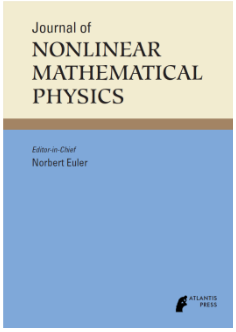

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INVARIANCE OF THE KAUP–KUPERSHMIDT EQUATION AND TRIANGULAR AUTO-BÄCKLUND TRANSFORMATIONS

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We report triangular auto-Bäcklund transformations for the solutions of a fifth-order evolution equation, which is a constraint for an invariance condition of the Kaup–Kupershmidt equation derived by E. G. Reyes in his paper titled "Nonlocal symmetries and the Kaup–Kupershmidt equation" [*J. Math. Phys.* **46** (2005) 073507, 19 pp.]. These auto-Bäcklund transformations can then be applied to generate solutions of the Kaup–Kupershmidt equation. We show that triangular auto-Bäcklund transformations result from a systematic multipotentialization of the Kupershmidt equation.

Keywords: Integrable evolution equations; auto-Bäcklund transformations; invariance.

Mathematics Subject Classification: 37K35, 35C05

1. Introduction

In his paper [7], Reyes reports an invariance of the Kaup–Kupershmidt equation

$$q_t = q_{xxxxx} + 5qq_{xxx} + \frac{25}{2}q_x q_{xx} + 5q^2 q_x, \tag{1.1}$$

by the following proposition.

Proposition 1 ([7]). *The Kaup–Kupershmidt equation*, (1.1), *is invariant under the transformation* $q \mapsto \bar{q}$, *in which*

$$\bar{q} = q + 3(\ln B)_{xx}, \tag{1.2}$$

where the variables q *and* B *are related by*

$$q(x,t) = -\frac{B_{xxx}}{B_x} + \frac{3}{4}\left(\frac{B_{xx}}{B_x}\right)^2 \tag{1.3}$$

and $B(x,t)$ is a solution to

$$B_t = B_{xxxxx} - 5\frac{B_{xx}B_{xxxx}}{B_x} - \frac{15}{4}\frac{B_{xxx}^2}{B_x} + \frac{65}{4}\frac{B_{xxx}B_{xx}^2}{B_x^2} - \frac{135}{16}\frac{B_{xx}^4}{B_x^3}. \tag{1.4}$$

In this letter we report a class of auto-Bäcklund transformations, known as the triangular auto-Bäcklund transformations (or $\triangle$-auto-Bäcklund transformations [5]), by a systematic multipotentialization of Eq. (1.4). Triangular auto-Bäcklund transformations have been defined and demonstrated in our recent paper [5]. For details on the multipotentialization procedure we refer the reader to our papers [2, 3]. We show that the equation,

$$v_t = v_{xxxxx} - 5\frac{v_{xx}v_{xxxx}}{v_x} + 5\frac{v_{xx}^2 v_{xxx}}{v_x^2} \tag{1.5}$$

plays a central role in the construction of solutions of the Kaup–Kupershmidt equation and consequently we find the general stationary solution of (1.5) by calculating its first integrals.

We remark that both Eqs. (1.4) and (1.5) first appeared in a paper by Weiss [9] on the Painlevé analysis in the form of singularity manifold constraints for the Kupershmidt equation (see [9, Eq. (3.32)]) and the Caudrey–Dodd–Gibbon equation (see [9, Eq. (3.21)]), respectively. For more details on the Painlevé analysis and singularity manifolds we refer the reader to [8]. Furthermore, both (1.4) and (1.5) are known symmetry-integrable equations and appear in [6] as part of the list of fifth-order semilinear integrable evolution equations (see [6, Eqs. (4.2.12) and (4.2.11)]). The recursion operators for both Eqs. (1.4) and (1.5) are given in [4].

2. Multipotentializations and $\triangle$-auto-Bäcklund Transformations

Let

$$u_t = F[u] \tag{2.1}$$

denote an evolution equation in u, where $F[u]$ denotes a given function that depends in general on x, t, u and x-derivatives of u. The procedure to potentialize (2.1) in the equation

$$v_t = G[v_x] \tag{2.2}$$

is well-known (see e.g. [1, 3]). Equation (2.2) is known as the potential equation of (2.1). For the benefit of the reader and to establish the notation we describe this procedure briefly: The potentialization of (2.1) in (2.2), if it exists, is established by a conserved current $\Phi^t[u]$ of (2.1) and the relation to the potential variable v is then

$$v_x = \Phi^t[u], \tag{2.3}$$

where

$$D_t\Phi^t[u] + D_x\Phi^x[u]\big|_{u_t=F[u]} = 0 \tag{2.4}$$

and Φ^x is the conserved flux of (2.1). Here D_t and D_x are the total t- and x-derivatives, respectively. Conserved currents, $\Phi^t[u]$, for (2.1) can be obtained by the relation

$$\Lambda[u] = \hat{E}[u]\Phi^t[u], \tag{2.5}$$

where Λ is an integrating factor (also called multiplier) of (2.1) that can be calculated by the relation

$$\hat{E}[u](\Lambda[u]u_t - \Lambda[u]F[u]) = 0. \tag{2.6}$$

Here $\hat{E}[u]$ is the Euler operator,

$$\hat{E}[u] = \frac{\partial}{\partial u} - D_x \circ \frac{\partial}{\partial u_x} - D_t \circ \frac{\partial}{\partial u_t} + D_x^2 \circ \frac{\partial}{\partial u_{2x}} - D_x^3 \circ \frac{\partial}{\partial u_{3x}} + \cdots .$$

A multipotentialization of (2.1) exists if (2.2) can also be potentialized. A $\triangle$-auto-Bäcklund transformation for (2.1) exists if (2.2) potentializes back into the original Eq. (2.1). In fact we have define three different types of $\triangle$-auto-Bäcklund transformations. Details are in [5], where several examples of $\triangle$-auto-Bäcklund transformations are given for some third-order and fifth-order evolution equations, as well as for systems in $(1+1)$ dimensions and higher-dimensional evolution equations.

Our starting point is the Kupershmidt equation

$$K_t = K_{xxxxx} + \lambda(K_x K_{xxx} + K_{xx}^2) - \frac{\lambda^2}{5}(K^2 K_{xxx} + 5K_x K_{xx} + K_x^3) + \frac{\lambda^4}{125}K^4 K_x \tag{2.7}$$

which potentializes under

$$U_x = K \tag{2.8}$$

in the first potential Kupershmidt equation,

$$U_t = U_{xxxxx} + \lambda U_{xx} U_{xxx} - \frac{\lambda^2}{5}(U_x^2 U_{xxx} + U_x U_{xx}^2) + \left(\frac{\lambda}{5}\right)^4 U_x^5, \tag{2.9}$$

and, in turn again potentializes under

$$u_x = -\frac{5}{2\lambda}\exp\left(-\frac{2\lambda}{5}U\right) \tag{2.10}$$

in a second-potential Kupershmidt equation,

$$u_t = u_{xxxxx} - \frac{5u_{xx}u_{xxxx}}{u_x} - \frac{15u_{xxx}^2}{4u_x} + \frac{65u_{xx}^2 u_{xxx}}{u_x^2} - \frac{135u_{xx}^4}{16u_x^3}. \tag{2.11}$$

In addition, the first potential Kupershmidt equation (2.9) also potentializes in

$$v_t = v_{xxxxx} - 5\frac{v_{xx}v_{xxxx}}{v_x} + 5\frac{v_{xx}^2 v_{xxx}}{v_x^2} \tag{2.12}$$

under

$$v_x = \frac{5}{\lambda}\exp\left(\frac{\lambda}{5}U\right). \tag{2.13}$$

A connection between the Kupershmidt equation, (2.7), and Eq. (2.11) is given by the differential substitution

$$K(x,t) = -\frac{5}{2\lambda}\frac{u_{xx}}{u_x}, \tag{2.14}$$

which is obtained by combining $U_x = K$ and $u_x = -5/(2\lambda)\exp(-2U/(5\lambda))$.

Furthermore, we can connect the first potential Kupershmidt equation, (2.9), and the Kaup–Kupershmidt equation

$$Q_t = Q_{xxxx} + \mu Q Q_{xxx} + \frac{5\mu}{2} Q_x Q_{xx} + \frac{\mu^2}{5} Q^2 Q_x \tag{2.15}$$

for arbitrary constant $\mu \neq 0$, by the differential substitution (given in [6] for the special case, $\mu = 5$ and $\lambda = 5$)

$$Q = \left(\frac{2\lambda}{\mu}\right) U_{xx} - \left(\frac{\lambda}{5\mu}\right) U_x^2. \tag{2.16}$$

We now focus our attention on Eq. (2.11). For this equation the most general second-order integrating factors are

$$\Lambda_1[u] = \alpha u_x^{-5/2} u_{xx}, \quad \Lambda_2[u] = \alpha(u u_x^{-5/2} u_{xx} - 2u_x^{-1/2}), \tag{2.17}$$

which leads to two potentializations in (2.12) with $\alpha = -3/4$, namely

$$v_{1,x} = u_x^{-1/2}, \quad v_{2,x} = u u_x^{-1/2}, \tag{2.18}$$

respectively. See the diagram below. Furthermore, the most general second-order integrating factors of Eq. (2.12) are

$$\Lambda_n[v] = v^n \frac{v_{xx}}{v_x^2} - \frac{n}{2} v^{n-1} \frac{1}{v_x^2}, \quad n = 0, 1, 2, 3, 4. \tag{2.19}$$

The integrating factor Λ_0 and Λ_4 lead to the potentialization of Eq. (2.12) in equation (2.11), namely

$$u_{3,x} = v_x^{-2}, \quad u_{4,x} = v^4 v_x^{-2}, \tag{2.20}$$

respectively. See the diagram below.

Diagram:

$$\begin{tikzcd}
\text{Eq. (2.11) in } u_1 \arrow[dr, "v_x=u_{1,x}^{-1/2}"] & & \text{Eq. (2.11) in } u_2 \arrow[dl, "v_x=u_2u_{2x}^{-1/2}"'] \\
& \boxed{v_t = v_{xxxxx} - 5v_x^{-1} v_{xx} v_{xxxx} + 5v_x^{-2} v_{xx}^2 v_{xxx}} \arrow[dl, "u_{3,x}=v_x^{-2}"'] \arrow[dr, "u_{4,x}=v^4v_x^{-2}"] & \\
\text{Eq. (2.11) in } u_3 & & \text{Eq. (2.11) in } u_4
\end{tikzcd}$$

Combining the above potentializations for the Eqs. (2.11) and (2.12), leads to the following $\triangle$-auto-Bäcklund transformations for these equations:

Proposition 2. *The Eq.* (2.11), *viz.*

$$u_t = u_{xxxxx} - 5\frac{u_{xx}u_{xxxx}}{u_x} - \frac{15}{4}\frac{u_{xxx}^2}{u_x} + \frac{65}{4}\frac{u_{xxx}u_{xx}^2}{u_x^2} - \frac{135}{16}\frac{u_{xx}^4}{u_x^3}$$

admits the following △-auto-Bäcklund transformations:

$$u_{j+1,xx} = u_{j+1,x}\left[\frac{u_{j,xx}}{u_{j,x}} - 2\frac{u_{j,x}}{u_j}\right] + 4\,u_{j+1,x}^{3/4}\left[\frac{u_j^{1/2}}{u_{j,x}^{1/4}}\right], \tag{2.21a}$$

$$u_{j+1,xx} = u_{j+1,x}\left[\frac{u_{j,xx}}{u_{j,x}}\right] + 4u_{j+1,x}^{3/4}\left[\frac{1}{u_{j,x}^{1/4}}\right], \tag{2.21b}$$

where u_j and u_{j+1} satisfy (2.11) *for all natural numbers j.*

Proposition 3. *The Eq.* (2.12), *viz.*

$$v_t = v_{xxxxx} - 5\frac{v_{xx}v_{xxxx}}{v_x} + 5\frac{v_{xx}^2 v_{xxx}}{v_x^2},$$

admits the following △-auto-Bäcklund transformations:

$$v_{j+1,xx} = v_{j+1,x}\left[\frac{v_{j,xx}}{v_{j,x}}\right] - \frac{1}{v_{j,x}}, \tag{2.22a}$$

$$v_{j+1,xx} = v_{j+1,x}\left[\frac{v_{j,xx}}{v_{j,x}}\right] + \frac{1}{v_{j,x}}, \tag{2.22b}$$

$$v_{j+1,xx} = v_{j+1,x}\left[\frac{v_{j,xx}}{v_{j,x}} - 2\frac{v_{j,x}}{v_j}\right] + \frac{v_j^2}{v_{j,x}}, \tag{2.22c}$$

where v_j and v_{j+1} satisfy (2.12) *for all natural numbers j.*

Remark 4. We note that in addition to the transformations given by Propositions 2 and 3, both (2.11) and (2.12) admit the symmetry transformation

$$u_{j+1} = -\frac{1}{u_j} + \text{constant}, \quad v_{j+1} = -\frac{1}{v_j} + \text{constant} \tag{2.23}$$

which are included in the diagram: combine $v_x = u_{1,x}^{-1/2}$ with $v_x = u_2 u_{2,x}^{-1/2}$, and $u_x = v_{1,x}^{-2}$ with $u_x = v_2^4 v_{2,x}^{-2}$, respectively. Note that these relations were also identified by Weiss (see relation [9, (3.37)]).

Remark 5. The △-auto-Bäcklund transformations (2.21a) and (2.21b) given in Proposition 2 can be linearized, as those are in the form of first-order Bernoulli equations under the substitution $u_{j+1,x} = w_{j+1}$.

3. Solutions

In the above diagram it is clear that Eq. (2.12), viz.

$$v_t = v_{xxxxx} - \frac{5v_{xx}v_{xxxx}}{v_x} + \frac{5v_{xx}^2 v_{xxx}}{v_x^2},$$

plays a central role in the connection between the Kupershmidt, the Kaup–Kupershmidt and Eq. (2.11).

We now find the most general stationary solution for Eq. (2.12) by calculating the first integrals and the general solution of the ordinary differential equation

$$v_{xxxxxx} - \frac{5v_{xx}v_{xxxxx}}{v_x} + \frac{5v_{xx}^2 v_{xxx}}{v_x^2} = 0. \tag{3.1}$$

We find that Eq. (3.1) admits five second-order integrating factors, $\{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\}$:

$$\Lambda_n(x, v, v_x, v_{xx}) = a_n \frac{v_{xx}}{v_x^2} - \frac{a_n'}{2}\frac{1}{v_x^2}, \tag{3.2}$$

where

$$a_n(v) = v^n \quad \text{with } n = 0, 1, 2, 3, 4.$$

Here the primes denote derivatives with respect to v. The corresponding first five integrals are then

$$\begin{aligned} I_n = &\left(a_n \frac{v_{xx}}{v_x^4} - \frac{1}{2}\frac{a_n'}{v_x^2}\right) v_{xxxx} - \frac{1}{2} a_n \frac{v_{xxx}^2}{v_x^4} + \left(\frac{1}{2} a_n' \frac{v_{xx}}{v_x^3} - a_n \frac{v_{xx}^2}{v_x^5} + \frac{1}{2}\frac{a_n''}{v_x}\right) v_{xxx} \\ &- \frac{1}{2} a_n''' v_{xx} + \frac{1}{4} a_n^{(4)} v_x^2, \quad n = 0, 1, 2, 3, 4. \end{aligned}$$

Combining $I_0,\ I_1,\ I_2,\ I_3$ and I_4, we obtain

$$v_x = (I_0 v^4 - 4I_1 v^3 + 6I_2 v^2 - 4I_3 v + I_4)^{1/2},$$

with the condition

$$I_0 I_4 - 4I_1 I_3 + 3I_2^2 = 0.$$

So the general solution of (3.1) is given by the quadrature

$$\int (I_0 v^4 - 4I_1 v^3 + 6I_2 v^2 - 4I_3 v + I_4)^{-1/2} dv = x + C, \tag{3.3}$$

where C is a constant of integration. Since the quadrature (3.3) contains five independent free constants, it represents the most general stationary solution of (2.12). This solution can now be used to generate nonstationary solutions for (2.12) by the $\triangle$-auto-Bäcklund transformations of Proposition 3 and, by applying the transformations in the above diagram, solutions of (2.11) can be constructed. Then by Proposition 1, the so constructed solutions of (2.11) lead to solutions of the Kaup–Kupershmidt equation, as well as solutions of the Kupershmidt equation by the differential substitution (2.14).

For example, by (3.3) with $I_0 = I_1 = I_2 = I_4 = C = 0$ and $I_3 = -2$ we obtain the solution

$$v(x, t) = 2x^2$$

for (2.12) and hence the solution

$$u(x,t) = 16x$$

for (2.11). Applying now, for example, the $\triangle$-auto-Bäcklund transformation (2.21a) leads to the solution

$$u(x,t) = \frac{1}{7}x^7 - \frac{4}{5}c_1x^5 + 2c_1^2x^3 - 4c_1^3x - c_1^4x^{-1} - 576c_1t + c_2,$$

where c_1 and c_2 are arbitrary constants. Applying again (2.21a), setting $c_1 = c_2 = 0$ for simplicity, we obtain the following solution for (2.11):

$$u(x,t) = \frac{1}{5^4 \cdot 7 \cdot 91}x^{13} + \frac{36}{5^3 \cdot 7}x^8t + \frac{2^7 \cdot 3^4}{5^2}x^3t^2 - 2^{12} \cdot 3^8 \cdot 7x^{-7}t^4 - \frac{2^{10} \cdot 3^6 \cdot 7}{5}x^{-2}t^3.$$

Proposition 1 can now be applied with the above solutions of (2.11) (put $u(x,t) \equiv B(x,t)$ in (1.2) and (1.3)) to gain solutions for the Kaup–Kupershmidt equation (1.1). Furthermore, solutions of the Kupershmidt equation, (2.7), are then given by the differential substitution (2.14).

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References

[1] C. S. Anco and G. W. Bluman, Direct construction method for conservation laws of partial differential equations Part II: General treatment, *European J. Appl. Math.* **13** (2002) 567–585.

[2] N. Euler and M. Euler, On nonlocal symmetries, nonlocal conservation laws and nonlocal transformations of evolution equations: Two linearisable hierarchies, *J. Nonlinear Math. Phys.* **16** (2009) 489–504.

[3] N. Euler and M. Euler, Multipotentialisation and iterating-solution formulae: The Krichever–Novikov equation, *J. Nonlinear Math. Phys.* **16**(Suppl. 1) (2009) 93–106.

[4] M. Euler and N. Euler, A class of semilinear fifth-order evolution equations: Recursion operators and multipotentialisations, *J. Nonlinear Math. Phys.* **18**(Suppl. 1) (2011) 61–75.

[5] N. Euler and M. Euler, The converse problem for the multipotentialisation of evolution equations and systems, *J. Nonlinear Math. Phys.* **18**(Suppl. 1) (2011) 77–105.

[6] A. V. Mikhailov, A. B. Shabat and V. V. Sokolov, The symmetry approach to classification of integrable equations, in *What is Integrability?*, ed. E. V. Zhakarov (Springer, 1991), pp. 115–184.

[7] E. G. Reyes, Nonlocal symmetries and the Kaup–Kupershmidt equation, *J. Math. Phys.* **46** (2005) 073507, 19 pp.

[8] W.-H. Steeb and N. Euler, *Nonlinear Evolution Equations and Painlevé Test* (World Scientific, 1988).

[9] J. Weiss, On classes of integrable systems and the Painlevé property, *J. Math. Phys.* **25** (1984) 13–24.