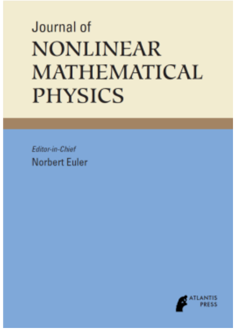

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BILLIARD ALGEBRA, INTEGRABLE LINE CONGRUENCES, AND DOUBLE REFLECTION NETS

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Billiard systems within quadrics, playing the role of discrete analogues of geodesics on ellipsoids, are incorporated into the theory of integrable quad-graphs. An initial observation is that the Six-pointed star theorem, as the operational consistency for the billiard algebra, is equivalent to an integrability condition of a line congruence. A new notion of the double reflection nets as a subclass of dual Darboux nets associated with pencils of quadrics is introduced, basic properties and several examples are presented. Corresponding Yang–Baxter maps, associated with pencils of quadrics are defined and discussed.

Keywords: Ellipsoidal billiard; confocal quadrics; quad-graphs; integrability.

Mathematics Subject Classification 2000: 37J35, 51N05, 52C35, 14H40, 14H70

1. Introduction

Although its historic roots may be traced back to Newtons *Principia*, discrete differential geometry emerged as a modern scientific discipline quite recently (see [6]), within a study of lattice geometry, where so-called integrability or consistency conditions for quad-graphs have been playing a fundamental role.

Geodesics on an ellipsoid present one of the most important and exciting basic examples of classical differential geometry. Billiard systems within quadrics (see [11] for a detailed study), being discretization of systems of geodesics on ellipsoids, should be seen as an important part of the foundations of discrete differential geometry.

The main aim of the present paper is to incorporate billiard systems within quadrics into the story of integrable quad-graphs. In Sec. 2, necessary notions of the theory of integrable systems on quad-graphs from works of Adler, Bobenko, Suris [1, 4, 6] are reviewed. In Sec. 3, we recall definition and main properties of Double reflection configuration from [9, 10] and in Proposition 3.9 we show that this configuration can play the geometric counterpart role of the quad-equation. In Sec. 4, we use the billiard algebra, which is developed in our paper [10] as an algebraic tool for synthetic realization of higher-genera addition theorems. Our main observation there is that the Six-pointed star theorem, i.e. Theorem 4.1 of this paper, on the consistency of a certain projective configuration, as the operational consistency for the billiard algebra, is equivalent to the integrability condition of a line congruence. In Sec. 5, a new notion of double reflection nets is introduced. We provide four illustrative examples of double reflection nets: two are based on the Poncelet–Darboux grids from [10], third one on our study of ellipsoidal billiards in pseudo-Euclidean spaces (see [12]), and in the last one we give a general construction of a double reflection net. After this, F-transformations and focal nets for double reflection nets are discussed. We show that double reflection nets induce a subclass of the Grassmannian Darboux nets from [3] associated with pencils of quadrics. As we know after Schief (see [18]) the Darboux nets are associated to discrete integrable hierarchies. In Sec. 6, we conclude by definition and study of corresponding Yang–Baxter maps, associated with pencils of quadrics, leading to a natural generalization of the situation analyzed in [2].

2. Line Congruences and Quad-Graphs Integrability

Now, we will start with basic ideas of the theory of integrable systems on quad-graphs from works of Adler, Bobenko, Suris [1, 4, 6].

Recall that the basic building blocks of systems on quad-graphs are equations on quadrilaterals of the form

$$Q(x_1, x_2, x_3, x_4) = 0, \tag{2.1}$$

where Q is a multiaffine polynomial, that is a polynomial of degree one in each argument.

Equations of type (2.1) are called *quad-equations*. The field variables x_i are assigned to four vertices of a quadrilateral as in Fig. 1. Equation (2.1) can be solved for each variable, and the solution is a rational function of the other three variables. A solution (a_1, a_2, a_3, a_4) of Eq. (2.1) is *singular* with respect to x_i if it also satisfies the equation:

$$\frac{\partial Q}{\partial x_i}(a_1, a_2, a_3, a_4) = 0.$$

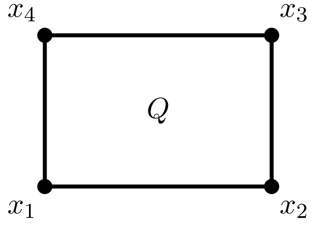

Fig. 1. Elementary quadrilateral; quad-equation $Q(x_1, x_2, x_3, x_4) = 0$.

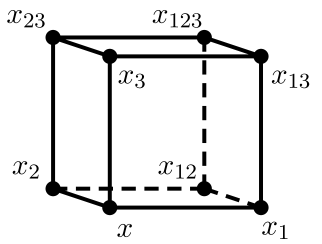

Fig. 2. 3D-consistency.

Following [4], we consider the idea of integrability as consistency, see Fig. 2. We assign six quad-equations to the faces of coordinate cube:

$$Q(x, x_1, x_{12}, x_2) = 0, \quad Q(x, x_1, x_{13}, x_3) = 0, \quad Q(x, x_2, x_{23}, x_3) = 0,$$
$$Q(x_1, x_{12}, x_{123}, x_{13}) = 0, \quad Q(x_2, x_{12}, x_{123}, x_{23}) = 0, \quad Q(x_3, x_{13}, x_{123}, x_{23}) = 0.$$

Notice that, for given x, x_1, x_2, x_3, values of x_{12}, x_{13}, x_{23} can be determined from the first three of these quad-equations. In general, the three values for x_{123} determined from the remaining three quad-equations may be distinct.

The system is said to be 3D-*consistent* if the three values for x_{123} obtained from these equations coincide for arbitrary initial data x, x_1, x_2, x_3.

We will be interested in geometric version of integrable quad-graphs, with lines in $\mathbf{P}^d$ playing the role of vertex fields. More precisely, denote by $\mathcal{L}^d$ the Grassmannian $\mathrm{Gr}(2, d+1)$ of two-dimensional vector subspaces of the $(d+1)$-dimensional vector space, $d \geq 2$.

For a map

$$\ell : \mathbf{Z}^m \to \mathcal{L}^d,$$

we denote: $\ell_i : \mathbf{Z}^m \to \mathcal{L}^d$, such that $\ell_i(u) = \ell(u + \mathbf{e}_i)$ for each $u \in \mathbf{Z}^m$, where $\mathbf{e}_i \in \mathbf{Z}^m$ is the ith unit coordinate vector, $i \in \{1, \ldots, m\}$.

Following [6], we say that ℓ is *a line congruence* if neighboring lines $\ell(u)$ and $\ell_i(u)$ intersect for each $u \in \mathbf{Z}^m$ and $i \in \{1, \ldots, m\}$.

Given a line congruence $\ell : \mathbf{Z}^m \to \mathcal{L}^d$, one may define its ith *focal net* as the map

$$F^{(i)} : \mathbf{Z}^m \to \mathbf{P}^{d+1}, \quad F^{(i)}(u) = \ell(u) \cap \ell_i(u).$$

Two given line congruences $\ell, \ell^+ : \mathbf{Z}^m \to \mathcal{L}^d$, are said to be related by an *F-transformation* if for each $u \in \mathbf{Z}^m$ the corresponding lines $\ell(u)$ and $\ell^+(u)$ intersect.

As we mentioned before, in the case of algebraic quad-graphs, there is a multiaffine function Q, which determines the fourth vertex of a quadrilateral, according to the other three. In the sequel, we are going to introduce the geometric configuration of double reflection, which is going to play the role of a multiaffine function Q in the case of line congruences we are going to study. In other words, given three lines in a configuration, the fourth line will be uniquely determined, as it will be proved in Proposition 3.9 at the end of Sec. 3. Moreover, using properties of billiard algebra developed in [10], we will prove that such a quad-relation is 3D-consistent, see Theorem 4.2.

3. Double Reflection Configuration

In this section, we review a fundamental projective geometry configuration of double reflection in the d-dimensional projective space $\mathbf{P}^d$ over an arbitrary field of characteristic not equal to 2. Detailed discussion on this can be found in [10, 11] (see also [7]).

The section is concluded by Proposition 3.9, where we show that double reflection configuration can take the role of the quad-equation, that is every line in such a configuration is determined by the remaining three.

Let us start with recalling the notions of quadrics and confocal families in the projective space.

A quadric in $\mathbf{P}^d$ is the set given by equation of the form:

$$(Q\xi, \xi) = 0,$$

where Q is a symmetric $(d+1) \times (d+1)$ matrix, and $\xi = [\xi_0 : \xi_1 : \cdots : \xi_d]$ are homogeneous coordinates of a point in the space.

Assume two quadrics are given:

$$\mathcal{Q}_1 : (Q_1\xi, \xi) = 0, \quad \mathcal{Q}_2 : (Q_2\xi, \xi) = 0.$$

A pencil of quadrics is the family of quadrics given by equations:

$$((Q_1 + \lambda Q_2)\xi, \xi) = 0, \quad \lambda \in \mathbf{P}^1.$$

A confocal system of quadrics is a family of quadrics such that its projective dual is a pencil of quadrics.

Now, let us recall the definition of reflection off a quadric in the projective space, where metrics is not defined. This definition, together with its crucial properties — the One reflection theorem and the Double reflection theorem, can be found in [7].

Denote by u the tangent plane to $\mathcal{Q}_1$ at point x and by z the pole of u with respect to $\mathcal{Q}_2$. Suppose lines ℓ_1 and ℓ_2 intersect at x, and the plane containing these two lines meet u along ℓ.

Definition 3.1. If lines ℓ_1, ℓ_2, xz, ℓ are coplanar and harmonically conjugated, we say that ℓ_1 *is reflected to* ℓ_2 off quadric $\mathcal{Q}_1$.

It can be proved that this definition does not depend on the choice of quadric $\mathcal{Q}_2$ from a given confocal system [7].

If we introduce a coordinate system in which quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are confocal in the usual sense, the reflection introduced by Definition 3.1 is the same as the standard, metric one.

Theorem 3.2 (One reflection theorem). *Suppose line ℓ_1 is reflected to ℓ_2 off $\mathcal{Q}_1$ at point x, with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let ℓ_1 intersect $\mathcal{Q}_2$ at y_1' and y_1, u be the tangent plane to $\mathcal{Q}_1$ at x, and z the pole of u with respect to $\mathcal{Q}_2$. Then lines $y_1'z$ and y_1z respectively contain intersecting points y_2' and y_2 of line ℓ_2 with $\mathcal{Q}_2$. The converse is also true.*

Corollary 3.3. *Let lines ℓ_1 and ℓ_2 reflect to each other off $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then ℓ_1 is tangent to $\mathcal{Q}_2$ if and only if ℓ_2*

is tangent to $\mathcal{Q}_2$; ℓ_1 *intersects* $\mathcal{Q}_2$ *at two points if and only if* ℓ_2 *intersects* $\mathcal{Q}_2$ *at two points.*

Next theorem is crucial for our further considerations.

Theorem 3.4 (Double reflection theorem). *Suppose that* $\mathcal{Q}_1$, $\mathcal{Q}_2$ *are given quadrics and* $x_1 \in \mathcal{Q}_1$, $y_1 \in \mathcal{Q}_2$. *Let* u_1 *be the tangent plane of* $\mathcal{Q}_1$ *at* x_1; z_1 *the pole of* u_1 *with respect to* $\mathcal{Q}_2$; v_1 *the tangent plane of* $\mathcal{Q}_2$ *at* y_1; *and* w_1 *the pole of* v_1 *with respect to* $\mathcal{Q}_1$. *Denote by* x_2 *the intersecting point of line* w_1x_1 *with* $\mathcal{Q}_1$, $x_2 \neq x_1$; *by* y_2 *the intersection of* y_1z_1 *with* $\mathcal{Q}_2$, $y_2 \neq y_1$; *and* $\ell_1 = x_1y_1$, $\ell_2 = x_1y_2$, $\ell'_1 = y_1x_2$, $\ell'_2 = x_2y_2$.

Then pair ℓ_1, ℓ_2 *obey the reflection law off* $\mathcal{Q}_1$ *at* x_1; ℓ_1, ℓ'_1 *obey the reflection law off* $\mathcal{Q}_2$ *at* y_1; ℓ_2, ℓ'_2 *obey the reflection law off* $\mathcal{Q}_2$ *at* y_2; *and* ℓ'_1, ℓ'_2 *obey the reflection law off* $\mathcal{Q}_1$ *at point* x_2.

Let us remark that in Theorem 3.4 the four tangent planes at the reflection points belong to a pencil.

Corollary 3.5. *If line* ℓ_1 *is tangent to a quadric* $\mathcal{Q}'$ *confocal with* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$, *then* ℓ_2, ℓ'_1, ℓ'_2 *also touch* $\mathcal{Q}'$.

The following definition of virtual reflection configuration and double reflection configuration is from [10]. These configurations played the central role in that work. In Theorem 3.7, some important properties of these configurations are given. This theorem is proved in [10].

The configurations are connected with so-called real and virtual reflections, but their properties remain in the projective case, too.

Let points x_1, x_2 belong to $\mathcal{Q}_1$ and y_1, y_2 to $\mathcal{Q}_2$.

Definition 3.6. We will say that the quadruple of points x_1, x_2, y_1, y_2 constitutes *a virtual reflection configuration* if pairs of lines x_1y_1, x_1y_2; x_2y_1, x_2y_2; x_1y_1, x_2y_1; x_1y_2, x_2y_2 satisfy the reflection law at points x_1, x_2 off $\mathcal{Q}_1$ and y_1, y_2 off $\mathcal{Q}_2$ respectively, with respect to the confocal system determined by $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If, additionally, the tangent planes to $\mathcal{Q}_1, \mathcal{Q}_2$ at x_1, x_2; y_1, y_2 belong to a pencil, we say that these points constitute *a double reflection configuration* (see Fig. 3).

Now, we list some of the basic facts about double reflection configurations.

Theorem 3.7. *Let* $\mathcal{Q}_1, \mathcal{Q}_2$ *be two quadrics in the projective space* $\mathbf{P}^d$, x_1, x_2 *points on* $\mathcal{Q}_1$ *and* y_1, y_2 *on* $\mathcal{Q}_2$. *If the tangent hyperplanes at these points to the quadrics belong to a pencil, then* x_1, x_2, y_1, y_2 *constitute a virtual reflection configuration.*

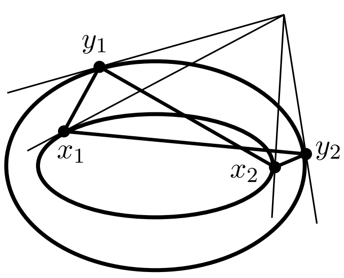

Fig. 3. Double reflection configuration.

Furthermore, suppose that the projective space is defined over the field of reals. Introduce a coordinate system, such that $\mathcal{Q}_1, \mathcal{Q}_2$ become confocal ellipsoids in the Euclidean space. If $\mathcal{Q}_1$ is placed inside $\mathcal{Q}_2$, then the sides of the quadrilateral $x_1y_1x_2y_2$ obey the real reflection from $\mathcal{Q}_2$ and the virtual reflection from $\mathcal{Q}_1$.

The statement converse to Theorem 3.7 is the following proposition.

Proposition 3.8. *In the Euclidean space $\mathbf{E}^d$, two confocal ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ are given. Let points X_1, X_2 belong to $\mathcal{E}_1$, Y_1, Y_2 to $\mathcal{E}_2$, and let α_1, α_2, β_1, β_2 be the corresponding tangent planes. If a quadruple X_1, X_2, Y_1, Y_2 is a virtual reflection configuration, then planes α_1, α_2, β_1, β_2 belong to a pencil.*

Proposition 3.9 shows that three lines of a double reflection configuration uniquely determine the fourth one.

Proposition 3.9. *Let ℓ, ℓ_1, ℓ_2 be lines and $\mathcal{Q}_1, \mathcal{Q}_2$ quadrics in the projective space. Suppose that ℓ, ℓ_1 reflect to each other off $\mathcal{Q}_1$, and ℓ, ℓ_2 off $\mathcal{Q}_2$, with respect to the confocal system determined by these two quadrics. Then there is a unique line ℓ_{12} such that four lines $\ell, \ell_1, \ell_2, \ell_{12}$ form a double reflection configuration.*

Proof. Let α be the tangent hyperplane to $\mathcal{Q}_1$ at the intersection point of ℓ and ℓ_1, and β the tangent hyperplane to $\mathcal{Q}_2$ at the intersection point of ℓ and ℓ_2. In the pencil of hyperplanes determined by α and β, take hyperplanes α_1, β_1 tangent to $\mathcal{Q}_1$, $\mathcal{Q}_2$ respectively, $\alpha_1 \neq \alpha$, $\beta_1 \neq \beta$. By Theorem 3.7, ℓ_{12} will be the line containing the touching points of α_1 and β_1 with the corresponding quadrics. □

Remark 3.10. Proposition 3.9 shows that double reflection configuration is playing the role of the quad-equation for lines in the projective space.

In Sec. 4, we are going to show that double reflection configuration is 3D-consistent.

4. Billiard Algebra and Construction of Quad-Graphs, and 3D-Consistency

In this section we make a quad-graph interpretation of some results obtained using billiard algebra. This algebra is constructed by the authors — for details of the construction, see [10, 11].

Let us start from a theorem on confocal families of quadrics from [10]:

Theorem 4.1 (Six-pointed star theorem). *Let $\mathcal{F}$ be a family of confocal quadrics in $\mathbf{P}^3$. There exist configurations consisting of twelve planes in $\mathbf{P}^3$ with the following properties:*

- *The planes may be organized in eight triplets, such that each plane in a triplet is tangent to a different quadric from $\mathcal{F}$ and the three touching points are collinear. Every plane in the configuration is a member of two triplets.*
- *The planes may be organized in six quadruplets, such that the planes in each quadruplet belong to a pencil and are tangent to two different quadrics from $\mathcal{F}$. Every plane in the configuration is a member of two quadruplets.*

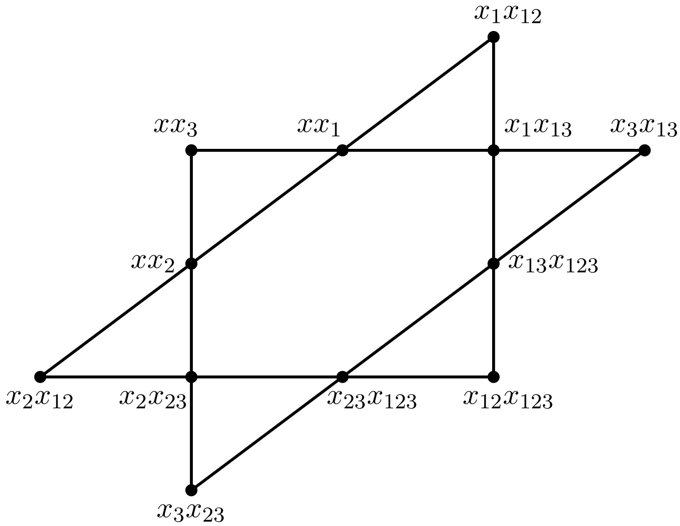

Fig. 4. A configuration of planes from Theorem 4.1.

Moreover, such a configuration is determined by three planes tangent to three different quadrics from $\mathcal{F}$, with collinear touching points.

Such a configuration of planes in the dual space $\mathbf{P}^{3*}$ is shown in Fig. 4: each plane corresponds to a vertex of the polygonal line.

To understand the notation used in Fig. 4, let us recall the construction leading to configurations from Theorem 4.1. Take $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ to be quadrics from $\mathcal{F}$, and α, β, γ respectively their tangent planes such that the touching points A, B, C are collinear. Denote by x the line containing these three points, and by x_1, x_2, x_3 the lines obtained from x by reflections off $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ at A, B, C respectively.

Now, as in Proposition 3.9, determine lines x_{12}, x_{13}, x_{23}, x_{123} such that they respectively complete triplets $\{x, x_1, x_2\}$, $\{x, x_1, x_3\}$, $\{x, x_2, x_3\}$, $\{x_3, x_{13}, x_{23}\}$ to double reflection configurations.[a]

Notice the following objects in Fig. 4:

Twelve vertices to each vertex, a plane tangent to one of the three quadrics $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and a pair of lines are assigned — the lines of any pair are reflected to each other off the quadric at the touching point with the assigned plane;

Eight triangles in any triangle, the planes assigned to the vertices are touching the corresponding quadrics at three collinear points — thus to each triangle, the line containing these points is naturally assigned;

Six edges each edge contains four vertices — four planes assigned to these vertices are in the same pencil; thus a double reflection configuration corresponds to each edge.

[a]Let us note that in [10], the lines x, x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} were respectively denoted by $\mathcal{O}$, p, q, s, $-x$, p_1, q_1, $x+s$, where the addition is defined in the billiard algebra introduced in that paper.

Now, we are ready to prove the 3D-consistency of the quad-relation introduced via double reflection configurations.

Theorem 4.2. *Let x, x_1, x_2, x_3 be lines in the projective space, such that x_1, x_2, x_3 are obtained from x by reflections off confocal quadrics $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ respectively. Introduce lines $x_{12}, x_{13}, x_{23}, x_{123}$ such that the following quadruplets are double reflection configurations*:

$$\{x, x_1, x_{12}, x_2\}, \quad \{x, x_1, x_{13}, x_3\}, \quad \{x, x_2, x_{23}, x_3\}, \quad \{x_1, x_{12}, x_{123}, x_{13}\}.$$

Then the following quadruplets are also double reflection configurations:

$$\{x_2, x_{12}, x_{123}, x_{23}\}, \quad \{x_3, x_{13}, x_{123}, x_{23}\}.$$

Proof. Let us remark that the configuration described in Theorem 4.1 has obviously a combinatorial structure of a cube, with planes corresponding to the edges of the cube. In this way, lines x, x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123} will correspond to the vertices of the cube as shown in Fig. 2. A pair of lines is represented by endpoints of an edge if they reflect to each other off the plane joined to this edge. Faces of the cube represent double reflection configurations. Notice also that planes joined to parallel edges of the cube are tangent to the same quadric. The statement follows from Theorem 4.1 and the construction given after, see Fig. 4. □

5. Double Reflection Nets

Assume a family of confocal quadrics is given in $\mathbf{P}^d$. Notice that, by the Chasles theorem [8], every line in $\mathbf{P}^d$ touches $d-1$ quadrics from the family.

Moreover, by Corollary 3.5, these $d-1$ quadrics are preserved by the billiard reflection. Confocal quadrics touched by a line are called *caustics* of this line, or consequently, caustics of the billiard trajectory that contains the line.

Now, fix $d-1$ quadrics from the pencil and take $\mathcal{A} \subset \mathcal{L}^d$ to be the set of all lines touching these $d-1$ quadrics.

Definition 5.1. *A double reflection net* is a map

$$\varphi : \mathbf{Z}^m \to \mathcal{A}, \tag{5.1}$$

such that there exist m quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ from the confocal pencil, satisfying the following conditions:

(1) sequence $\{\varphi(\mathbf{n}_0 + i\mathbf{e}_j)\}_{i\in\mathbf{Z}}$ represents a billiard trajectory within $\mathcal{Q}_j$, for each $j \in \{1, \dots, m\}$ and $\mathbf{n}_0 \in \mathbf{Z}^m$;
(2) lines $\varphi(\mathbf{n}_0)$, $\varphi(\mathbf{n}_0+\mathbf{e}_i)$, $\varphi(\mathbf{n}_0+\mathbf{e}_j)$, $\varphi(\mathbf{n}_0+\mathbf{e}_i+\mathbf{e}_j)$ form a double reflection configuration, for all $i, j \in \{1, \dots, m\}$, $i \neq j$ and $\mathbf{n}_0 \in \mathbf{Z}^m$.

In other words, for each edge in $\mathbf{Z}^m$ of direction $\mathbf{e}_i$, the lines corresponding to its vertices intersect at $\mathcal{Q}_i$, while the four tangent planes at the intersection points, associated to an elementary quadrilateral, belong to a pencil.

In the following Secs. 5.1–5.4, we describe some examples of double reflection nets. After that, in Sec. 5.5, we construct F-transformations of double reflection nets and conclude this section by establishing connection with the Grassmannian Darboux nets from [3] in Sec. 5.6.

5.1. *Example of a double reflection net in the Minkowski space*

Consider three-dimensional Minkowski space $\mathbf{E}^{2,1}$, that is $\mathbf{R}^3$ with the Minkowski scalar product:

$$\langle (x_1, y_1, z_1), (x_2, y_2, z_2)\rangle = x_1x_2 + y_1y_2 - z_1z_2.$$

In this space, let a general confocal family be given (see [14]):

$$\mathcal{Q}_\lambda : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c+\lambda} = 1, \quad \lambda \in \mathbf{R}, \quad (a > b > 0, c > 0), \tag{5.2}$$

In this family, one can observe four different geometric types of non-degenerated quadrics:

- one-sheeted hyperboloids oriented along z-axis, for $\lambda \in (-\infty, -c)$;
- ellipsoids, corresponding to $\lambda \in (-c, b)$;
- one-sheeted hyperboloids oriented along y-axis, for $\lambda \in (b, a)$ (see Fig. 5);
- two-sheeted hyperboloids, for $\lambda \in (a, +\infty)$ — these hyperboloids are oriented along z-axis.

In addition, there are four degenerated quadrics: $\mathcal{Q}_a$, $\mathcal{Q}_b$, $\mathcal{Q}_{-c}$, $\mathcal{Q}_\infty$, that is planes $x = 0$, $y = 0$, $z = 0$, and the plane at the infinity respectively.

On each quadric, notice the *tropic curves* — the set of points where the induced metrics on the tangent plane is degenerate.

It is known that (see [12]):

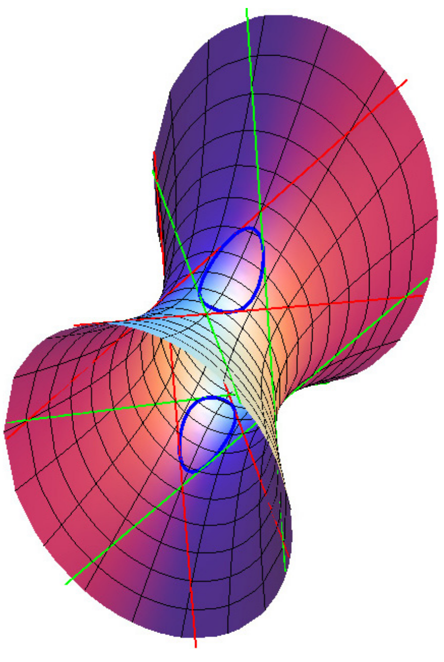

Fig. 5. The tropic curves and their light-like tangents on a one-sheeted hyperboloid oriented along y-axis from confocal family (5.2).

- a tangent line to the tropic curve of a non-degenerate quadric of the family (5.2) is always space-like, except on a one-sheeted hyperboloid oriented along y-axis;
- tangent lines of a tropic on one-sheeted hyperboloids oriented along y-axis are light-like exactly at four points, while at other points of the tropic curve, the tangents are space-like;
- a tangent line to the tropic of a quadric from (5.2) belongs to the quadric if and only if it is light-like;
- each one-sheeted hyperboloid oriented along y-axis has exactly eight light-like generatrices, as shown in Fig. 5.

Fix $\lambda_0 \in (b, a)$, and consider hyperboloid $\mathcal{Q}_{\lambda_0}$. Denote by a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 the light-like generatrices of $\mathcal{Q}_{\lambda_0}$, in the following way (see Fig. 6):

- lines a_i belong to one, and b_i to the other family of generatrices of $\mathcal{Q}_{\lambda_0}$; that is, a_i and a_j are always skew for $i \neq j$, while a_i and b_j are coplanar for all i, j;
- a_1, a_2, b_3, b_4 are tangent to the tropic curve contained in the half-space $z > 0$, while a_3, a_4, b_1, b_2 are touching the other tropic curve;
- a_i is parallel to b_i for each i;
- pairs (a_1, b_2), (a_2, b_1), (a_3, b_4), (a_4, b_3) have intersection points in the xy-plane;
- pairs (a_1, b_3), (a_2, b_4), (a_3, b_1), (a_4, b_2) have intersection points in the xz-plane;
- pairs (a_1, b_4), (a_2, b_3), (a_3, b_2), (a_4, b_1) have intersection points in the yz-plane.

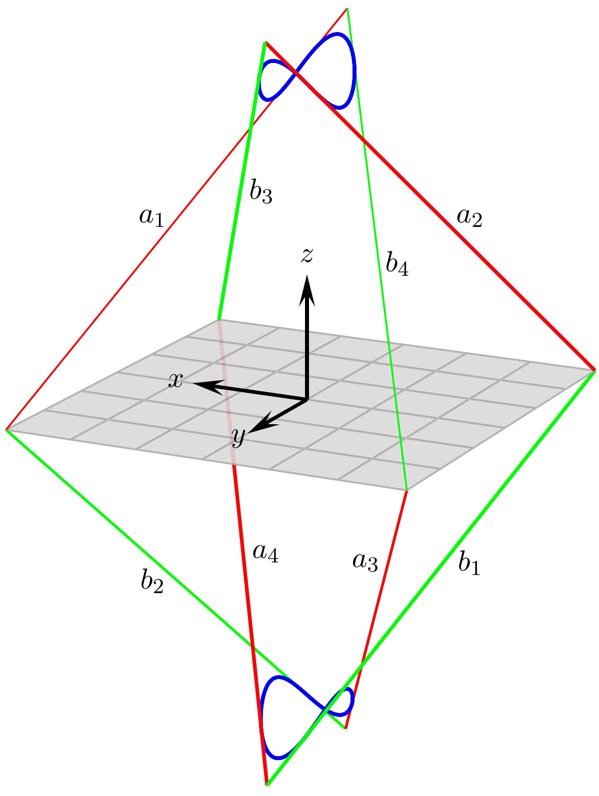

Fig. 6. The tropic curves of $\mathcal{Q}_{\lambda_0}$ and their light-like tangents.

Take $\mathcal{A}$ to be the set of all generatrices of hyperboloid $\mathcal{Q}_{\lambda_0}$, i.e. the set of all lines having $\mathcal{Q}_{\lambda_0}$ as a double caustic. In particular, $\mathcal{A}$ contains all lines a_i, b_i.

It is possible to define a map

$$\varphi_M : \mathbf{Z}^4 \to \mathcal{A},$$

such that the image of φ_M is the set $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ and for each $\mathbf{n} \in \mathbf{Z}^4$ lines $\varphi_M(\mathbf{n} + \mathbf{e}_1)$, $\varphi_M(\mathbf{n} + \mathbf{e}_2)$, $\varphi_M(\mathbf{n} + \mathbf{e}_3)$, $\varphi_M(\mathbf{n} + \mathbf{e}_4)$ are obtained from $\varphi_M(\mathbf{n})$ by reflection off $\mathcal{Q}_a$, $\mathcal{Q}_b$, $\mathcal{Q}_{-c}$, $\mathcal{Q}_\infty$ respectively.

More precisely, φ_M will be periodic with period 2 in each coordinate and:

$$\begin{aligned}
&\varphi_M(0,0,0,0) = \varphi_M(1,1,1,1) = a_1, \quad &&\varphi_M(1,1,0,0) = \varphi_M(0,0,1,1) = a_2,\\
&\varphi_M(1,0,1,0) = \varphi_M(0,1,0,1) = a_3, \quad &&\varphi_M(0,1,1,0) = \varphi_M(1,0,0,1) = a_4,\\
&\varphi_M(0,0,0,1) = \varphi_M(1,1,1,0) = b_1, \quad &&\varphi_M(1,1,0,1) = \varphi_M(0,0,1,0) = b_2,\\
&\varphi_M(1,0,1,1) = \varphi_M(0,1,0,0) = a_3, \quad &&\varphi_M(0,1,1,1) = \varphi_M(1,0,0,0) = b_4,
\end{aligned}$$

It is shown in Fig. 7 how vertices of the unit tesseract in $\mathbf{Z}^4$ are mapped by φ_M.

It is straightforward to prove the following proposition.

Proposition 5.2. *φ_M is a double reflection net.*

5.2. *Poncelet–Darboux grids and double reflection nets*

Let $\mathcal{E}$ be an ellipse in the Euclidean plane:

$$\mathcal{E} : \frac{x^2}{a} + \frac{y^2}{b}, \quad a > b > 0,$$

and $(a_i)_{i\in\mathbf{Z}}$ a billiard trajectory within $\mathcal{E}$.

As it is well-known, all lines a_i are touching the same conic $\mathcal{C}$ confocal with $\mathcal{E}$. Here, we will additionally suppose that $\mathcal{C}$ is an ellipse. Denote by $\mathcal{A}$ the set of tangents of $\mathcal{C}$.

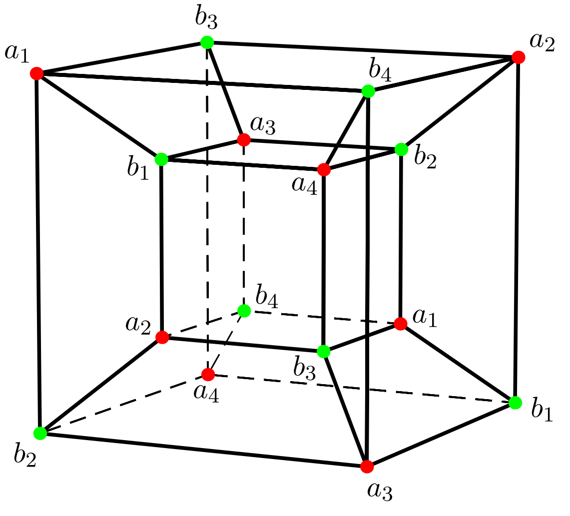

Fig. 7. Mapping φ_M on the unit tesseract.

Fix m positive integers $k_1, \ldots, k_m$ and define the mapping:

$$\varphi_D : \mathbf{Z}^m \to \mathcal{A}, \quad \varphi_D(n_1, \ldots, n_m) = a_{n_1 k_1 + \cdots + n_m k_m}.$$

Proposition 5.3. *Map φ_D is a double reflection net.*

Proof. Since $\varphi_D(\mathbf{n} + i\mathbf{e}_j) = a_{n_1 k_1 + \cdots + n_m k_m + i k_j}$, ($\mathbf{n} = (n_1, \ldots, n_m)$), it follows by [12, Theorem 18] that sequence $(\varphi_D(\mathbf{n} + i\mathbf{e}_j))_{i \in \mathbf{Z}}$ represents a billiard trajectory within some ellipse $\mathcal{E}_j$, confocal with $\mathcal{E}$ and $\mathcal{C}$.

Immediately, by Definition 3.6, lines $\varphi(\mathbf{n}_0)$, $\varphi(\mathbf{n}_0 + \mathbf{e}_i)$, $\varphi(\mathbf{n}_0 + \mathbf{e}_j)$, $\varphi(\mathbf{n}_0 + \mathbf{e}_i + \mathbf{e}_j)$ form a virtual reflection configuration for each $\mathbf{n}_0 \in \mathbf{Z}^m$, $i, j \in \{1, \ldots, m\}$.

Moreover, by Proposition 3.8, they also form a double reflection configuration. $\square$

Remark 5.4. It is interesting to consider only nets where m ellipses $\mathcal{E}_j$ appearing in the proof of Proposition 5.3 are distinct. If some of them coincide, then we may consider a corresponding subnet.

Suppose that (a_i) is a non-periodic trajectory. Then, choosing any m, and any set of distinct positive numbers $k_1, \ldots, k_m$, we get substantially different double reflection nets.

For (a_i) being n-periodic, it is enough to consider the case $k_i = i$, $i \in \{1, \ldots, [n/2]\}$, $(m = [n/2])$.

Example 5.5. Suppose (a_i) is a 5-perodic billiard trajectory within $\mathcal{E}$, see Fig. 8.

The corresponding double reflection net is:

$$\varphi_D : \mathbf{Z}^2 \to \mathcal{A}, \quad \varphi_D(n_1, n_2) = a_{n_1 + 2n_2}.$$

5.3. *s-skew lines and double reflection nets*

Now, let us consider a family of confocal quadrics in $\mathbf{E}^d$ $(d \geq 3)$ and fix its $d-1$ quadrics. As usually, $\mathcal{A}$ is the set of all lines tangent to the fixed quadrics.

It is shown in [10] that, from a line in $\mathcal{A}$, we can obtain any other line from that set in at most $d-1$ reflections on quadrics from the confocal family. We called lines a, b from $\mathcal{A}$ *s-skew* if s is the smallest number such that they can be obtained by $s+1$ such reflections.

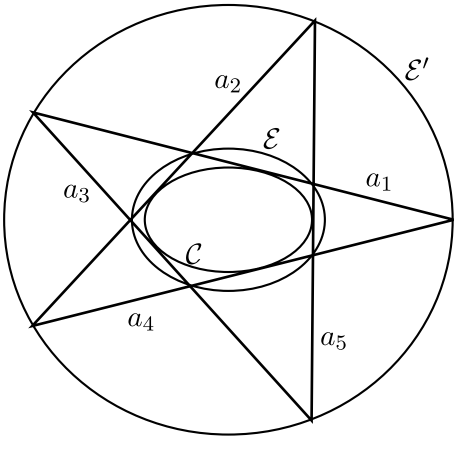

Fig. 8. A Poncelet pentagon.

Now, suppose lines a, b are s-skew ($s \ge 1$), and let $\mathcal{Q}_1, \dots, \mathcal{Q}_{s+1}$ be the corresponding quadrics from the confocal family.

Theorem 5.6. *There is a unique double reflection net*

$$\varphi_s : \mathbf{Z}^{s+1} \to \mathcal{A}$$

which satisfies the following:

- $\varphi_s(0, \dots, 0) = a$;
- $\varphi_s(1, \dots, 1) = b$;
- $\{\varphi(\mathbf{n}_0 + i\mathbf{e}_j)\}_{i \in \mathbf{Z}}$ *represents a billiard trajectory within* $\mathcal{Q}_j$, *for each* $j \in \{1, \dots, s+1\}$ *and* $\mathbf{n}_0 \in \mathbf{Z}^{s+1}$.

Proof. First, using the construction of the billiard algebra from [10], we are going to define mapping φ_s on $\{0,1\}^{s+1}$.

For a permutation $\mathbf{p} = (p_1, \dots, p_{s+1})$ of the set $\{1, \dots, s+1\}$, we take a sequence of lines $(\ell_0^{\mathbf{p}}, \dots, \ell_{s+1}^{\mathbf{p}})$ such that $\ell_0^{\mathbf{p}} = a$, $\ell_s^{\mathbf{p}} = b$, and $\ell_{i-1}^{\mathbf{p}}$, $\ell_i^{\mathbf{p}}$ satisfy the reflection law off $\mathcal{Q}_{p_i}$ for each $i \in \{1, \dots, s+1\}$. Such a sequence exists and it is unique. Moreover, if $k \in \{1, \dots, s+1\}$ is given, and permutations $\mathbf{p}$, $\mathbf{p}'$ coincide in the first k coordinates, then $\ell_i^{\mathbf{p}} = \ell_i^{\mathbf{p}'}$ for $i \le k$. Take $\{i_1, \dots, i_k\}$ to be a subset of $\{1, \dots, s+1\}$, and $\mathbf{p}$ any permutation of set $\{1, \dots, s+1\}$ with $p_1 = i_1, \dots, p_k = i_k$. We define:

$$\varphi_s(\chi(1), \dots, \chi(s+1)) = \ell_k^{\mathbf{p}},$$

where $\chi = \chi_{\{i_1, \dots, i_k\}}$ is the corresponding characteristic function on $\{1, \dots, s+1\}$:

$$\chi : \{1, \dots, s+1\} \to \{0,1\}, \quad \chi(j) = \begin{cases} 1, & j \in \{i_1, \dots, i_k\}; \\ 0, & j \notin \{i_1, \dots, i_k\}. \end{cases}$$

In this way, we constructed φ_s on $\{0,1\}^{s+1}$.

Subsequently, φ_s can be extended to the rest of $\mathbf{Z}^{s+1}$, so that $\{\varphi_s(\mathbf{n}_0 + i\mathbf{e}_j)\}_{i \in \mathbf{Z}}$ will represent billiard trajectories within $\mathcal{Q}_j$.

This construction is correct and unique due to Theorem 4.2. □

5.4. *Construction of double reflection nets*

Let $\mathcal{Q}_1, \dots, \mathcal{Q}_m$ be distinct quadrics belonging to a confocal family and ℓ a line in $\mathbf{P}^d$. Let us choose lines ℓ_i satisfying with ℓ the reflection law off $\mathcal{Q}_i$, $1 \le i \le m$.

Theorem 5.7. *There is a unique double reflection net* $\varphi : \mathbf{Z}^m \to \mathcal{A}_\ell$, *with the following properties*:

- $\varphi(0, \dots, 0) = \ell$;
- $\varphi(\mathbf{e}_i) = \ell_i$, *for each* $i \in \{1, \dots, m\}$.

By $\mathcal{A}_\ell$, *we denoted the set of all lines in* $\mathbf{P}^d$ *touching the same* $d-1$ *quadrics from the confocal family as* ℓ.

Proof. First, we define φ on $\{0,1\}^m$, from the condition that lines corresponding to each 2-face of the unit cube need to form a double reflection configuration. This construction is unique because of Proposition 3.9 and correct, due to the 3D-consistency property proved in Theorem 4.2.

At all other points of $\mathbf{Z}^m$, φ is uniquely defined from the request that $\{\varphi(\mathbf{n}_0 + i\mathbf{e}_j)\}_{i\in\mathbf{Z}}$ will be billiard trajectories within $\mathcal{Q}_j$.

Consistency of the construction follows again from Theorem 4.2. □

5.5. *Focal nets and F-transformations of double reflection nets*

Let $\varphi : \mathbf{Z}^m \to \mathcal{A}$ be a double reflection net.

For given $\mathbf{n}_0 \in \mathbf{Z}^m$ and distinct indices $i, j, k \in \{1, \dots, m\}$, consider the following points of its ith focal net:

$$\begin{aligned} F_i &= F^{(i)}(\mathbf{n}_0) = \varphi(\mathbf{n}_0) \cap \varphi(\mathbf{n}_0 + \mathbf{e}_i),\\ F_{ij} &= F^{(i)}(\mathbf{n}_0 + \mathbf{e}_j) = \varphi(\mathbf{n}_0 + \mathbf{e}_j) \cap \varphi(\mathbf{n}_0 + \mathbf{e}_j + \mathbf{e}_i),\\ F_{ik} &= F^{(i)}(\mathbf{n}_0 + \mathbf{e}_k) = \varphi(\mathbf{n}_0 + \mathbf{e}_k) \cap \varphi(\mathbf{n}_0 + \mathbf{e}_k + \mathbf{e}_i),\\ F_{ijk} &= F^{(i)}(\mathbf{n}_0 + \mathbf{e}_j + \mathbf{e}_k) = \varphi(\mathbf{n}_0 + \mathbf{e}_j + \mathbf{e}_k) \cap \varphi(\mathbf{n}_0 + \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_i). \end{aligned}$$

Proposition 5.8. *Points $F_i, F_{ij}, F_{ik}, F_{ijk}$ are coplanar.*

Proof. This is a consequence of the theorem of focal nets from [6]. However, we will show the direct proof, from configurations considered in Sec. 4.

The four points belong to quadric $\mathcal{Q}_i$. The tangent planes to $\mathcal{Q}_i$ at these points, divided into two pairs, determine two pencils of planes. According to Theorem 4.1, the two pencils are coplanar; thus they intersect. As a consequence, the lines of poles with respect to the quadric $\mathcal{Q}_i$, which correspond to these two pencils of planes, also intersect. It follows that the four points are coplanar. □

We are going to construct an F-transformation of the double reflection net.

First, we select a quadric $\mathcal{Q}_\delta$ from the confocal family and introduce line ℓ' which satisfies with $\varphi(\mathbf{n}_0)$ the reflection law on $\mathcal{Q}_\delta$.

By Theorem 5.7, it is possible to construct a double reflection net $\bar{\varphi} : \mathbf{Z}^{m+1} \to \mathcal{A}$, such that:

- $\bar{\varphi}(\mathbf{n}, 0) = \varphi(\mathbf{n})$;
- $\bar{\varphi}(\mathbf{n}_0, 1) = \ell'$.

Now, we define:

$$\varphi^+ : \mathbf{Z}^m \to \mathcal{A}_\ell, \quad \varphi^+(\mathbf{n}) = \bar{\varphi}(\mathbf{n}, 1).$$

Proposition 5.9. *Map φ^+ is an F-transformation of φ.*

Proof. Lines $\varphi^+(\mathbf{n})$ and $\varphi(\mathbf{n})$ intersect, since they satisfy the reflection law off $\mathcal{Q}_\delta$. □

5.6. *Double reflection nets and Grassmannian Darboux nets*

Let us recall the definition of a Grassmannian Darboux net from [3]: a map from the edges of a regular square lattice $\mathbf{Z}^m$ to the Grassmannian $\mathbf{G}_r^d$ of r-dimensional projective subspaces of the d-dimensional projective space is *a Grassmannian Darboux net* if the four r-spaces of an elementary quadrilateral belong to a $(2r+1)$-space. For $r = 0$, the ordinary Darboux nets from [18] are obtained, where the four points of intersection associated to a quadrilateral, belong to a line.

Using the isomorphism between the Grassmannian $\mathbf{G}_r^d$ and Grassmannian $\mathrm{Gr}(r+1, d+1)$ of $(r+1)$-dimensional vector subspaces of the $(d+1)$-dimensional vector space and following the analytical description of the Darboux nets from [3], we come to the equations:

$$x_j^i = \rho^{ij} x^i + (I - \rho^{ij}) x^j, \tag{5.3}$$

where x are $(d+1, r+1)$-matrices with appropriate normalization, and ρ are invertible $(r+1, r+1)$-matrices with I as the identity. Developing further, we come to

$$x_{jk}^i = \rho_k^{ij}(\rho^{ik} x^i + (I - \rho^{ik}) x^k) + (I - \rho_k^{ij})(\rho^{jk} x^j + (I - \rho^{jk}) x^k),$$

and finally to a condition for a matrix-valued one-form to be closed

$$\rho_k^{ij} \rho^{ik} = \rho_j^{ik} \rho^{ij},$$

see [3]. Moreover, the closed one-form ρ can be represented as an exact one: $\rho^{ij} = s_j^i (s^i)^{-1}$ and the rotation coefficients are $b^{ij} = ((s^i)^{-1} - (s_j^i)^{-1}) s^j$.

Now, consider a general double reflection net (5.1).

To each edge $(\mathbf{n_0}, \mathbf{n_0} + \mathbf{e}_i)$ of $\mathbf{Z}^m$, we can associate the plane which is tangent to $\mathcal{Q}_i$ at point $\varphi(\mathbf{n_0}) \cap \varphi(\mathbf{n_0} + \mathbf{e}_i)$. Since the lines corresponding to the vertices of a face form a double reflection configuration, the four planes associated to the edges belong to a pencil.

In this way, we see that a double reflection net induces a map:

$$E(\mathbf{Z}^m) \to \mathbf{G}_{d-1}^d,$$

where $E(\mathbf{Z}^m)$ is the set of all edges of the integer lattice $\mathbf{Z}^m$.

In this way, double reflection nets induce a subclass of dual Darboux nets.

It was shown in [18] how to associate discrete integrable hierarchies to the Darboux nets.

6. Yang–Baxter Map

A Yang–Baxter map is a map $R : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$, satisfying the Yang–Baxter equation:

$$R_{23} \circ R_{13} \circ R_{12} = R_{12} \circ R_{13} \circ R_{23},$$

where $R_{ij} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X} \times \mathcal{X}$ acts as R on the ith and jth factor in the product, and as the identity on the remaining one, see [2] and references therein.

Here, we are going to construct an example of Yang–Baxter map associated to confocal families of quadrics. To begin, we fix a family of confocal quadrics in $\mathbf{CP}^n$:

$$\mathcal{Q}_\lambda : \frac{z_1^2}{a_1 - \lambda} + \cdots + \frac{z_d^2}{a_d - \lambda} = z_{n+1}^2, \tag{6.1}$$

where $a_1, \ldots, a_d$ are constants in $\mathbf{C}$, and $[z_1 : z_2 : \cdots : z_{n+1}]$ are homogeneous coordinates in $\mathbf{CP}^n$.

Take $\mathcal{X}$ to be the space $\mathbf{CP}^{n*}$ dual to the n-dimensional projective space, i.e. the variety of all hyperplanes in $\mathbf{CP}^n$. Note that a general hyper-plane in the space is tangent to exactly one quadric from family (6.1). Besides, in a general pencil of hyperplanes, there are exactly two of them tangent to a fixed general quadric.

Now, consider a pair x, y of hyperplanes. They are touching respectively unique quadrics $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ from (6.1). Besides, these two hyperplanes determine a pencil of hyperplanes. This pencil contains unique hyperplanes x', y', other than x, y, that are tangent to $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$ respectively.

We define $R : \mathbf{CP}^{n*} \times \mathbf{CP}^{n*} \to \mathbf{CP}^{n*} \times \mathbf{CP}^{n*}$, in such a way that $R(x, y) = (x', y')$ if (x', y') are obtained from (x, y) as just described.

Maps

$$R_{12}, R_{13}, R_{23} : \mathbf{CP}^{n*} \times \mathbf{CP}^{n*} \times \mathbf{CP}^{n*} \to \mathbf{CP}^{n*} \times \mathbf{CP}^{n*} \times \mathbf{CP}^{n*}$$

are then defined as follows:

$$\begin{aligned}
R_{12}(x, y, z) &= (x', y', z) \quad \text{for } (x', y') = R(x, y);\\
R_{13}(x, y, z) &= (x', y, z') \quad \text{for } (x', z') = R(x, z);\\
R_{23}(x, y, z) &= (x, y', z') \quad \text{for } (y', z') = R(y, z).
\end{aligned}$$

To prove the Yang–Baxter equation for map R, we will need the following lemma.

Lemma 6.1. *Let $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$, $\mathcal{Q}_\gamma$ be three non-degenerate quadrics from family* (6.1) *and x, y, z respectively their tangent hyperplanes. Take*:

$$(x_2, y_1) = R(x, y), \quad (x_3, z_1) = R(x, z), \quad (y_3, z_2) = R(y, z).$$

Let x_{23}, y_{13}, z_{12} be the joint hyperplanes of pencils determined by pairs (x_3, y_3) and (x_2, z_2), (x_3, y_3) and (y_1, z_1), (y_1, z_1) and (x_2, z_2) respectively.

Then x_{23}, y_{13}, z_{12} touch quadrics $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$, $\mathcal{Q}_\gamma$ respectively.

Proof. This statement, formulated for the dual space in dimension $n = 2$ is proved as [2, Theorem 5].

Consider the dual situation in an arbitrary dimension n. The dual quadrics $\mathcal{Q}_\alpha^*$, $\mathcal{Q}_\beta^*$, $\mathcal{Q}_\gamma^*$ belong to a linear pencil, and points x^*, y^*, z^*, dual to hyperplanes x, y, z, are respectively placed on these quadrics. Take the two-dimensional plane containing these three points. The intersection of the pencil of quadrics with this, and any other plane as well, represents a pencil of conics. Thus, Theorem 5 from [2] will remain true in any dimension.

This lemma is dual to this statement, thus the proof is complete. $\square$

Theorem 6.2. *Map R satisfies the Yang–Baxter equation.*

Proof. Let x, y, z be hyperplanes in $\mathbf{CP}^n$. We want to prove that

$$R_{23} \circ R_{13} \circ R_{12}(x, y, z) = R_{12} \circ R_{13} \circ R_{23}(x, y, z).$$

Denote by $\mathcal{Q}_\alpha$, $\mathcal{Q}_\beta$, $\mathcal{Q}_\gamma$ the quadrics from (6.1) touching x, y, z respectively.

Let:

$$(x, y, z) \xrightarrow{R_{12}} (x_2, y_1, z) \xrightarrow{R_{13}} (x_{23}, y_1, z_1) \xrightarrow{R_{23}} (x_{23}, y_{13}, z_{12}),$$
$$(x, y, z) \xrightarrow{R_{23}} (x, y_3, z_2) \xrightarrow{R_{13}} (x_3, y_3, z'_{12}) \xrightarrow{R_{12}} (x'_{23}, y'_{13}, z'_{12}).$$

Now, apply Lemma 6.1 to hyperplanes x, y, z_2. Since:

$$(x_2, y_1) = R(x, y), \quad (x_3, z'_{12}) = R(x, z_2), \quad (y_3, z) = R(y, z_2),$$

we have that the joint hyper-plane of pencils (x_3, y_3) and (x_2, z) is touching $\mathcal{Q}_\alpha$ — therefore, this plane must coincide with x_{23} and x'_{23}, i.e. $x_{23} = x'_{23}$. Also, the joint hyper-plane of pencils (y_1, z'_{12}) and (x_2, z) is touching $\mathcal{Q}_\gamma$ — therefore, this is z_1 and $z_{12} = z'_{12}$. Finally, the joint hyper-plane of pencils (x_3, y_3) and (y_1, z'_{12}) is tangent to $\mathcal{Q}_\beta$ — it follows $y_{13} = y'_{13}$, which completes the proof. $\square$

Remark 6.3. Instead of defining R to act on the whole space $\mathbf{CP}^{n*} \times \mathbf{CP}^{n*}$, we can restrict it to the product of two non-degenerate quadrics from (6.1), namely:

$$R(\alpha, \beta) \colon \mathcal{Q}^*_\alpha \times \mathcal{Q}^*_\beta \to \mathcal{Q}^*_\alpha \times \mathcal{Q}^*_\beta,$$

where pair (x, y) of tangent hyperplanes is mapped into pair (x_1, y_1) in such a way that x, y, x_1, y_1 belong to the same pencil.

The corresponding Yang–Baxter equation is:

$$R_{23}(\beta, \gamma) \circ R_{13}(\alpha, \gamma) \circ R_{12}(\alpha, \beta) = R_{12}(\alpha, \beta) \circ R_{13}(\alpha, \gamma) \circ R_{23}(\alpha, \beta),$$

where both sides of the equation represent maps from $\mathcal{Q}^*_\alpha \times \mathcal{Q}^*_\beta \times \mathcal{Q}^*_\gamma$ to itself.

In [2], for irreducible algebraic varieties $\mathcal{X}_1$ and $\mathcal{X}_2$, *a quadrirational mapping* $F \colon \mathcal{X}_1 \times \mathcal{X}_2$ is defined. For such a map F and any fixed pair $(x, y) \in \mathcal{X}_1 \times \mathcal{X}_2$, except from some closed subvarieties of codimension at least 1, the graph $\Gamma_F \subset \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \times \mathcal{X}_2$ intersects each of the sets $\{x\} \times \{y\} \times \mathcal{X}_1 \times \mathcal{X}_2$, $\mathcal{X}_1 \times \mathcal{X}_2 \times \{x\} \times \{y\}$, $\mathcal{X}_1 \times \{y\} \times \{x\} \times \mathcal{X}_2$, $\{x\} \times \mathcal{X}_2 \times \mathcal{X}_1 \times \{y\}$ exactly at one point (see [2, Definition 3]). In other words, Γ_F is the graph of four rational maps: F, F^{-1}, $\bar{F}$, $\bar{F}^{-1}$.

The following Proposition is a generalization of [2, Proposition 4].

Proposition 6.4. *Map* $R(\alpha, \beta) \colon \mathcal{Q}^*_\alpha \times \mathcal{Q}^*_\beta \to \mathcal{Q}^*_\alpha \times \mathcal{Q}^*_\beta$, *is quadrirational. It is an involution and it coincides with its companion* $\bar{R}(\alpha, \beta)$.

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