ of dimension 3.

Remark 3.1. Of course, we also have $S \in \mathfrak{h}_S^1 \cap \mathcal{O}_J(\mathrm{SU}(3))$.

Thus the function $S(x)$ takes values in $\mathcal{P}$ and we shall denote the set of the smooth functions with values in $\mathcal{P}$ that tend rapidly to constant values when $x \mapsto \pm\infty$ by $\mathcal{Q}$. Clearly, $\mathcal{Q}$ is a real submanifold of the manifold $\mathcal{N}$. If τ is a tangent vector at a point $q \in \mathcal{P}$ then $\tau \in \mathfrak{g}_1 \cap T_q(\mathcal{O}_J(\mathrm{SU}(3)))$ and since

$$T_q(\mathcal{O}_J(\mathrm{SU}(3))) = \mathfrak{h}_q^\perp \cap \mathrm{isu}(3) = \mathfrak{f}_q^1 \tag{3.14}$$

we have that

$$T_q(\mathcal{P}) = \mathfrak{f}_q^1 \cap \mathrm{isu}(3). \tag{3.15}$$

Then if $S \in \mathcal{Q}$ is a function taking values in $\mathcal{P}$ we shall have that

$$T_S(\mathcal{Q}) = \mathfrak{F}(\mathfrak{f}_S^1) \cap \mathrm{i}\mathfrak{F}(\mathrm{su}(3)), \quad T_S^*(\mathcal{Q}) = \mathfrak{F}(\mathfrak{f}_S^1) \cap \mathrm{i}\mathfrak{F}(\mathrm{su}(3)), \tag{3.16}$$

where by $\mathfrak{F}(\mathrm{su}(3))$ is denoted the space of smooth, rapidly decreasing functions with values in su(3). As before the pairing between the tangent and cotangent space spaces is given by $\langle\langle\, , \,\rangle\rangle$. This pairing (inner product) is nondegenerate, because on su(3) the Killing form reduces to a negatively defined inner product (su(3) is a compact real form) and so on isu(3) the Killing form reduces to positively defined inner product. Summarizing, we have the following thoerem.

Theorem 3.2. *The space $\mathcal{P} = \mathcal{O}_J(\mathrm{SU}(3))$ is a real manifold of dimension* 3, *the space $\mathcal{Q} \subset \mathcal{N}$ of the smooth functions with values in $\mathcal{P}$ that tend rapidly to constant values is the manifold of potentials for the GMV system. For $S \in \mathcal{Q}$ the tangent space and cotangent spaces at S are given by* (3.16).

Finally, we underline, that all the formulae we had in the above about the operators N, N^* and their restrictions on the spaces $\mathfrak{F}(\mathfrak{f}_S^0), \mathfrak{F}(\mathfrak{f}_S^1)$ remain true in the new situation when these spaces are substituted by $\mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3))$ and $\mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3))$.

4. The P–N Structure for the GMV System

Let us consider now the problem of reducing the P–N structure from $\mathcal{N}$ to $\mathcal{Q}$. For this reduction we shall apply another result, which we call Second Restriction Theorem, see [24].

Theorem 4.1. *Let $\mathcal{M}$ be a P–N manifold endowed with Poisson tensor P and Nijenhuis tensor N. Let $\bar{\mathcal{M}} \subset \mathcal{M}$ be a submanifold of $\mathcal{M}$ and suppose that we have*:

(i) *P allows a restriction $\bar{P}$ on $\bar{\mathcal{M}}$ such that if $\mathrm{j} : \bar{\mathcal{M}} \mapsto \mathcal{M}$ is the inclusion map then $\bar{P}$ is* j*-related with P, that is*

$$P_m = \mathrm{dj}_m \circ \bar{P}_m \circ (\mathrm{dj}_m)^*, \quad m \in \bar{\mathcal{M}}. \tag{4.1}$$

(ii) *The tangent spaces of $\bar{\mathcal{M}}$, considered as subspaces of the tangent spaces of $\mathcal{M}$ are invariant under N, so that N allows a natural restriction $\bar{N}$ to $\bar{\mathcal{M}}$, that is $\bar{N}$ is* j*-related with N.*

Then $(\bar{P}, \bar{N})$ endow $\bar{\mathcal{M}}$ with a P–N structure.

In order to apply the above result we need to find suitable tensors on our manifold of potentials $\mathcal{N}$ for GZS system in general position. The most natural will be to take ad_S and N but this is not possible. In fact, from what we have in the previous section now follows that the Poisson tensor ad_S becomes identically zero when restricted to $T^*_S(\mathcal{Q})$. Indeed, for $\alpha, X \in \mathfrak{F}(\mathfrak{f}^1_S) \cap \mathrm{i}\mathfrak{F}(\mathrm{su}(3))$

$$\langle\!\langle \mathrm{ad}_S(\alpha), X \rangle\!\rangle = 0 \tag{4.2}$$

because ad_S interchanges the spaces $\mathfrak{f}^0_S$ and $\mathfrak{f}^1_S$. So we need to find another Poisson structure to restrict on $\mathcal{Q}$. The pair $\bar{P}$, N is also a bad candidate — one sees that $N(T_S(\mathcal{Q})) \subset T_S(\mathcal{Q})$ is not true and therefore the Nijenhuis tensor does not allow a restriction on $T_S(\mathcal{Q})$.

However, from the general theory of the Nijenhuis tensors follows, see for example [12], that all the powers N^k, $k = 1, 2, \dots$ are also Nijenhuis tensors and if N^{-1} exists all the tensors N^k, $k = \pm 1, \pm 2, \dots$ are Nijenhuis tensors. Also, in case (P, N) is a P–N structure on $\mathcal{M}$, then for $k, s = 1, 2, \dots$ all the tensors pairs $(N^s P, N^k)$ are also P–N structures on $\mathcal{M}$. Since the tensors N^{2r} for $r = 1, 2, \dots$ satisfy the condition

$$N^{2r}(T_S(\mathcal{Q})) \subset T_S(\mathcal{Q}), \tag{4.3}$$

we have the following candidates for restriction — the Nijenhuis tensor N^2 and the Poisson tensor $\bar{P} = N \circ \mathrm{ad}_S = \mathrm{ad}_S \circ N^*$. Let us take $\bar{P}$ and try to restrict it.

We want to apply the First Restriction Theorem. $\mathcal{X}^*(\bar{P})_S$ consists of smooth functions β, going rapidly to zero when $|x| \to \infty$ such that $\beta(x) \in \mathfrak{h}^{\perp}_S(x) \cap \mathrm{isu}(3)$ and $\bar{P}(\beta) \in T_S(\mathcal{Q})$. The last means that

$$\mathrm{i}\pi_S \partial_x \beta - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x \beta(x), S(x)\rangle - \mathrm{i}\frac{(S_1)_x}{6}\partial_x^{-1}\langle \partial_x\beta(x), S_1(x)\rangle \in \mathfrak{f}^1_S(x) \cap \mathrm{isu}(3)(x).$$

In other words, for arbitrary smooth function $X(x)$ such that $X \in \mathfrak{f}^0_S \cap \mathrm{isu}(3)$ and going rapidly to zero when $|x| \to \infty$ we have

$$\left\langle \mathrm{i}\pi_S \partial_x \beta - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x \beta(x), S(x)\rangle - \mathrm{i}\frac{(S_1)_x}{6}\partial_x^{-1}\langle \partial_x\beta(x), S_1(x)\rangle, X(x) \right\rangle = 0. \tag{4.4}$$

The space $T^{\perp}(\mathcal{Q})_S$ consists of smooth functions $\beta(x)$ such that: $\beta \in \mathfrak{h}^{\perp}_S \cap \mathrm{isu}(3)$, $\beta(x)$ goes rapidly to zero when $|x| \to \infty$ and satisfies

$$\left\langle\!\!\left\langle \mathrm{i}\pi_S \partial_x \beta - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x \beta(x), S(x)\rangle - \mathrm{i}\frac{(S_1)_x}{6}\partial_x^{-1}\langle \partial_x\beta(x), S_1(x)\rangle, Y(x) \right\rangle\!\!\right\rangle = 0,$$

for each smooth function $Y(x)$, $Y \in \mathfrak{f}_S^1(x) \cap \mathrm{isu}(3)(x)$ going rapidly to zero when $|x| \to \infty$. Arguments identical to that used in the Haar's lemma from the Variational Calculus show that this means that for each x we have

$$\left\langle \mathrm{i}\pi_S \partial_x \beta - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x \beta(x), S(x)\rangle - \mathrm{i}\frac{(S_1)_x}{6}\partial_x^{-1}\langle \partial_x \beta(x), S_1(x)\rangle, Y(x)\right\rangle = 0, \tag{4.5}$$

where $Y(x)$ is as above. Then if $\beta \in \mathcal{X}^*(\bar{P})_S \cap T^\perp(\mathcal{Q})_S$ we shall have simultaneously (4.4) and (4.5) which means that $\beta \in \ker \bar{P}_S$. Thus the first requirement of First Restriction Theorem is fulfilled.

In order to see that the second requirement also holds, we introduce the following lemma.

Lemma 4.2. *The operator $\bar{P}$ has the properties*:

$$\bar{P}_S(\mathfrak{F}(\mathfrak{f}_S^0)) \subset \mathfrak{F}(\mathfrak{f}_S^0), \quad \bar{P}_S(\mathfrak{F}(\mathfrak{f}_S^1)) \subset \mathfrak{F}(\mathfrak{f}_S^1), \tag{4.6}$$

$$\bar{P}_S(\mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3))) \subset \mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3)), \tag{4.7}$$

$$\bar{P}_S(\mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3))) \subset \mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3)). \tag{4.8}$$

Proof. It is enough to see that the spaces $\mathfrak{f}_S^0$ and $\mathfrak{f}_S^1$ are invariant with respect to π_S. □

Using the Lemma, suppose that $\beta(x)$ is a smooth function $\beta \in \mathfrak{h}_S^\perp \cap \mathrm{isu}(3)$ going rapidly to zero when $|x| \to \infty$. Clearly, we can write it uniquely into the form:

$$\beta = \beta_0 + \beta_1, \quad \beta_0 \in \mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3)), \quad \beta_1 \in \mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3)).$$

Then $\bar{P}_S(\beta_0) \in \mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3))$, $\bar{P}_S(\beta_1) \in \mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3))$ and we see that $\bar{P}_S\beta_0 \in \mathcal{X}^*(\bar{P})_S$, $\bar{P}_S\beta_1 \in T^\perp(\mathcal{Q})_S$. So the second requirement of the First Restriction Theorem is also satisfied and $\bar{P}$ allows restriction. If γ is 1-form on $\mathcal{Q}$, that is $\gamma \in \mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3))$, the restriction $\bar{P}'$ is given by

$$\bar{P}'(\gamma) = \mathrm{i}\pi_S \partial_x \gamma - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x \gamma(x), S(x)\rangle. \tag{4.9}$$

Finally, we summarize our considerations into the following theorem.

Theorem 4.3. *The manifold of potentials $\mathcal{Q}$ is endowed with a P–N structure, defined by the Poisson tensor $\bar{P}'$ and the restriction of the Nijenhuis tensor N^2. Explicitly, the restriction of N^2 is given by $N_0'N_1'$ where*:

$$N_0'(X) = \mathrm{i}\pi_S \partial_x(\mathrm{ad}_S^{-1}X) - \mathrm{i}\frac{(S_1)_x}{4}\partial_x^{-1}\langle \partial_x(\mathrm{ad}_S^{-1}X), S_1(x)\rangle,$$

$$X \in \mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3)), \quad N_0'(X) \in \mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3)), \tag{4.10}$$

$$N_1'(X) = \mathrm{i}\pi_S \partial_x(\mathrm{ad}_S^{-1}X) - \mathrm{i}\frac{S_x}{12}\partial_x^{-1}\langle \partial_x(\mathrm{ad}_S^{-1}X), S(x)\rangle,$$

$$X \in \mathfrak{F}(\mathfrak{f}_S^1 \cap \mathrm{isu}(3)), \quad N_0'(X) \in \mathfrak{F}(\mathfrak{f}_S^0 \cap \mathrm{isu}(3)). \tag{4.11}$$

A short calculation shows that $(N_0')^* = \Lambda_1$, $(N_1')^* = \Lambda_2$ and the above Theorem gives geometric interpretation of the operators $\Lambda_{1,2}$ introduced in [11].

Finally, let us write the Poisson bracket for the Poisson structure $\bar{P}'$. If H_1, H_2 are two allowed functionals

$$\{H_1, H_2\} = \mathrm{i}\left\langle\!\left\langle \partial_x\left(\frac{\delta H_1}{\delta S}\right), \frac{\delta H_2}{\delta S}\right\rangle\!\right\rangle + \frac{\mathrm{i}}{24}\int_{-\infty}^{+\infty}\left[\left\langle S_x, \frac{\delta H_2}{\delta S}\right\rangle(x)\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right)\left\langle S_y, \frac{\delta H_1}{\delta S}\right\rangle(y)\mathrm{d}y\right]\mathrm{d}x. \quad (4.12)$$

The geometric theory for the GMV system will be complete after we are able to find the fundamental fields for the P–N structure on $\mathcal{Q}$. This is not hard to achieve. Indeed, let us consider first the fundamental fields for the P–N structure in general position (without reductions). As it is known, see [12], the fields of the type $X_H(S) = [H, S]$, $H \in \mathfrak{h}$ are fundamental for the P–N structure in general position. The reason for that is the original tensors P, Q (see Eqs. (2.2) and (2.3)) are covariant with respect to the one-parametric group of transformations

$$\varphi_t^H(S) = \mathrm{Ad}(\exp tH)S, \quad H \in \mathfrak{h}, \quad t \in \mathbb{R}. \quad (4.13)$$

and the submanifold $\mathcal{N}$ is invariant with respect to φ_t^H. The one-parametric group φ_t^H corresponds to the vector fields $X_H = [H, S]$ which are tangent to $\mathcal{N}$. According to the theory of the P–N manifolds if X is a fundamental field then all the fields $N^p X$, $p = 1, 2, \ldots$ are also fundamental and commute, that is they have zero Lie brackets. Thus to each $H \in \mathfrak{h}$ corresponds a hierarchy of fundamental fields $N^k[H, S]$ and we obtain two independent families of fundamental fields. However the fields X_H are not tangent to the manifold $\mathcal{Q}$ so the hierarchies of equations related to the P–N structure on $\mathcal{Q}$ is different. It is enough however to check which of the fields $N^k X_H$ are tangent to $\mathcal{Q}$ in order to find the needed hierarchies. If one calculates $N(X_H)$ (this field is already tangent to $\mathcal{Q}$) one can check that from the above 2-parametric family "survive" only the fields

$$N^{2p}S_x = (N_0N_1)^p S_x, \quad N^{2p+1}(S^2)_x = (N_0N_1)^p N_0(S^2)_x, \quad p = 0, 1, 2 \ldots. \quad (4.14)$$

These fields are tangent to $\mathcal{Q}$ and are fundamental for the P–N structure on this manifold. Therefore they give rise to hierarchies of integrable equations of the type $S_t = F(S)$ where $F(S)$ is a finite linear combination of the fields (4.14), see [11].

5. Conclusion

We have found the P–N structure on the manifold of potentials $\mathcal{Q}$ for the GMV system thus obtaining geometric interpretation of the Recursion Operators introduced earlier for that system. This have been achieved performing two subsequent reductions. After the first reduction is obtained the P–N structure for the GZS system in pole gauge for the algebra $\mathrm{sl}(3, \mathbb{C})$, consisting of a Nijenhuis tensor N and the Kirillov tensor Q on the corresponding orbit. After the second reduction (that is on $\mathcal{Q}$) the Kirillov tensor trivializes and N cannot be restricted. The tensors that can be restricted are the second Poisson structure for the GZS system, namely NQ and the square of the Nijenhuis tensor N^2 and they endow the manifold of potentials $\mathcal{Q}$ for the GMV system with a P–N structure.

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