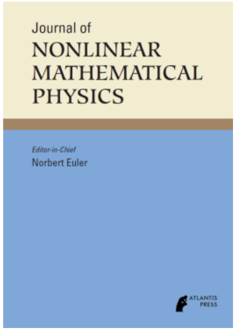

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LAGRANGIANS FOR BIOLOGICAL MODELS

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We show that a method presented in [S. L. Trubatch and A. Franco, Canonical Procedures for Population Dynamics, *J. Theor. Biol.* **48** (1974) 299–324] and later in [G. H. Paine, The development of Lagrangians for biological models, *Bull. Math. Biol.* **44** (1982) 749–760] for finding Lagrangians of classic models in biology, is actually based on finding the Jacobi Last Multiplier of such models. Using known properties of Jacobi Last Multiplier we show how to obtain linear Lagrangians of systems of two first-order ordinary differential equations and nonlinear Lagrangian of the corresponding single second-order equation that can be derived from them, even in the case where those authors failed such as the host-parasite model. Also we show that the Lagrangians of certain second-order ordinary differential equations derived by Volterra in [V. Volterra, Calculus of variations and the logistic curve, *Hum. Biol.* **11** (1939) 173–178] are particular cases of the Lagrangians that can be obtained by means of the Jacobi Last Multiplier. Actually we provide more than one Lagrangian for those Volterra's equations.

Keywords: Jacobi Last Multiplier; Lagrangian; population dynamics.

PACS: 02.30.Hq, 02.30.Xx, 45.20.Jj, 87.23.Cc

1. Introduction

In 1974, nearly forty years ago, Trubatch and Franco published a paper [42] in which they presented an explicit algorithm for constructing Lagrangians of some biological systems, namely the classical Volterra–Lotka's model [48], the Gompertz's model [10], the Verhulst's model [44], and an host-parasite model [25].

Their method for finding a Lagrangian of a second-order equation is, as they state, that by Havas [13] who based his method on Helmholtz's work [14]. Neither Helmholtz nor Havas ever acknowledged the use of the Jacobi Last Multiplier in order to find Lagrangians of a second-order equation [19],[a] [50].

Indeed, the method by Trubatch and Franco is based on finding a function f that satisfies their Eq. (6) and is nothing else than the Jacobi Last Multiplier. Because they did not know the properties of the Jacobi Last Multiplier they were unable to find a Lagrangian for the host-parasite model. In fact they found just a linear Lagrangian of this model and stated explicitly "In general, there is no relation between the linear Lagrangians of this section and the nonlinear ones of the previous section for the same model systems." In this paper we prove that they were wrong since a relation exists and it is given by means of the Jacobi Last Multiplier.

It is interesting to note that the method by Trubatch and Franco for finding linear Lagrangians is that introduced by Kerner [23]. Again Trubatch and Franco did not realize that their key-function W that satisfies their Eq. (50a) is nothing else than the Jacobi Last Multiplier.[b]

Eight years later, in 1982, Paine [37] published a paper on the same subject and based his work on the method introduced by Kerner [23], and cited Helmholtz's work [14] as well. Of course, the method proposed by Paine is based on a function g that is actually the Jacobi Last Multiplier of the two-dimensional systems that he studies.

Paine posed the following questions: "What are the criteria that a system of ordinary differential equations must satisfy to assure the existence of a Lagrangian?" (omissis) "Does there exist an algorithm that enables one to construct the Lagrangian from the dynamical equations?"

Strangely enough, he did not mention the previous work by Trubatch and Franco [42]. Paine's examples are the Volterra–Lotka's model similar to that studied by Trubatch and Franco [42], and two trivial linear systems of two first-order equations.

In this paper we show that recognizing that the key-function for finding a Lagrangian is the Jacobi Last Multiplier permits to obtain all the results in [42] in a simple and complete way and furthermore where Trubatch and Franco fail, namely the model of host-parasite, Jacobi Last Multiplier prevails by yielding a suitable Lagrangian.

Also we show that the Lagrangians of certain second-order ordinary differential equations derived by Volterra in [49] are particular cases of the Lagrangians that can be obtained by means of the Jacobi Last Multiplier. Indeed here we provide more than just one Lagrangian for those Volterra's equations thanks to the link between Lie symmetries and Jacobi Last Multiplier that was found by Lie himself [26].

The paper is organized in the following way. In Sec. 2 we recall the properties of the Jacobi Last Multiplier, its connection with Lagrangians of second-order equations [19, 50], and that with the Lagrangians of systems of two first-order equations that we have found; also Noether's theorem [30] is presented, and the link between Lie symmetries and Jacobi Last Multiplier is recalled [26, 27, 5]. In Sec. 3 we apply the method of the Jacobi Last

[a] An English translation is now available [20].

[b] Actually Havas and Kerner never acknowledged each other work, although, at least once, they were presenting at the same meeting in the same Section [12, 22].

Multiplier to the same systems as given in [42], and their equivalent single second-order equations. In Sec. 4 we apply the method of the Jacobi Last Multiplier to the second-order equations considered by Volterra in [49]. Section 5 contains some final remarks.

2. The Method by Jacobi

The method of the Jacobi Last Multiplier[c] [16–19] provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial f}{\partial x_i} = 0 \tag{2.1}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \cdots = \frac{\mathrm{d}x_n}{a_n}. \tag{2.2}$$

In fact, if one knows the Jacobi Last Multiplier and all but one of the solutions, namely $n-2$ solutions, then the last solution can be obtained by a quadrature. The Jacobi Last Multiplier M is given by

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = M\mathcal{A}f, \tag{2.3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial \omega_1}{\partial x_1} & & \frac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \frac{\partial \omega_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{2.4}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (2.1) or, equivalently, first integrals of (2.2) independent of each other. This means that M is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi Last Multiplier are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of Eq. (2.1), then the corresponding last multiplier N is linked to M by the relationship:

$$N = M\frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$

[c]Many authors have dealt with the Jacobi Last Multiplier, and an up to date (2004) nearly complete list can be found in [31]. It ranges from the 1871-paper by Laguerre [24] and the seminal 1874-paper by Lie [26] to the 2003-review paper by Berrone and Giacomini [4]. A missed reference in [31] is Sec. 2.11 in the 2001-book by Goriely [11].

(b) Given a nonsingular transformation of variables

$$\tau : (x_1, x_2, \ldots, x_n) \to (x'_1, x'_2, \ldots, x'_n),$$

then the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x'_1, x'_2, \ldots, x'_n)},$$

where M obviously comes from the $n - 1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(M a_i)}{\partial x_i} = 0; \tag{2.5}$$

vice versa every solution M of this equation is a Jacobi Last Multiplier.

(d) If one knows two Jacobi Last Multipliers M_1 and M_2 of Eq. (2.1), then their ratio is a solution ω of (2.1), or, equivalently, a first integral of (2.2). Naturally the ratio may be quite trivial, namely a constant. Viceversa the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie [26, 27]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [5]. If we know $n - 1$ symmetries of (2.1)/(2.2), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n)\partial_{x_j}, \quad i = 1, n-1, \tag{2.6}$$

a Jacobi Last Multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}. \tag{2.7}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing the variable associated with that component, i.e. $\partial a_i/\partial x_i = 0$, the last multiplier is a constant, and any other Jacobi Last Multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation

$$\ddot{x} = \phi(t, x, \dot{x}) \tag{2.8}$$

i.e. [19, Lecture 10; 50]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}, \tag{2.9}$$

where $M = M(t, x, \dot{x})$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial \phi}{\partial \dot{x}} = 0. \tag{2.10}$$

Then Eq. (2.8) becomes the Euler–Lagrangian equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \tag{2.11}$$

The proof is given by taking the derivative of (2.11) by $\dot{x}$ and showing that this yields (2.10). If one knows a Jacobi Last Multiplier, then L can be obtained by a double integration, i.e.:

$$L = \int \left(\int M \,\mathrm{d}\dot{x} \right) \mathrm{d}\dot{x} + \ell_1(t, x)\dot{x} + \ell_2(t, x), \tag{2.12}$$

where ℓ_1 and ℓ_2 are functions of t and x which have to satisfy a single partial differential equation related to (2.8) [33]. As it was shown in [33], ℓ_1, ℓ_2 are related to the gauge function $F = F(t, x)$. In fact, we may assume

$$\begin{aligned} \ell_1 &= \frac{\partial F}{\partial x}, \\ \ell_2 &= \frac{\partial F}{\partial t} + \ell_3(t, x), \end{aligned} \tag{2.13}$$

where ℓ_3 has to satisfy the mentioned partial differential equation and F is obviously arbitrary.

In [42] it was shown that a system of two first-order ordinary differential equations

$$\begin{aligned} \dot{u}_1 &= \phi_1(t, u_1, u_2), \\ \dot{u}_2 &= \phi_2(t, u_1, u_2) \end{aligned} \tag{2.14}$$

always admits a linear Lagrangian of the form

$$L = U_1(t, u_1, u_2)\dot{u}_1 + U_2(t, u_1, u_2)\dot{u}_2 - V(t, u_1, u_2). \tag{2.15}$$

The key is a function W such that[d]

$$W = -\frac{\partial U_1}{\partial u_2} = \frac{\partial U_2}{\partial u_1} \tag{2.16}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log W) + \frac{\partial \phi_1}{\partial u_1} + \frac{\partial \phi_2}{\partial u_2} = 0. \tag{2.17}$$

[d]In [42] this formula contains an inessential multiplicative constant, namely the integer 2.

It is obvious that Eq. (2.17) is Eq. (2.5) of the Jacobi Last Multiplier for system (2.14). Therefore once a Jacobi Last Multiplier $M(t, u_1, u_2)$ has been found, then a Lagrangian of system (2.14) can be obtained by two integrations, i.e.:

$$L = \left(\int M \, \mathrm{d}u_1\right) \dot{u}_2 - \left(\int M \, \mathrm{d}u_2\right) \dot{u}_1 + g(t, u_1, u_2) + \frac{\mathrm{d}}{\mathrm{d}t} G(t, u_1, u_2), \tag{2.18}$$

where $g(t, u_1, u_2)$ satisfies two linear differential equations of first-order that can be always integrated, and $G(t, u_1, u_2)$ is the arbitrary gauge function[e] that should be taken into consideration in order to correctly apply Noether's theorem [30]. If a Noether's symmetry

$$\Gamma = \xi(t, u_1, u_2)\partial_t + \eta_1(t, u_1, u_2)\partial_{u_1} + \eta_2(t, u_1, u_2)\partial_{u_2} \tag{2.19}$$

exists for the Lagrangian L in (2.18) then a first integral of system (2.14) is

$$-\xi L - \frac{\partial L}{\partial \dot{u}_1}(\eta_1 - \xi \dot{u}_1) - \frac{\partial L}{\partial \dot{u}_2}(\eta_2 - \xi \dot{u}_2) + G(t, u_1, u_2). \tag{2.20}$$

We underline that $\dot{u}_1$ and $\dot{u}_2$ always disappear from the expression of the first integral (2.20) thanks to the linearity of the Lagrangian (2.18) and formula (2.18).

3. Some Biological Examples from [42]

3.1. *Volterra–Lotka's model*

The Volterra–Lotka's model considered in [42] is the following:

$$\begin{aligned} \dot{w}_1 &= w_1(a + bw_2), \\ \dot{w}_2 &= w_2(A + Bw_1). \end{aligned} \tag{3.1}$$

In order to simplify system (3.1) we follow [42] and introduce the change of variables

$$w_1 = \exp(r_1), \quad w_2 = \exp(r_2) \tag{3.2}$$

and then system (3.1) becomes

$$\begin{aligned} \dot{r}_1 &= b \exp(r_2) + a, \\ \dot{r}_2 &= B \exp(r_1) + A. \end{aligned} \tag{3.3}$$

An obvious Jacobi Last Multiplier of this system is a constant, say 1, and consequently by means of (2.18) a linear Lagrangian of system (3.3) is

$$L_{[r]} = r_1\dot{r}_2 - r_2\dot{r}_1 + 2(-B\exp(r_1) + b\exp(r_2) - Ar_1 + ar_2) + \frac{\mathrm{d}}{\mathrm{d}t} G(t, r_1, r_2) \tag{3.4}$$

which (minus the gauge function G) was found in [42]. Moreover we can derive a Jacobi Last Multiplier for the Volterra–Lotka system (3.1) by using property (b). In fact we have

[e]The gauge function was not taken into consideration in [42].

to calculate the Jacobian of the transformation (3.2) between (w_1, w_2) and (r_1, r_2) and this yields a Jacobi Last Multiplier of system (3.1), i.e.

$$M_{[w]} = M_{[r]} \frac{\partial(r_1, r_2)}{\partial(w_1, w_2)} = \begin{vmatrix} \frac{1}{w_1} & 0 \\ 0 & \frac{1}{w_2} \end{vmatrix} = \frac{1}{w_1 w_2}. \tag{3.5}$$

Finally, formula (2.18) yields a linear Lagrangian of system (3.1)

$$\begin{aligned} L_{[w]} = \log(w_1)\frac{\dot{w}_2}{w_2} - \log(w_2)\frac{\dot{w}_1}{w_1} + 2(-A\log(w_1) \\ + a\log(w_2) - Bw_1 + bw_2) + \frac{\mathrm{d}}{\mathrm{d}t}G(t, w_1, w_2). \end{aligned} \tag{3.6}$$

This Lagrangian was not obtained in [42]. We note that (3.1) is autonomous and therefore invariant under time translation, namely ∂_t. It is easy to show that the Lagrangian $L_{[w]}$ in (3.6) yields a time-invariant first integral through Noether's theorem [30], i.e.:

$$-L_{[w]} + \dot{w}_1 \frac{\partial L_{[w]}}{\partial \dot{w}_1} + \dot{w}_2 \frac{\partial L_{[w]}}{\partial \dot{w}_2} = A\log(w_1) - a\log(w_2) + Bw_1 - bw_2 = \text{const}. \tag{3.7}$$

We may mention that in 1998 a particular case of the Volterra–Lotka's model (3.1) was investigated with the purpose of finding its possible underlying Lagrangian structure [9]. The author was unaware of Trubatch and Franco's paper [42]: his Lagrangian is a particular case of the Lagrangian (3.6).

Following [42] we can transform system (3.3) into an equivalent second-order ordinary differential equation by eliminating, say, r_1. In fact from the second equation in (3.3) one gets

$$r_1 = \log\left(\frac{\dot{r}_2 - A}{B}\right), \tag{3.8}$$

and the equivalent second-order equation in r_2 is the following

$$\ddot{r}_2 = -(b\exp(r_2) + a)(A - \dot{r}_2). \tag{3.9}$$

A Jacobi Last Multiplier for this equation has to satisfy Eq. (2.10), i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + b\exp(r_2) + a = 0 \tag{3.10}$$

namely

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \dot{r}_1 = 0, \tag{3.11}$$

by taking into account the first equation in (3.3), and consequently we get the following Jacobi Last Multiplier for Eq. (3.9):

$$M_1 = \exp(-r_1) = \frac{B}{\dot{r}_2 - A}, \tag{3.12}$$

the last equality thanks to (3.8). Then a Lagrangian can be obtained by a double integration as in (2.12), i.e.

$$L_1 = B((\dot{r}_2 - A)\log(A - \dot{r}_2) - \dot{r}_2 + b\exp(r_2) + ar_2) + \frac{\mathrm{d}}{\mathrm{d}t}F(t, r_2). \tag{3.13}$$

The same Lagrangian (minus the gauge function F) was obtained in [42]. In order to show the power of the Jacobi's method we derive at least another Lagrangian for Eq. (3.9).

We note that (3.9) is autonomous and therefore invariant under time translation. It is easy to show that the Lagrangian L_1 in (3.13) yields a time-invariant first integral, through Noether's theorem [30], i.e.:

$$I_1 = -ar_2 + \dot{r}_2 + A\log(A - \dot{r}_2) - b\exp(r_2) = \text{const.} \tag{3.14}$$

As a consequence of the property (d) of the Jacobi last multiplier, the product of a Jacobi last multiplier M_1 as in (3.12) and a first integral I_1 as in (3.14) of Eq. (3.9) yields another Jacobi last multiplier, i.e.

$$M_2 = M_1 I_1 = \frac{B}{A - \dot{r}_2}(ar_2 - \dot{r}_2 - A\log(A - \dot{r}_2) + b\exp(r_2)) \tag{3.15}$$

and therefore we can obtain a second Lagrangian of Eq. (3.9), i.e.

$$\begin{aligned} L_2 = -\frac{B}{2}&((A\log(A - \dot{r}_2) - 2ar_2)(A - \dot{r}_2)\log(A - \dot{r}_2) \\ &- (2ar_2 + \dot{r}_2)\dot{r}_2 - 2b\exp(r_2)((A - \dot{r}_2)\log(A - \dot{r}_2) + \dot{r}_2) \\ &+ b^2\exp(2r_2) + 2abr_2\exp(r_2) + a^2r_2^2) + \frac{\mathrm{d}}{\mathrm{d}t}F(t, r_2). \end{aligned} \tag{3.16}$$

This Lagrangian yields another time invariant first integral which is just the square of I_1 in (3.14).

We can keep using property (d) to derive more and more Jacobi last multipliers and therefore Lagrangians of Eq. (3.9). In fact other Jacobi last multipliers can be obtained by simply taking any function of the first integral I_1 in (3.14) and multiplying it for either M_1 in (3.12) or M_2 in (3.15), and so on *ad libitum.*

3.2. *Gompertz's model*

The Gompertz's model considered in [42] is the following[f]:

$$\begin{aligned} \dot{w}_1 &= w_1\left(A\log\left(\frac{w_1}{m_1}\right) + Bw_2\right), \\ \dot{w}_2 &= w_2\left(a\log\left(\frac{w_2}{m_2}\right) + bw_1\right). \end{aligned} \tag{3.17}$$

In order to simplify system (3.17) we follow [42] and introduce the change of variables

$$w_1 = m_1\exp(r_1), \quad w_2 = m_2\exp(r_2) \tag{3.18}$$

[f] In [42] some parentheses are missing: an obvious misprint.

and then system (3.17) becomes

$$\begin{aligned} \dot{r}_1 &= m_2 B \exp(r_2) + A r_1, \\ \dot{r}_2 &= m_1 b \exp(r_1) + a r_2. \end{aligned} \tag{3.19}$$

It is easy to derive a Jacobi Last Multiplier for this system from (2.5), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \log(M_{[r]}) = -(a + A) \Rightarrow M_{[r]} = \exp[-(a + A)t] \tag{3.20}$$

and therefore the following Lagrangian

$$\begin{aligned} L_{[r]} = \exp[-(a + A)t][r_1 \dot{r}_2 - r_2 \dot{r}_1 - 2m_1 b \exp(r_1) \\ + 2m_2 B \exp(r_2) + (A - a) r_1 r_2] + \frac{\mathrm{d}}{\mathrm{d}t} G(t, r_1, r_2), \end{aligned} \tag{3.21}$$

which (minus the gauge function G) was found in [42]. Then, property (b) yields a Jacobi Last Multiplier for the Gompertz's system (3.17). The product of $M_{[r]}$ in (3.20) with the Jacobian of the transformation (3.18) between (w_1, w_2) and (r_1, r_2) yields the following Jacobi Last Multiplier of system (3.17), i.e.

$$M_{[w]} = M_{[r]} \frac{\partial(r_1, r_2)}{\partial(w_1, w_2)} = \exp[-(a + A)t] \begin{vmatrix} \frac{1}{w_1} & 0 \\ 0 & \frac{1}{w_2} \end{vmatrix} = \exp[-(a + A)t] \frac{1}{w_1 w_2}, \tag{3.22}$$

and consequently a Lagrangian of the original system (3.17)

$$\begin{aligned} L_{[w]} = \exp[-(a + A)t] \left[\log(w_1) \frac{\dot{w}_2}{w_2} - \log(w_2) \frac{\dot{w}_1}{w_1} - 2a \log\left(\frac{w_2}{m_2}\right) \log(w_1) + 2B w_2 - 2b w_1 \right. \\ \left. + 2A \log\left(\frac{w_1}{m_1}\right) \log(w_2) - (A - a) \log(w_1) \log(w_2) \right] + \frac{\mathrm{d}}{\mathrm{d}t} G(t, w_1, w_2). \end{aligned} \tag{3.23}$$

This Lagrangian was not obtained in [42].

We can transform system (3.19) into an equivalent second-order ordinary differential equation by eliminating, say, r_2. In fact from the second equation in (3.19) one gets

$$r_2 = \log\left(\frac{\dot{r}_1 - A r_1}{B m_2}\right), \tag{3.24}$$

and the equivalent second-order equation in r_2 is the following

$$\ddot{r}_1 = \left(b m_1 \exp(r_1) + a \log\left(\frac{\dot{r}_1 - A r_1}{B m_2}\right)\right) (\dot{r}_1 - A r_1) + A \dot{r}_1. \tag{3.25}$$

Using property (b) a Jacobi Last Multiplier for this equation can be obtained. In fact we have to calculate the Jacobian of the transformation between (r_1, r_2) and $(r_1, \dot{r}_1)$, namely

(3.24) and this yields a Jacobi Last Multiplier of Eq. (3.25), i.e.[g]

$$M_1 = M_{[r]}\frac{\partial(r_1, r_2)}{\partial(r_1, \dot{r}_1)} = \exp[-(a+A)t]\frac{1}{\dot{r}_1 - Ar_1}. \tag{3.26}$$

Then a Lagrangian can be obtained by a double integration as in (2.12), i.e.

$$\begin{aligned} L_1 = {}& \exp[-(a+A)t]((\dot{r}_1 - Ar_1)\log(\dot{r}_1 - Ar_1) \\ & + m_1 b\exp(r_1) - ar_1\log(Bm_2) - ar_1) + \frac{\mathrm{d}}{\mathrm{d}t}F(t, r_1). \end{aligned} \tag{3.27}$$

The same Lagrangian (minus the gauge function F) was obtained in [42].

3.3. *Verhulst's model*

The Verhulst's model considered in [42] is the following:

$$\begin{aligned} \dot{w}_1 &= w_1(A + Bw_1 + f_1 w_2), \\ \dot{w}_2 &= w_2(a + bw_2 + f_2 w_1). \end{aligned} \tag{3.28}$$

In order to derive a Jacobi Last Multiplier for this system from (2.5), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(M_{[w]}) + (2B + f_2)w_1 + (2b + f_1)w_2 + a + A = 0 \tag{3.29}$$

we assume that $M_{[w]}$ has the following form:

$$M_{[w]} = w_1^{b_1} w_2^{b_2}\exp(b_3 t), \tag{3.30}$$

where $b_i, (i = 1, 2, 3)$ are constants to be determined. Replacing this $M_{[w]}$ into (3.29) yields

$$b_1 = \frac{-2Bb + bf_2 + f_1 f_2}{Bb - f_1 f_2}, \tag{3.31}$$

$$b_2 = \frac{-2Bb + Bf_1 + f_1 f_2}{Bb - f_1 f_2}, \tag{3.32}$$

$$b_3 = \frac{ABb - Abf_2 + aBb - aBf_1}{Bb - f_1 f_2}, \tag{3.33}$$

if $Bb - f_1 f_2 \neq 0$, and therefore if no condition is imposed on the parameters in Verhulst's model. Consequently a Lagrangian of system (3.28) is

$$\begin{aligned} L_{[w]} = {}& \exp(b_3 t)\left(w_2^{b_2} w_1^{b_1+1}\frac{\dot{w}_2}{b_1+1} - w_2^{b_2+1} w_1^{b_1}\frac{\dot{w}_1}{b_2+1}\right. \\ & \left. - w_2^{b_2+1} w_1^{b_1+1}\left(2\frac{f_2 w_1}{b_1+2} + 2\frac{bw_2}{b_1+1} + \frac{2a(b_2+1) + b_3}{(b_1+1)(b_2+1)}\right)\right) \\ & + \frac{\mathrm{d}}{\mathrm{d}t}G(t, w_1, w_2) \end{aligned} \tag{3.34}$$

that was not obtained in [42].

[g]Of course, we do not consider any multiplicative constants because they are inessential.

We follow [42] and introduce the change of variables[h]

$$w_1 = \exp(r_1), \quad w_2 = \exp(r_2) \tag{3.35}$$

and then system (3.28) becomes

$$\begin{aligned} \dot r_1 &= A + B\exp(r_1) + f_1\exp(r_2),\\ \dot r_2 &= a + b\exp(r_2) + f_2\exp(r_1). \end{aligned} \tag{3.36}$$

We can transform this system into an equivalent second-order ordinary differential equation by eliminating, say, r_2. In fact from the second equation in (3.36) one gets

$$r_2 = \log\left(\frac{\dot r_1 - B\exp(r_1) - A}{f_1}\right), \tag{3.37}$$

and the equivalent second-order equation in r_1 is the following

$$\begin{aligned} \ddot r_1 = \frac{1}{f_1}&[(af_1 + b\dot r_1)\dot r_1 + A^2 b + B\exp(2r_1)(Bb - f_1 f_2) - A(af_1 + 2b\dot r_1)\\ &- \exp(r_1)(f_1(aB - f_2\dot r_1) + B(2b - f_1)\dot r_1 - A(2bB - f_1 f_2))]. \end{aligned} \tag{3.38}$$

Using property (b) a Jacobi Last Multiplier for this equation can be obtained. In fact we have to calculate the Jacobian of the transformation between (w_1, w_2) and $(r_1, \dot r_1)$, and this yields a Jacobi Last Multiplier of Eq. (3.38), i.e.[i]

$$M_1 = M_{[w]}\frac{\partial(w_1, w_2)}{\partial(r_1, \dot r_1)} = \exp(b_1 r_1 + b_3 t)[\dot r_1 - A - B\exp(r_1)]^{b_2}(b_2 + 2)(b_2 + 1). \tag{3.39}$$

Then a Lagrangian can be obtained by a double integration as in (2.12), i.e.

$$L_1 = \exp(b_1 r_1)\exp(b_3 t)[\dot r_1 - A - B\exp(r_1)]^{b_2+2} + \frac{\mathrm{d}}{\mathrm{d}t}F(t, r_1). \tag{3.40}$$

The same Lagrangian (minus the gauge function F) was obtained in [42].

Since a Jacobi Last Multiplier of system (3.36) is

$$M_{[r]} = M_{[w]}\frac{\partial(w_1, w_2)}{\partial(r_1, r_2)} = \exp[(b_1 + 1)r_1 + (b_2 + 1)r_2 + b_3 t], \tag{3.41}$$

analogously a Lagrangian of system (3.36) is

$$\begin{aligned} L_{[r]} = \exp((b_1 + 1)r_1 + (b_2 + 1)r_2 + b_3 t)&\left(\frac{\dot r_2}{b_1 + 1} - \frac{\dot r_1}{b_2 + 1}\right.\\ &\left. - \left(2\frac{f_2\exp(r_1)}{b_1 + 2} + 2\frac{b\exp(r_2)}{b_1 + 1} + \frac{2a(b_2 + 1) + b_3}{(b_1 + 1)(b_2 + 1)}\right)\right) + \frac{\mathrm{d}}{\mathrm{d}t}G(t, r_1, r_2). \end{aligned} \tag{3.42}$$

This Lagrangian (minus the gauge function G) was also obtained in [42].

[h]It is not clear the reason of this change of variables that was performed in [42].

[i]Of course, we do not consider any multiplicative constants because they are inessential.

3.4. *Host-Parasite model*

As stated in [42], "a simple mathematical model which describes the interaction between a host and its parasite and which takes into account the nonlinear effects of the host population size on the growth rate of the parasite population is given by the equations [25]"

$$\begin{aligned}\dot{w}_1 &= (a - bw_2)w_1,\\ \dot{w}_2 &= \left(A - B\frac{w_2}{w_1}\right)w_2.\end{aligned} \tag{3.43}$$

As in the previous example it is easy to derive that a Jacobi Last Multiplier is

$$M_{[w]} = \frac{\exp(At)}{w_1 w_2^2}, \tag{3.44}$$

and consequently a Lagrangian of system (3.47)

$$\begin{aligned}L_{[w]} = \exp[At]\Bigg[&\log(w_1)\frac{\dot{w}_2}{w_2^2} + \frac{\dot{w}_1}{w_1 w_2} - 2\frac{a}{w_2} - 2\frac{B}{w_1}\\ &- \log(w_1)\frac{A}{w_2} - 2b\log(w_2)\Bigg] + \frac{\mathrm{d}}{\mathrm{d}t}G(t, w_1, w_2).\end{aligned} \tag{3.45}$$

This Lagrangian was obtained in [42].

In order to simplify system (3.43) we introduce the change of variables[j]

$$w_1 = r_1\exp(at), \quad w_2 = r_2\exp(At) \tag{3.46}$$

and then system (3.43) becomes

$$\begin{aligned}\dot{r}_1 &= -b\exp(At)r_1 r_2,\\ \dot{r}_2 &= -\frac{B\exp(At)r_2^2}{\exp(at)r_1}.\end{aligned} \tag{3.47}$$

Since a Jacobi Last Multiplier of system (3.47) is

$$M_{[r]} = M_{[w]}\frac{\partial(w_1, w_2)}{\partial(r_1, r_2)} = \frac{1}{r_1 r_2^2}, \tag{3.48}$$

analogously a Lagrangian of system (3.47) is

$$L_{[r]} = \frac{\log(r_1)\dot{r}_2}{r_2^2} + \frac{\dot{r}_1}{r_1 r_2} - 2\exp(At)\frac{br_1\log(r_2)\exp(at) + B}{r_1\exp(at)} + \frac{\mathrm{d}}{\mathrm{d}t}G(t, r_1, r_2). \tag{3.49}$$

This Lagrangian was obviously not obtained in [42].

[j]This change of variables was not performed in [42].

We can transform system (3.47) into an equivalent second-order ordinary differential equation by eliminating, say, r_2. In fact from the first equation in (3.47) one gets

$$r_2 = -\frac{\dot{r}_1}{b\exp(At)r_1}, \tag{3.50}$$

and the equivalent second-order equation in r_1 is the following

$$\ddot{r}_1 = \frac{b\exp(at)r_1 + B}{b\exp(at)r_1^2}\dot{r}_1^2 + A\dot{r}_1. \tag{3.51}$$

Using property (b) a Jacobi Last Multiplier for this equation can be obtained. In fact we have to calculate the Jacobian of the transformation between (r_1, r_2) and $(r_1, \dot{r}_1)$, and this yields a Jacobi Last Multiplier of Eq. (3.51), i.e.[k]

$$M_1 = M_{[r]}\frac{\partial(r_1, r_2)}{\partial(r_1, \dot{r}_1)} = \frac{b^2\exp(2At)r_1}{\dot{r}_1^2}\begin{vmatrix} 1 & 0 \\ \frac{\dot{r}_1}{b\exp(At)r_1^2} & -\frac{1}{b\exp(At)r_1} \end{vmatrix} = -\frac{b\exp(At)}{\dot{r}_1^2}. \tag{3.52}$$

Then a Lagrangian can be obtained by a double integration as in (2.12), i.e.

$$L_1 = b\exp(At)\log(\dot{r}_1) - b\exp(At)\log(r_1) + \frac{B\exp(At)}{\exp(at)r_1} + \frac{\mathrm{d}}{\mathrm{d}t}F(t, r_1). \tag{3.53}$$

This Lagrangian was not obtained in [42].

4. Volterra's Lagrangians

In this section we present the Volterra's Lagrangians for two biological equations [49] and show that they can be obtained by means of Jacobi Last Multiplier.

Volterra knew Jacobi's work, especially [19] that in 1887 he mentions in one of his earlier works [46, p. 280]. Therefore Volterra knew Lecture 10 of [19] since he cites p. 78 of precisely this Lecture in [46]. May be he did overlooked the following pages especially p. 82 where Jacobi wrote his formula (2.9) that links the last multiplier to the Lagrangian of any second-order equation. Also in his 1906 address at the Congress of Italian Naturalists [47] Volterra wrote: "una delle più celebri scoperte del matematico Jacobi, quella del principio dell'ultimo moltiplicatore".[l]

In 1939 Volterra wrote [49]: "I have been able to show that the equations of the struggle for existence depend on a question of Calculus of Variations (*omissis*). In order to obtain this result, I have replaced the notion of *population* by that of *quantity of life* [48]. In this manner I have also obtained some results by which dynamics is brought into relation to problems of the struggle for existence." The quantity of life X and the population N of a species are connected by the relation

$$N = \frac{\mathrm{d}X}{\mathrm{d}t}. \tag{4.1}$$

[k] Of course, we do not consider any multiplicative constants because they are inessential.
[l] "One of the most celebrated discovery by the mathematician Jacobi, that of the principle of the last multiplier" translated by M. C. Nucci.

It is immediately obvious that this idea of raising the order of each equation is totally different from that by Trubach and Franco who provided a method for finding a linear Lagrangian for systems of first-order equations. Also Volterra's method is different from that of deriving a single second-order equation from a system of two first-order equations: indeed Volterra takes a system of first-order equations and transform it into a system of second-order equations.

We now show that Volterra's Lagrangians for the two second-order equations that he considered in [49] can be obtained by means of Jacobi Last Multiplier (2.9). The first equation is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = a\prod_{i=1}^{n}(N - a_i) \tag{4.2}$$

that through (4.1) becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = a\prod_{i=1}^{n}\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_i\right), \tag{4.3}$$

and the second equation is the Verhulst–Pearl equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N(\varepsilon - \lambda N) \tag{4.4}$$

that through (4.1) becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = \frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right). \tag{4.5}$$

Substituting the unknown function with its first derivative in the original autonomous first-order ordinary differential equation yields a second-order ordinary differential equation which admits a two-dimensional abelian transitive Lie symmetry algebra generated by the operators of translations in both the independent and dependent variables. Therefore a Jacobi Last Multiplier is obtained and consequently a Lagrangian. In particular both Eqs. (4.3) and (4.5) admit a two-dimensional Lie symmetry algebra generated by the operators ∂_t and ∂_X. Then a Jacobi Last multiplier for (4.3) and (4.5) can be obtained by means of (2.7), i.e.

$$\Delta_{(4.3)} = \det\begin{bmatrix}1 & \frac{\mathrm{d}X}{\mathrm{d}t} & a\prod_{i=1}^{n}\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_i\right)\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \Rightarrow M_{(4.3)} = \frac{1}{\Delta_{(4.3)}} = \frac{1}{a\prod_{i=1}^{n}\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_i\right)} \tag{4.6}$$

and

$$\Delta_{(4.5)} = \det\begin{bmatrix}1 & \frac{\mathrm{d}X}{\mathrm{d}t} & \frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \Rightarrow M_{(4.5)} = \frac{1}{\Delta_{(4.5)}} = \frac{1}{\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)} \tag{4.7}$$

respectively, and consequently from Jacobi's formula (2.9) we obtain the Lagrangians

$$L_{(4.3)} = -\frac{1}{a}\sum_{i=1}^{n}\frac{1}{\prod_{j=1,j\neq i}^{n}(a_i - a_j)}\left(a_i - \frac{\mathrm{d}X}{\mathrm{d}t}\right)\log\left(a_i - \frac{\mathrm{d}X}{\mathrm{d}t}\right) + X, \tag{4.8}$$

and

$$L_{(4.5)} = \frac{1}{\varepsilon}\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \frac{1}{\varepsilon\lambda}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)\log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) + X \tag{4.9}$$

respectively, which are indeed the Volterra's Lagrangians in [49].

Equation (4.5) admits an eight-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{gathered}\Gamma_1 = \exp(\lambda X - \varepsilon t)\partial_t, \quad \Gamma_2 = \exp(\lambda X)\left(\partial_t + \frac{\varepsilon}{\lambda}\partial_X\right), \quad \Gamma_3 = \exp(-\lambda X + \varepsilon t)\partial_X,\\ \Gamma_4 = \exp(-\lambda X)\partial_X, \quad \Gamma_5 = \exp(\varepsilon t)\left(\frac{\lambda}{\varepsilon}\partial_t + \partial_X\right), \quad \Gamma_6 = \partial_X,\\ \Gamma_7 = \exp(-\varepsilon t)\partial_t, \quad \Gamma_8 = \partial_t.\end{gathered} \tag{4.10}$$

Therefore Eq. (4.5) is linearizable by means of a point transformation. In order to find the linearizing transformation we have to look for a two-dimensional abelian intransitive subalgebra, and, following Lie's classification of two-dimensional algebras in the real plane [27], we have to transform it into the canonical form

$$\partial_u, \quad y\partial_u$$

with u and y the new dependent and independent variables, respectively. We found that one such subalgebra is that generated by X_3 and X_4. Then it is easy to derive that

$$y = \exp(-\varepsilon t), \quad u = \frac{1}{\lambda}\exp(\lambda X - \varepsilon t) \tag{4.11}$$

and Eq. (4.5) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0. \tag{4.12}$$

As we have shown above the Volterra's Lagrangian (4.9) of Eq. (4.5) comes from the Jacobi Last Multiplier that can be obtained by means of (2.7) with the two symmetries Γ_8 and Γ_6 in (4.10). This Lagrangian (4.9) admits two Noether symmetries and therefore two first integrals of Eq. (4.5) can be derived by Noether's theorem [30], i.e.

$$\begin{aligned}&\Gamma_6 \Rightarrow \mathrm{In}_6 = \log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) - \log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \varepsilon t,\\ &\Gamma_8 \Rightarrow \mathrm{In}_8 = \log\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right) + \lambda X.\end{aligned} \tag{4.13}$$

Other nine Jacobi Last Multipliers and therefore Lagrangians can be obtained by means of (2.7) and any other combination of two symmetries in (4.10). The nine Jacobi Last Multipliers are:

$$\begin{aligned}
&\mathrm{JLM}_{14} = \frac{\exp(\varepsilon t)}{\lambda\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2}, \quad \mathrm{JLM}_{15} = -\frac{\varepsilon\exp(-\lambda X)}{\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2}, \quad \mathrm{JLM}_{17} = -\frac{\exp(2\varepsilon t-\lambda X)}{\lambda\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^3},\\
&\mathrm{JLM}_{18} = \frac{\exp(\varepsilon t-\lambda X)}{\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)}, \quad \mathrm{JLM}_{23} = \frac{\lambda\exp(-\varepsilon t)}{\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2}, \quad \mathrm{JLM}_{25} = \frac{\varepsilon\lambda\exp(-\varepsilon t-\lambda X)}{\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^3},\\
&\mathrm{JLM}_{34} = -\frac{\exp(-\varepsilon t-2\lambda X)}{\varepsilon}, \quad \mathrm{JLM}_{36} = -\frac{\exp(-\varepsilon t+\lambda X)}{\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}}, \quad \mathrm{JLM}_{37} = \frac{\exp(\lambda X)}{\frac{\mathrm{d}X}{\mathrm{d}t}},
\end{aligned} \tag{4.14}$$

and the indices indicate which two of the symmetries in (4.10) have been used. Consequently the nine Lagrangians are:

$$\begin{aligned}
\mathrm{Lag}_{14} &= -\exp(\varepsilon t)\left(\frac{1}{\lambda}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)+X\right),\\
\mathrm{Lag}_{15} &= \exp(-\lambda X)\left(\frac{1}{\varepsilon}\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)+\frac{1}{\varepsilon}\log\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t}-\varepsilon\right)\frac{\mathrm{d}X}{\mathrm{d}t}+\frac{1}{\lambda}\right),\\
\mathrm{Lag}_{17} &= -\frac{1}{2\lambda\frac{\mathrm{d}X}{\mathrm{d}t}}\exp(2\varepsilon t-\lambda X),\\
\mathrm{Lag}_{18} &= \frac{1}{\varepsilon^2}\exp(\varepsilon t-\lambda X)\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t}-\varepsilon\right)\left(\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)-\varepsilon\log\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t}-\varepsilon\right)\right),\\
\mathrm{Lag}_{23} &= -\frac{1}{\lambda}\exp(-\varepsilon X)\left(\log\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)+\lambda X\right),\\
\mathrm{Lag}_{25} &= \frac{\varepsilon\exp(-\varepsilon t-\lambda X)}{2\lambda\left(\varepsilon t-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)},\\
\mathrm{Lag}_{34} &= -\frac{1}{2\varepsilon}\exp(-\varepsilon t+2\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2,\\
\mathrm{Lag}_{36} &= \frac{1}{\lambda^2}\exp(-\varepsilon t+\lambda X)\left(\left(\lambda\frac{\mathrm{d}X}{\mathrm{d}t}-\varepsilon\right)\log\left(\varepsilon-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)-\lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),\\
\mathrm{Lag}_{37} &= \frac{1}{\varepsilon}\exp(\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\log\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)-\frac{\mathrm{d}X}{\mathrm{d}t}+\frac{\varepsilon}{\lambda}\right).
\end{aligned} \tag{4.15}$$

These Lagrangians admit a different number of Noether symmetries. The Lagrangians $\mathrm{Lag}_{17}, \mathrm{Lag}_{25}, \mathrm{Lag}_{34}$ admit five Noether symmetries, the possible higher number. For example the Lagrangian Lag_{34} in (4.15) yields the following five Noether symmetries and corresponding first integrals of Eq. (4.5)

$$\begin{aligned}
\Gamma_3 &\Rightarrow \mathrm{Int}_3 = \exp(\lambda X)\left(-\varepsilon + \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),\\
\Gamma_4 &\Rightarrow \mathrm{Int}_4 = \exp(-\varepsilon t + \lambda X)\frac{\mathrm{d}X}{\mathrm{d}t},\\
\Gamma_5 &\Rightarrow \mathrm{Int}_5 = \exp(2\lambda X)\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2,\\
\Gamma_6 + 2\frac{\lambda}{\varepsilon}\Gamma_8 &\Rightarrow \mathrm{Int}_6 = \exp(-\varepsilon t + 2\lambda X)\frac{\mathrm{d}X}{\mathrm{d}t}\left(\varepsilon - \lambda\frac{\mathrm{d}X}{\mathrm{d}t}\right),\\
\Gamma_7 &\Rightarrow \mathrm{Int}_7 = \exp(-2\varepsilon t + 2\lambda X)\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)^2.
\end{aligned} \tag{4.16}$$

Since in this paper we are mainly interested to show that the Volterra's Lagrangians $L_{(4.3)}$ and $L_{(4.5)}$ can be obtained by means of Jacobi Last Multiplier, the Noether symmetries of the other Lagrangians in (4.15) will be reported elsewhere.

About Eq. (4.3) we have found out that it possesses a third Lie point symmetry if all the a_i are equal, say $a_i = b, (i = 1, \ldots, n)$, i.e. the equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = a\left(\frac{\mathrm{d}X}{\mathrm{d}t} - b\right)^n \tag{4.17}$$

admits the following third symmetry

$$t\partial_t + \frac{1}{n-1}\left(bt + (n-2)X\right)\partial_X. \tag{4.18}$$

Then two other Jacobi Last Multipliers[m] can be obtained, i.e.

$$M_1 = \frac{n-1}{a\left(\frac{\mathrm{d}X}{\mathrm{d}t} - b\right)^n (bt + (n-2)X) + \left(\frac{\mathrm{d}X}{\mathrm{d}t} - b\right)\frac{\mathrm{d}X}{\mathrm{d}t}} \tag{4.19}$$

that comes from (2.7) with the two symmetries (4.18), ∂_t, and

$$M_2 = -\frac{n-1}{a(n-1)t\left(\frac{\mathrm{d}X}{\mathrm{d}t} - b\right) + \frac{\mathrm{d}X}{\mathrm{d}t} - b} \tag{4.20}$$

that comes from (2.7) with the two symmetries (4.18), ∂_X. Property (d) in Sec. 2 implies that the ratio of M_1 and M_2, i.e. M_1/M_2, is a first integral of Eq. (4.17), and since in this case

$$M_3 \equiv M_{(4.3)} = \frac{1}{a\left(\frac{\mathrm{d}X}{\mathrm{d}t} - b\right)^n}, \tag{4.21}$$

also M_1/M_3 and M_2/M_3 are first integrals[n] of Eq. (4.17).

[m] The corresponding Lagrangian cannot be obtained in closed form since the integral of both M_1 and M_2 with respect to $\frac{\mathrm{d}X}{\mathrm{d}t}$ can be evaluated for $n = 1, 2, 3$ only.
[n] Obviously only two of these three first integrals are functionally independent.

In the case when the right-hand side of Eq. (4.3) is either a second degree (i.e. all $a_j = 0, j > 2$) or a third degree (i.e. all $a_j = 0, j > 3$) polynomial then Eq. (4.3) admits an eight-dimensional Lie symmetry algebra and therefore is linearizable and many more Lagrangians can be determined as it was shown above for Eq. (4.5). In particular equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = a\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_1\right)\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_2\right), \tag{4.22}$$

admits an eight-dimensional Lie point symmetry algebra generated by the following operators:

$$\begin{gathered}\partial_t, \quad \partial_X, \quad \exp(a(a_1t - X))(\partial_t + a_2\partial_X), \quad \exp(a(a_2t - X))(\partial_t + a_1\partial_X),\\ \exp(-a(a_1t - X))\partial_X, \quad \exp(-a(a_2t - X))\partial_X,\\ \exp((a_1 - a_2)at)(\partial_t + a_2\partial_X), \quad \exp((a_1 - a_2)at)(\partial_t + a_1\partial_X),\end{gathered} \tag{4.23}$$

and equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}t^2} = a\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_1\right)\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_2\right)\left(\frac{\mathrm{d}X}{\mathrm{d}t} - a_3\right) \tag{4.24}$$

admits an eight-dimensional Lie point symmetry algebra generated by the following operators:

$$\begin{gathered}\partial_t, \quad \partial_X, \quad \exp(a(a_1 - a_2)(a_3t - X))(\partial_t + a_1\partial_X), \quad \exp(a(a_1 - a_3)(a_2t - X))(\partial_t + a_1\partial_X),\\ \exp(a(a_2 - a_1)(a_3t - X))(\partial_t + a_2\partial_X), \quad \exp(a(a_2 - a_3)(a_1t - X))(\partial_t + a_2\partial_X),\\ \exp(a(a_3 - a_1)(a_2t - X))(\partial_t + a_3\partial_X), \quad \exp(a(a_3 - a_2)(a_1t - X))(\partial_t + a_3\partial_X).\end{gathered} \tag{4.25}$$

Obviously both equations are linearizable. The transformation that takes Eq. (4.24) into the free particle Eq. (4.12) is

$$y = \exp(at(a_1 - a_2)), \quad u = -\frac{1}{a}\exp(a(a_1t - X)), \tag{4.26}$$

thanks to the two-dimensional abelian intransitive subalgebra generated by the two operators:

$$\exp(-a(a_1t - X))\partial_X, \quad \exp(-a(a_2t - X))\partial_X,$$

while that for Eq. (4.24) is

$$y = \exp(a(a_2 - a_3)(X - a_1t)), \quad u = \frac{\exp(a(a_1 - a_3)(X - a_2t))}{a(a_1 - a_2)(a_1 - a_3)}, \tag{4.27}$$

thanks to the two-dimensional abelian intransitive subalgebra generated by the two operators:

$$\exp(a(a_1 - a_2)(a_3t - X))(\partial_t + a_1\partial_X), \quad \exp(a(a_1 - a_3)(a_2t - X))(\partial_t + a_1\partial_X).$$

It is a simple application of (2.7) to find all the Jacobi Last Multipliers and then from (2.9) the corresponding Lagrangians for both Eqs. (4.22) and (4.24). Since it is outside the purpose of the present paper, we will report them elsewhere.

5. Final Remarks

This paper deals with Jacobi Last Multiplier and its connection to the inverse problem of calculus of variation for certain biological systems, namely finding one or more Lagrangians for either systems of two first-order equations or single second-order equations.

It was shown in [32] that Jacobi Last Multiplier yields the Lagrangian for any equation of even order[o]

$$u^{(2n)} = F(x, u, u', u'', \dots, u^{(2n-1)}), \tag{5.1}$$

since it can be derived from the following formula

$$M^{1/n} = \frac{\partial^2 L}{\partial (u^{(n)})^2}, \tag{5.2}$$

where M is the Jacobi Last Multiplier of Eq. (5.1) and L is its Lagrangian. This formula was given by Jacobi himself in [18, p. 364].

We recall that Fels has proven [8] that the Lagrangian is unique in the case of fourth-order equation if it exists. In the case of equations of sixth- and eighth-order the uniqueness was proven by Juráš [21].

In [41] Tonti provided a brief historical survey of the inverse problem of calculus of variations. He begins with the year 1887 when both Helmholtz [14] and Volterra [46] published their work. Unfortunately no mention is made of Jacobi's work. That historical survey should have at least begun — if not with the year 1845 when Jacobi's paper [18] was published in Crelle's journal — with the year 1884 when Jacobi's Dynamics Lectures, delivered at the University of Königsberg in the Winter Semester 1842–1843, were finally published [19] with a foreword by Weierstrass.

As pointed out by Tonti [41] many authors have dealt with the inverse problem of calculus of variations by either using a formal approach or an operatorial approach following on the steps of either Helmholtz or Volterra: for example [40, 29, 43, 38, 45, 1] and many others.

We do not underestimate the research of these very distinguished authors. Yet when possible we prefer to follow Jacobi since his Last Multiplier has a direct link to conservation laws and symmetries that are the essential elements that in our opinion make the difference between a mathematical abstraction and a physical concreteness.

We provide two examples of such a difference and may call them two missed opportunities [7].

The first example is the following ordinary differential equation of fourth-order

$$u^{(iv)} + 2uu'' + u'^2 = 0 \tag{5.3}$$

[o] We use a prime to indicate the derivative with respect to x.

that can be found in Olver's book [36, p. 364]. This equation comes from the Lagrangian

$$L = \frac{1}{2}u''^2 - uu'^2. \tag{5.4}$$

In [36] the homotopy formula was used in order to find the Lagrangian of Eq. (5.3). Instead of a Lagrangian of order two it yielded the following Lagrangian of order four

$$\frac{1}{2}uu^{(iv)} + \frac{2}{3}u^2u'' + \frac{1}{3}uu'^2 \tag{5.5}$$

that, as shown by Olver himself, while not the same as the original one, is still equivalent to it since it differs from (5.4) by a total derivative, i.e:

$$\frac{1}{2}uu^{(iv)} + \frac{2}{3}u^2u'' + \frac{1}{3}uu'^2 = \frac{1}{2}u''^2 - uu'^2 + D_t\left(\frac{1}{2}uu''' - \frac{1}{2}u'u'' + \frac{2}{3}u^2u'\right). \tag{5.6}$$

Instead the Jacobi Last Multiplier straightforwardly yields the Lagrangian (5.4). In fact, a Jacobi Last Multiplier of Eq. (5.3) is trivially a constant, say 1, thus formula (5.2) after two simple integrations, i.e.

$$L_J = \int\left(\int \sqrt{1}\mathrm{d}u''\right)\mathrm{d}u'' = \int u''\mathrm{d}u'' + F_1(x,u) = \frac{1}{2}u''^2 + F_1(x,u,u')u'' + F_2(x,u,u')$$
$$= \frac{1}{2}u''^2 + f(x,u,u') + \frac{\mathrm{d}G}{\mathrm{d}x}(x,u,u'), \tag{5.7}$$

yields (5.4). In fact imposing that the Euler–Lagrange equation of (5.7) be (5.3) implies the following equation for the function $f(x,u,u')$

$$-u'\frac{\partial f}{\partial u'} + \frac{\partial f}{\partial u} - \frac{\partial^2 f}{\partial x\partial u'} - u''\frac{\partial^2 f}{\partial u'^2} - 2uu'' - u'^2 = 0 \tag{5.8}$$

i.e. the overdetermined system

$$\frac{\partial^2 f}{\partial u'^2} + 2u = 0, \quad u'\frac{\partial f}{\partial u'} - \frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial x\partial u'} + u'^2 = 0, \tag{5.9}$$

and its solution[p] is indeed $f(x,u,u') = -uu'^2$. Other examples of the application of Jacobi's formula (5.2) to fourth-order equations can be found in [32].

The second example is the following system of two second-order ordinary differential equations

$$\ddot{u}_1 = -\frac{u_2}{u_1^2+u_2^2}, \quad \ddot{u}_2 = \frac{u_1}{u_1^2+u_2^2}. \tag{5.10}$$

that can be found in [1, 2]. In [34] it was shown that the method of the Jacobi Last Multiplier yields the Lagrangian

$$L = \frac{1}{2}(\dot{u}_1^2 + \dot{u}_2^2) - \arctan\left(\frac{u_2}{u_1}\right) + \frac{\mathrm{d}}{\mathrm{d}t}g(t,u_1,u_2) \tag{5.11}$$

[p]The additive linear function in u' can be omitted without loss of generality.

that is t-translational invariant while in [1] the method based on the variational bicomplex yielded the Lagrangian

$$L_1 = \frac{1}{2}(\dot{u}_1^2 + \dot{u}_2^2) + t\frac{u_2\dot{u}_1 - u_1\dot{u}_2}{u_1^2 + u_2^2} \tag{5.12}$$

that is not t-translational invariant. The two Lagrangians are obviously connected by giving a particular value to the otherwise arbitrary gauge function $g(t, u_1, u_2)$ in (5.11). More details can be found in [34].

We would like also to mention that many systems do not admit a Lagrangian. Nevertheless they may admit a Lagrangian if put in a different form as suggested by Bateman [3], namely "finding a set of equations equal in number to a given set, compatible with it and derivable from a variational principle". In [35] it was demonstrated how to construct many different Lagrangians for two famous examples which were deemed by Douglas [6] not to have a Lagrangian. Following Bateman's dictat different sets of equations compatible with those by Douglas and derivable from a variational principle were found.

Finally on systems of more than two first-order equations: the general formula was given by Trubatch and Franco [42] but no nontrivial examples were provided. Work is in progress to address this instance.

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