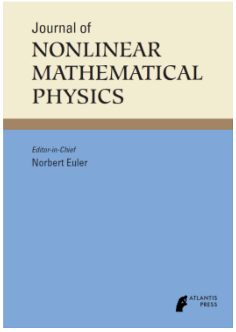

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Ting Su, Huihui Dai, Xian Guo Geng

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ON THE APPLICATION OF A GENERALIZED VERSION OF THE DRESSING METHOD TO THE INTEGRATION OF VARIABLE COEFFICIENT N-COUPLED NONLINEAR SCHRÖDINGER EQUATION

TING SU

Department of Mathematical and Physical Science
Henan Institute of Engineering
Zhengzhou, Henan 451191, P. R. China
suting1976@163.com

HUIHUI DAI

Department of Mathematics, City University of Hong Kong
83 Tat Chee Avenue, Kowloon, Hong Kong, P. R. China
mahhdai@math.cityu.edu.hk

XIAN GUO GENG

Department of Mathematics, Zhengzhou University
Zhengzhou, Henan 450052, P. R. China
xggeng@zzu.edu.cn

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N-coupled nonlinear Schrödinger (NLS) equations have been proposed to describe N-pulse simultaneous propagation in optical fibers. When the fiber is nonuniform, N-coupled variable-coefficient NLS equations can arise. In this paper, a family of N-coupled integrable variable-coefficient NLS equations are studied by using a generalized version of the dressing method. We first extend the dressing method to the versions with $(N+1) \times (N+1)$ operators and $(2N+1) \times (2N+1)$ operators. Then, we obtain three types of N-coupled variable-coefficient equations (N-coupled NLS equations, N-coupled Hirota equations and N-coupled high-order NLS equations). Then, the compatibility conditions are given, which insure that these equations are integrable. Finally, the explicit solutions of the new integrable equations are obtained.

Keywords: Variable-coefficient; the generalized dressing method; integrability.

Mathematics Subject Classification 2000: 22E46, 53C35, 57S20

1. Introduction

The dressing method is a power tool for studying integrable nonlinear evolution equations. This method was first presented by Zakharov and Shabat [15] and used to solve some nonlinear evolution equations. Subsequently many authors developed the dressing method and obtained explicit solutions of a number of nonlinear evolution equations [3, 5, 14, 16]. In [4, 9], Dai and Jeffrey extended the dressing method to a generalized version for solving nonlinear evolution equations with variable-coefficients, in which a key is that variable-coefficient dressing operators are transformed to different variable-coefficient ones.

As the technology of optical fibers for long distance communication and signal processing has rapidly developed, coupled nonlinear Schrödinger (NLS) equations have become a hot research topic. Hasegawa and Tappert [7] theoretically proved that the principle of soliton in optical fiber is based on the exact balance between group velocity dispersion (GVD) and self-phase modulation (SPM). Mollenauer [12] verified the principle with the aid of experiments. N-coupled NLS equations (homogeneous and inhomogeneous) model N-pulse simultaneous propagation. Nakkeeran in [13] studied a family of N-coupled NLS by using Bäcklund transformation method. In [8], Hioe discussed the solitary wave solutions for N-coupled NLS equations. Actually, some NLS equations with constant coefficients are so idealized that they fail to model some complex physics situations, thus a class of variable-coefficient Higher-order NLS (HNLS) equations have arisen in physics fibers. Kodama and Hasegawa first presented the integrable variable-coefficient HNLS equation [10]

$$\begin{aligned} &iu_z + a(z)u_{tt} + b(z)u|u|^2 + (c(z,t) + ic_1)u \\ &\quad + ih_1(z,t)u_t + pu(|u|^2)_t + il(z)(u|u|^2)_t + ib(z)u_{ttt} = 0. \end{aligned} \tag{1.1}$$

In the sequel, many investigators carried out research in this topic from different aspects [1, 6, 11].

In this paper, we first extend the generalized dressing method to $(N+1)\times(N+1)$ matrix operators, from which we propose the integrable variable-coefficient N-coupled cylindrical NLS equations:

$$\begin{aligned} &Q_{jt} + \frac{i\rho_2}{2}Q_{jxx} + \left(\rho_1 + ix^2\rho_1 + a_0\rho_2 - \frac{i\rho_2}{2}\right)Q_j + x(\rho_1 - \rho_2)Q_{jx} \\ &\quad - i\epsilon\rho_2 Q_j \sum_{l=1}^{N} |Q_l|^2 = 0, \quad j = 1, \ldots, N, \end{aligned} \tag{1.2}$$

which govern simulatance propagation of fields in our homogeneous fiber medium, with the effect of GVD, the inverse velocity and self-phase modulation, where, ρ_1, ρ_2 and a_0 are arbitrary functions of t, the coefficients of Q_j, $Q_j \sum_{l=1}^{N} |Q_l|^2$, Q_{jxx}, Q_{jx}, are related to gain (loss), phase modulation, GVD and the inverse velocity.

And the integrable variable-coefficient N-coupled Hirota equations:

$$\begin{aligned} &q_{j,y} + \frac{i\rho_2}{2}q_{j,xx} - i\epsilon\rho_2 \sum_{n=1}^{N} |q_n|^2 q_j + 2ix\rho_0 q_j + \rho_1(xq_j)_x \\ &\quad + \frac{\rho_3}{4}\left[q_{j,xxx} - 3\epsilon\left(\sum_{n=1}^{N} |q_n|^2 q_{j,x} + \sum_{n=1}^{N} q_n^* q_{n,x} q_j\right)\right] = 0, \end{aligned} \tag{1.3}$$

with ρ_1, ρ_2 being arbitrary functions of y.

In addition, we apply the generalization to the $(2N+1)\times(2N+1)$ matrix operators, we derive the integrable variable-coefficient N-coupled high-order NLS equations:

$$\begin{aligned} &u_{kY}+\left(1-\frac{\rho_3}{4}\right)\frac{u_{kX}}{12}+\left(\rho_1-\frac{i}{216}+\frac{3\rho_3}{4}\right)u_k+\rho_1\left(X-\frac{Y}{12}\right)\left(u_{kX}-\frac{i}{6}u_k\right)\\ &\quad+\frac{\rho_3}{4}\left[u_{kXXX}+3u_k\sum_{s=1}^{N}|u_s|_X^2+6u_{kX}\sum_{s=1}^{N}|u_s|^2\right]\\ &\quad-\frac{i\rho_3}{8}u_{kXX}-\frac{i\rho_3}{4}u_k\sum_{s=1}^{N}|u_s|^2=0, \end{aligned} \tag{1.4}$$

where ρ_1, ρ_3 are arbitrary functions of Y.

This paper is organized as follows. In Sec. 2, we briefly describe the generalized dressing method and its properties. In Sec. 3, as an application, this generalization is applied to a family N-coupled NLS equations. Their solutions and Lax pairs are also given. In Sec. 4, we give some simplest cases for reductions and discuss their solutions form.

2. A Generalized Dressing Method

First, we summarize the outline of the generalized dressing method. To this end, we consider three integral operators $\mathbf{F}(x,z,y), \mathbf{K}_+(x,z,y), \mathbf{K}_-(x,z,y)$ defined by

$$\begin{aligned} \mathbf{K}_+(x,z,y)\psi(x) &\equiv \int_x^{\infty} K_+(x,z,y)\psi(z)dz,\\ \mathbf{K}_-(x,z,y)\psi(x) &\equiv \int_{-\infty}^{x} K_-(x,z,y)\psi(z)dz,\\ \mathbf{F}(x,z,y)\psi(x) &\equiv \int_{-\infty}^{\infty} F(x,z,y)\psi(z)dz, \end{aligned} \tag{2.1}$$

where $\mathbf{F}(x,z,y), \mathbf{K}_+(x,z,y), \mathbf{K}_-(x,z,y)$ are $n\times n$ matrices, $\psi(x)$ is any $n\times 1$ matrix. $\mathbf{K}_+(x,z,y)$ and $\mathbf{K}_-(x,z,y)$ are the Volterra operators, so that $K_+(x,z,y)=0$ for $z<x$ and $K_-(x,z,y)=0$ for $z>x$. We assume that $(\mathbf{I}+\mathbf{K}_+)^{-1}$ exists and $\mathbf{F}$ admits the triangular factorization

$$\mathbf{I}+\mathbf{F}=(\mathbf{I}+\mathbf{K}_+)^{-1}(\mathbf{I}+\mathbf{K}_-), \tag{2.2}$$

where $\mathbf{I}$ is the identity operator. From (2.2), a direct calculation shows that $\mathbf{F}$ and $\mathbf{K}_+$ satisfy the Gel'fand–Levitan–Marchenko (GLM) equation [15]

$$K_+(x,z,y)+F(x,z,y)+\int_x^{\infty}K_+(x,s,y)F(s,z,y)ds=0,\quad z>x. \tag{2.3}$$

Similarly, we have

$$F(x,z,y)-K_-(x,z,y)+\int_x^{\infty}K_+(x,s,y)F(s,z,y)ds=0,\quad z<x.$$

Here it is supposed that

$$\sup \int_{x_0}^{+\infty} |K_{\pm}(x,z,y)|\psi(z)dz < +\infty, \quad \sup \int_{x_0}^{+\infty} |F(x,z,y)|\psi(z)dz < +\infty, \quad x_0 > -\infty.$$

We now introduce two differential operators $\mathbf{M_1}$ and $\mathbf{M_m}$ defined by

$$\begin{aligned} \mathbf{M}_1 &= \alpha\partial_x + A(x,t_m), \\ \mathbf{M}_m &= \beta\partial_{t_m} + \mathbf{L}(x,t_m), \quad \mathbf{L}(x,t_m) = \sum_{j=0}^{m} b_j \frac{\partial^j}{\partial_x^j}, \end{aligned} \tag{2.4}$$

where α, β, b_j are matrix functions of their arguments. Suppose that the operator $\mathbf{F}$ commutes with $\mathbf{M_1}$ and $\mathbf{M_m}$, that is

$$[\mathbf{M_1}, \mathbf{F}] = \mathbf{M_1F} - \mathbf{FM_1} = 0, \quad [\mathbf{M_m}, \mathbf{F}] = \mathbf{M_mF} - \mathbf{FM_m} = 0. \tag{2.5}$$

Equation (2.5) together with (2.4) implies the following equations

$$\alpha\mathbf{F}_x + \mathbf{F}_z\alpha + A(x,t_m)\mathbf{F} - \mathbf{F}A(z,t_m) = 0, \tag{2.6}$$

$$\beta\mathbf{F}_{t_m} + \mathbf{LF} - \mathbf{FL}^{+}(z,t_m) = 0, \tag{2.7}$$

where

$$\mathbf{FL}^{+}(z,t_m) = \sum_{j=0}^{m} (-1)^j \frac{\partial^j}{\partial_z^j}(\mathbf{F}b_j(z,t_m)).$$

Now we "dress" the two differential operators $\mathbf{M_1}$ and $\mathbf{M_m}$ to obtain the dressed operators $\mathbf{N_1}$ and $\mathbf{N_m}$. The dressing procedure is accomplished through the relations

$$\begin{aligned} \mathbf{N_1}(\mathbf{I} + \mathbf{K}_+) - (\mathbf{I} + \mathbf{K}_+)\mathbf{M_1} &= 0, \\ \mathbf{N_m}(\mathbf{I} + \mathbf{K}_+) - (\mathbf{I} + \mathbf{K}_+)\mathbf{M_m} &= 0, \end{aligned} \tag{2.8}$$

where $\mathbf{N_1}$ and $\mathbf{N_m}$ can be ensured to be simple differential operators by the above equations.

A key of the generalized dressing method is to let the differential operators $\mathbf{M_1}$ and $\mathbf{M_m}$ satisfy the relation

$$[\mathbf{M_1}, \mathbf{M_m}] = \phi_1\mathbf{M_1} + \phi_2\mathbf{M_m}, \tag{2.9}$$

where ϕ_1 and ϕ_2 are arbitrary functions of their arguments. According to [4, 9], the corresponding dressing operators obey the equation

$$[\mathbf{N_1}, \mathbf{N_m}] = \phi_1\mathbf{N_1} + \phi_2\mathbf{N_m}. \tag{2.10}$$

We consider the case of $m = 3$ and denote $t_3 = y$, $\mathbf{N_1} = \mathbf{M_1} + \mathbf{D_1}$, $\mathbf{N_2} = \mathbf{M_3} + \mathbf{D_2}$. For convenience, we denote $\hat{K} = K_+(x,z)|_{z=x}$. From the first expression of (2.8), we obtain

$$\begin{aligned} \mathbf{D_1} &= \alpha\widehat{K} - \widehat{K}\alpha, \\ \alpha K_x + A(x,y)K + \mathbf{D_1}K + K_z\alpha - KA(z,y) &= 0. \end{aligned} \tag{2.11}$$

Using the second expression of (2.8), we have $\mathbf{D_2} = C_1 + C_2\partial_x$, where

$$\begin{gathered} C_2 - b_2\widehat{K} + \widehat{K}b_2 - 2b_3\widehat{K}_x - b_3K_x|_{z=x} - (Kb_3(z))_z|_{z=x} = 0, \\ \widehat{K}b_3 - b_3\widehat{K} = 0 \end{gathered} \tag{2.12}$$

and C_1 is determined from (2.10) by the following two equations:

$$\begin{aligned} &\alpha C_{1x} + AC_1 - C_1A - C_2A_x + \mathbf{D_1}b_0 - b_0\mathbf{D_1} - \mathbf{D}_{1,y} - b_1\mathbf{D}_{1x} - b_2\mathbf{D}_{1xx} \\ &\quad - b_3\mathbf{D}_{1xxx} + \mathbf{D_1}C_1 - C_1\mathbf{D_1} - C_2\mathbf{D}_{1x} = \phi_1\mathbf{D_1}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\alpha C_1 - C_1\alpha + \alpha C_{2x} + AC_2 - C_2A + \mathbf{D_1}b_1 - b_1\mathbf{D_1} \\ &\quad - 2b_2\mathbf{D}_{1x} - 3b_3\mathbf{D}_{1xx} + \mathbf{D_1}C_2 - C_2\mathbf{D_1} = 0. \end{aligned} \tag{2.14}$$

Further, from (2.9), we derive $\phi_2 = 0$ and

$$\begin{gathered} \alpha b_0 - b_0\alpha + Ab_1 - b_1A + \alpha b_{1x} - 2b_2A_x - 3b_3A_{xx} = \phi_1\alpha, \\ \alpha b_{0x} - \beta A_y - b_0A + Ab_0 - b_1A_{0x} - b_2A_{xx} - b_3A_{xxx} = \phi_1A. \end{gathered} \tag{2.15}$$

Actually, we obtain nonlinear evolution equations from (2.13). In what follows, we give the solution formula of the obtained equations.

Assume that (2.6) and (2.7) have solutions in the form of separation of variables

$$F(x,z,y) = \sum_{j=1}^{N} f_j(x,y)g_j(z,y), \tag{2.16}$$

where $f_j(x,y), g_j(z,y)$ are some $n \times n$ matrices. Moreover, we suppose that

$$K(x,z,y) = \sum_{j=1}^{N} k_j(x,y)g_j(z,y). \tag{2.17}$$

Substituting (2.16) and (2.17) into the GLM equation (2.3) yields that

$$\widehat{K} = \sum_{j=1}^{N} k_j(x,y)g_j(x,y) = -(f_1, f_2, \ldots, f_N)L^{-1}(g_1, g_2, \ldots, g_N)^T, \tag{2.18}$$

where L is defined by

$$L_{jl} = \delta_{jl} + \int_x^\infty g_j(s,y)f_l(s,y)ds, \quad 1 \le j, \;\; l \le N \tag{2.19}$$

and δ_{jl} is the Kronecker's delta.

For the case of $N = 1$ in (2.16), we have known that

$$\widehat{K} = -f_1\left[1 + \int_x^\infty g_1(s)f_1(s)ds\right]^{-1} g_1. \tag{2.20}$$

In view of (2.20), the one-soliton solutions of the obtained equation are given.

3. Applications to the Integrable Variable-Coefficient N-Coupled NLS Equations

In this section, we will discuss a family of the integrable variable-coefficient N-coupled NLS equations, including integrable variable-coefficient N-coupled cylindrical NLS equations, integrable variable-coefficient N-coupled Hirota equations, integrable variable-coefficient N-coupled mKdV-type equations and high-order NLS equations. Further, some new conclusions of these equations are given. For convenience, the same symbols stand for different means in different sections.

3.1. *Variable-coefficient N-coupled cylindrical NLS equations*

Let $\mathbf{M_1}$ and $\mathbf{M_2}$ be

$$\mathbf{M_1} = \alpha\partial_x + x a_0 I, \tag{3.1}$$

$$\mathbf{M_2} = I\partial_t + \rho_2\alpha\partial_x^2 + x\rho_1 I\partial_x, \tag{3.2}$$

where ρ_1, ρ_2 and a_0 are functions of t, and

$$\alpha = i\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix}_{(N+1)\times(N+1)}, \quad I = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{(N+1)\times(N+1)}. \tag{3.3}$$

Substituting (3.1) and (3.2) into (2.15), we have

$$\rho_1 - 2\rho_2 a_0 = \phi_1, \quad a_{0,t} + 2\rho_1 a_0 - 2\rho_2 a_0^2 = 0.$$

From which, it yields that $a_0 = \frac{1}{e^{2\int\rho_1 dt}(c_0 - \int 2\rho_2 e^{-2\int\rho_1 dt}dt)}$, where c_0 is an integration constant.

Similarly, we derive from (2.11)

$$\mathbf{D_1} = i\begin{pmatrix} 0 & \cdots & 0 & q_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & q_N \\ -\epsilon q_1^* & \cdots & -\epsilon q_N^* & 0 \end{pmatrix},$$

with $\hat{k}^{(jN+1)} = \frac{q_j}{2}$, $\hat{k}^{(N+1j)} = \epsilon\frac{q_j^*}{2}$, $\widehat{k}_x^{(lj)} = -\epsilon\frac{q_l q_j^*}{2}$, $\widehat{k}_x^{(N+1N+1)} = -\frac{\epsilon}{2}\sum_{k=1}^N |q_k|^2$, $\epsilon^2 = 1$, $(l, j = 1, \ldots, N)$. It yields from (2.12) that $C_2 = \rho_2\mathbf{D_1}$. For the sake of simplicity, we denote $C_1 = (C_1^{(ks)})_{(N+1)\times(N+1)}$, $C_2 = (C_2^{(ks)})_{(N+1)\times(N+1)}$, $(k, s = 1, \ldots, N+1)$.

From (2.13) and (2.14), we obtain

$$C_1^{(jl)} = -i\frac{\epsilon\rho_2}{2}q_j q_l^*, \quad C_1^{(jN+1)} = i\frac{\rho_2}{2}q_{jx},$$
$$C_1^{(N+1j)} = -i\frac{\epsilon\rho_2}{2}q_{jx}^*, \quad C_1^{(N+1N+1)} = i\frac{\epsilon\rho_2}{2}\sum_{l=1}^N |q_l|^2 \tag{3.4}$$

and the integrable variable-coefficient N-coupled cylindrical NLS equations:

$$q_{jt} + \rho_1(xq_j)_x + \rho_2 a_0 q_j + i\frac{\rho_2}{2}q_{jxx} - i\epsilon\rho_2 q_j \sum_{l=1}^{N} |q_l|^2 = 0, \quad j = 1, \ldots, N. \tag{3.5}$$

Under the transformation $q_j = Q_j e^{\frac{i}{2}x^2}$, the above equations are reduced to different integrable variable-coefficient N-coupled cylindrical NLS equations:

$$\begin{aligned} &Q_{jt} + \frac{i\rho_2}{2}Q_{jxx} + \left(\rho_1 + ix^2\rho_1 + a_0\rho_2 - \frac{i\rho_2}{2}\right)Q_j + x(\rho_1 - \rho_2)Q_{jx} \\ &\quad - i\epsilon\rho_2 Q_j \sum_{l=1}^{N} |Q_l|^2 = 0, \quad j = 1, \ldots, N. \end{aligned} \tag{3.6}$$

Equation (3.5) have Lax pair $\mathbf{N_1}$ and $\mathbf{N_2}$ defined by

$$\mathbf{N_1} = \mathbf{M_1} + \mathbf{D_1}, \quad \mathbf{N_2} = \mathbf{M_2} + \mathbf{D_2}, \quad \mathbf{D_2} = C_1 + C_2\partial_x,$$

where

$$C_1 = -\frac{i\rho_2}{2}\begin{pmatrix} \epsilon|q_1|^2 & \cdots & \epsilon q_1 q_N^* & -q_{1x} \\ \epsilon q_2 q_1^* & \cdots & \epsilon q_2 q_N^* & -q_{2x} \\ \vdots & \ddots & \vdots & \vdots \\ \epsilon q_N q_1^* & \cdots & \epsilon|q_N|^2 & -q_{Nx} \\ \epsilon q_{1x}^* & \cdots & \epsilon q_{Nx}^* & -\epsilon\sum_{l=1}^{N}|q_l|^2 \end{pmatrix}, \quad C_2 = i\rho_2\begin{pmatrix} 0 & \cdots & 0 & q_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & q_N \\ -\epsilon q_1^* & \cdots & -\epsilon q_N^* & 0 \end{pmatrix}.$$

In what follows, we will derive the one-soliton solution of (3.5).

From (2.6) and (2.7), we have the equations for F

$$\alpha F_x + F_z\alpha + (x - z)a_0 F = 0, \tag{3.7}$$

$$F_t + \rho_1(t)(xF_x + zF_z) + \rho_1(t)F + \rho_2(t)(\alpha F_{xx} - F_{zz}\alpha) = 0. \tag{3.8}$$

In view of (3.7), we set F as follows

$$\begin{aligned} &F^{(jN+1)} = w_{jN+1}e^{i\frac{a_0}{2}(x^2+z^2)+\mu_{jN+1}(x+z)}, \qquad F^{(jl)} = 0, \\ &F^{(N+1j)} = w_{N+1j}e^{-i\frac{a_0}{2}(x^2+z^2)+\mu_{N+1j}(x+z)}, \quad F^{(N+1N+1)} = 0, \end{aligned} \tag{3.9}$$

with $F^{(jl)}$ being elements of F.

Substitution of (3.9) into (3.8), we derive

$$\begin{aligned} &\partial_t\mu_{jN+1} + [\rho_1 - 2a_0\rho_2]\mu_{jN+1} = 0, \\ &\partial_t\mu_{N+1j} + [\rho_1 - 2a_0\rho_2]\mu_{N+1j} = 0, \\ &\partial_t w_{jN+1} + [\rho_1 - 2a_0\rho_2 + 2i\rho_2\mu_{jN+1}^2]w_{jN+1} = 0, \\ &\partial_t w_{N+1j} + [\rho_1 - 2a_0\rho_2 - 2i\rho_2\mu_{N+1j}^2]w_{N+1j} = 0. \end{aligned} \tag{3.10}$$

We assume that $\mu_{jN+1} = \mu_{1N+1}$, $\mu_{N+1j} = \mu_{N+11}$ and $F = f(x,t)g(z,t)$, with

$$f(x,t) = \begin{pmatrix} 0 & \cdots & 0 & w_{1N+1}e^{i\frac{a_0x^2}{2}+\mu_{1N+1}x} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & w_{NN+1}e^{i\frac{a_0x^2}{2}+\mu_{1N+1}x} \\ w_{N+11}e^{-i\frac{a_0x^2}{2}+\mu_{N+11}x} & \cdots & w_{N+1N}e^{-i\frac{a_0x^2}{2}+\mu_{N+11}x} & 0 \end{pmatrix},$$

$$g(z,t) = \begin{pmatrix} e^{-i\frac{a_0z^2}{2}+\mu_{N+11}z} & 0 & \cdots & 0 & 0 \\ 0 & e^{-i\frac{a_0z^2}{2}+\mu_{N+11}z} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{-i\frac{a_0z^2}{2}+\mu_{N+11}z} & 0 \\ 0 & 0 & \cdots & 0 & e^{i\frac{a_0z^2}{2}+\mu_{1N+1}z} \end{pmatrix}.$$

Let $K = k(x,t)g(z,t)$. We take $N = 1$ in (2.16). From (2.19), we obtain

$$L = \begin{pmatrix} 1 & \cdots & 0 & -\frac{w_{1N+1}e^{(\mu_{1N+1}+\mu_{N+11})x}}{\mu_{1N+1}+\mu_{N+11}} \\ 0 & \cdots & 0 & -\frac{w_{2N+1}e^{(\mu_{1N+1}+\mu_{N+11})x}}{\mu_{1N+1}+\mu_{N+1,1}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\frac{w_{NN+1}e^{(\mu_{1N+1}+\mu_{N+11})x}}{\mu_{1N+1}+\mu_{N+11}} \\ -\frac{w_{N+11}e^{(\mu_{1N+1}+\mu_{N+11})x}}{\mu_{1N+1}+\mu_{N+11}} & \cdots & -\frac{w_{N+1N}e^{(\mu_{1N+1}+\mu_{N+11})x}}{\mu_{1N+1}+\mu_{N+11}} & 1 \end{pmatrix},$$

where $|L| = 1 - \sum_{j=1}^{N} \frac{w_{jN+1}w_{N+1j}}{(\mu_{N+11}+\mu_{1N+1})^2} e^{2(\mu_{N+11}+\mu_{1N+1})x}$, $L^*_{(N+1N+1)} = 1$, $L = (L_{(lk)})_{(N+1)\times(N+1)}$, $|L|$ is determinant of L, $L^*_{(lk)}$ is algebraic cofactor of L.

From (2.20), we can derive

$$\widehat{K} = -\frac{1}{|L|} \times \begin{pmatrix} w_{1N+1}L^*_{(1N+1)}e^{(\mu_{1N+1}+\mu_{N+11})x} & \cdots & w_{1N+1}e^{2\mu_{1N+1}x+ia_0x^2} \\ w_{2N+1}L^*_{(1N+1)}e^{(\mu_{1N+1}+\mu_{N+11})x} & \cdots & w_{2N+1}e^{2\mu_{1N+1}x+ia_0x^2} \\ \vdots & \ddots & \vdots \\ w_{NN+1}L^*_{(1N+1)}e^{(\mu_{1N+1}+\mu_{N+11})x} & \cdots & w_{NN+1}e^{2\mu_{1N+1}x+ia_0x^2} \\ \sum_{k=1}^{N} w_{N+1k}L^*_{(1k)}e^{2\mu_{N+11}x-ia_0x^2} & \cdots & \sum_{k=1}^{N} w_{N+1k}L^*_{(N+1k)}e^{(\mu_{1N+1}+\mu_{N+11})x} \end{pmatrix}.$$

From which, it yields the one-soliton solution of (3.5):

$$q_j = -\frac{2}{|L|}w_{jN+1}e^{ia_0x^2+2\mu_{1N+1}x}, \quad q_j^* = -\frac{2}{|L|}w_{N+1j}e^{-ia_0x^2+2\mu_{N+11}x},$$

where

$$\sum_{k=1}^{N} w_{N+1k}L^*_{(jk)} = w_{N+1j}, \quad w_{jN+1} = \epsilon w^*_{N+1j}, \quad \mu_{N+11} = \mu^*_{1N+1},$$

$$L^*_{(jk)} = w_{N+1j}w_{kN+1}, \quad L^*_{(jj)} = 1 - \sum_{k=1,k\neq j}^{N} w_{N+1k}w_{kN+1},$$

are used.

Further, we obtain the one-soliton solution of (3.6):

$$Q_j = -\frac{2q_j}{|L|}w_{jN+1}e^{i(a_0-\frac{1}{2})x^2+2\mu_{1N+1}x}.$$

According to the form $|L|$, we can find that $\frac{1}{|L|}$ has not singular solution for ($\epsilon = -1$) and has singular solution for ($\epsilon = 1$). Therefore, the one-soliton solution of (3.5) for has singular solution for ($\epsilon = 1$). Theoretically, we can give N-soliton solution of (3.5) and (3.6). However, it is necessary to complex calculations with the aid of mathematics.

3.2. *Variable-coefficient N-coupled Hirota equations*

We set $\mathbf{M_1}$ and $\mathbf{M_3}$ to be

$$\mathbf{M_1} = \alpha\partial_x + a_0(y)I, \tag{3.11}$$

$$\mathbf{M_3} = I\partial_y + I\rho_3(y)\partial_x^3 + \rho_2(y)\alpha\partial_x^2 + Ix\rho_1(y)\partial_x + x\rho_0(y)\alpha, \tag{3.12}$$

with α and I are given in (3.3).

Substituting (3.11) and (3.12) into (2.15), which gives that

$$\rho_1 = \phi_1, \quad a_{0,y} + \rho_1 a_0 + \rho_0 = 0,$$

from which, we have $a_0 = e^{-\int \rho_1 dy}(c - \int \rho_0 e^{\int \rho_1 dy}dy)$, with c being an arbitrary constant.

In the same way, we obtain

$$\mathbf{D_1} = i\begin{pmatrix} 0 & \cdots & 0 & q_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & q_N \\ -\epsilon q_1^* & \cdots & -\epsilon q_N^* & 0 \end{pmatrix},$$

where, $\hat{k}^{(jN+1)} = \frac{q_j}{2}$, $\hat{k}^{(N+1j)} = \frac{\epsilon q_j^*}{2}$, $\hat{k}_x^{(N+1N+1)} = -\epsilon\sum_{j=1}^{N}\frac{|q_j|^2}{2}, \hat{k}_x^{(lj)} = -\epsilon\frac{q_l q_j^*}{2}$, $l, j = 1, \ldots, N$.

Using (2.12)–(2.14), we have $C_2 = \rho_2\mathbf{D_1} + 3\rho_3\hat{K}_x$. C_1 is to be determined by (3.13) and (3.14).

Substitution of (3.11) and (3.12) into (2.13), we have

$$\alpha C_1 - C_1\alpha - \rho_2\alpha\mathbf{D}_{1,x} + 3\rho_3\alpha\widehat{K}_{xx} - 3\rho_3\mathbf{D}_{1,xx} + 3\rho_3(\mathbf{D_1}\widehat{K}_x - \widehat{K}_x\mathbf{D_1}) = 0, \tag{3.13}$$

$$\alpha C_{1,x} + x\rho_0(\mathbf{D_1}\alpha - \alpha\mathbf{D_1}) - x\rho_1\mathbf{D}_{1,x} - \rho_2\mathbf{D}_{1,xx} - \rho_3\mathbf{D}_{1,xxx}$$
$$+ \mathbf{D_1}C_1 - C_1\mathbf{D_1} - C_2\mathbf{D}_{1,x} - \mathbf{D}_{1,y} = \phi_1\mathbf{D_1}. \tag{3.14}$$

From which, we derive

$$C_1^{(lj)} = -\frac{3\epsilon\rho_3}{4}(q_l q_j^*)_x - \frac{i\epsilon\rho_2}{2}q_l q_j^*, \quad C_1^{(N+1N+1)} = -\frac{3\epsilon\rho_3}{4}\sum_{j=1}^{N}(|q_j|^2)_x + \frac{i\epsilon\rho_2}{2}\sum_{j=1}^{N}|q_j|^2,$$
$$C_1^{(jN+1)} = \frac{i\rho_2}{2}q_{jx} + \frac{3\rho_3}{4}q_{j,xx}, \quad C_1^{(N+1j)} = -\epsilon\frac{i\rho_2}{2}q_{jx}^* + \frac{3\epsilon\rho_3}{4}q_{j,xx}^* \tag{3.15}$$

and the integrable variable-coefficient N-coupled Hirota equations:

$$q_{j,y} + \frac{i\rho_2}{2}q_{j,xx} - i\epsilon\rho_2\sum_{n=1}^{N}|q_n|^2 q_j + 2ix\rho_0 q_j + \rho_1(xq_j)_x$$
$$+ \frac{\rho_3}{4}\left[q_{j,xxx} - 3\epsilon\left(\sum_{n=1}^{N}|q_n|^2 q_{j,x} + \sum_{n=1}^{N}q_n^* q_{n,x} q_j\right)\right] = 0. \tag{3.16}$$

The nonlinear wave propagation of simultaneous fields in an optical fiber with core medium for not homogeneous, with the effects, various GVD, SPM, higher-order dispersion (HOD) and Kerr dispersion is by governed by the integrable variable-coefficients N-coupled by (3.16).

The Lax pair is $\mathbf{N_1}$ and $\mathbf{N_2}$, given by

$$\mathbf{N_1} = \mathbf{M_1} + \mathbf{D_1}, \quad \mathbf{N_2} = \mathbf{M_2} + \mathbf{D_2}, \quad \mathbf{D_2} = C_1 + C_2\partial_x,$$

with

$$C_1 = \begin{pmatrix} -\frac{3\epsilon\rho_3}{4}|q_1|_x^2 - \frac{i\epsilon\rho_2}{2}|q_1|^2 & \cdots & \frac{3\rho_3}{4}q_{1xx} + \frac{i\rho_2}{2}q_{1x} \\ -\frac{3\epsilon\rho_3}{4}(q_2q_1^*)_x - \frac{i\epsilon\rho_2}{2}q_2q_1^* & \cdots & \frac{3\rho_3}{4}q_{2xx} + \frac{i\rho_2}{2}q_{2x} \\ \vdots & \ddots & \vdots \\ -\frac{3\epsilon\rho_3}{4}(q_Nq_1^*)_x - \frac{i\epsilon\rho_2}{2}q_Nq_1^* & \cdots & \frac{3\rho_3}{4}q_{Nxx} + \frac{i\rho_2}{2}q_{Nx} \\ \frac{3\epsilon\rho_3}{4}q_{1xx}^* - \frac{i\epsilon\rho_2}{2}q_1^* & \cdots & -\frac{3\epsilon\rho_3}{4}\sum_{j=1}^{N}|q_j|_x^2 + \frac{i\epsilon\rho_2}{2}\sum_{j=1}^{N}|q_j|^2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -\frac{3\epsilon\rho_3}{2}|q_1|^2 & \cdots & \frac{3\rho_3}{2}q_{1x} + i\rho_2 q_1 \\ -\frac{3\epsilon\rho_3}{3}q_2q_1^* & \cdots & \frac{3\rho_3}{2}q_{2x} + i\rho_2 q_2 \\ \vdots & \ddots & \vdots \\ -\frac{3\epsilon\rho_3}{2}q_Nq_1^* & \cdots & \frac{3\rho_3}{2}q_{Nx} + i\rho_2 q_N \\ -\frac{3\epsilon\rho_3}{2}q_{1x}^* - i\epsilon\rho_2 q_1^* & \cdots & -\frac{i3\epsilon\rho_3}{2}\sum_{j=1}^{N}|q_j|^2 \end{pmatrix}.$$

In the sequel, we shall discuss the solution of (3.16).

From (2.6) and (2.7), we have the equations for F

$$\alpha F_x + F_z\alpha = 0, \tag{3.17}$$

$$\begin{aligned} F_y + \rho_0(x\alpha F - zF\alpha) + \rho_1(xF_x + zF_z) + \rho_1 F \\ + \rho_2(\alpha F_{xx} - F_{zz}\alpha) + \rho_3(F_{xxx} + F_{zzz}) = 0. \end{aligned} \tag{3.18}$$

From (3.17), it is easy to derive

$$F_x^{(jN+1)} - F_z^{(jN+1)} = 0, \quad F_x^{(N+1j)} - F_z^{(N+1j)} = 0, \tag{3.19}$$

$$F_x^{(lj)} + F_z^{(lj)} = 0, \quad F_x^{(N+1N+1)} + F_z^{(N+1N+1)} = 0. \tag{3.20}$$

Let

$$\begin{aligned} F^{(jN+1)} = w_{jN+1}e^{s_{jN+1}(x+z)}, \quad F^{(N+1j)} = w_{N+1j}e^{s_{N+1j}(x+z)}, \\ F^{(lj)} = 0, \quad F^{(N+1N+1)} = 0, \quad l, j = 1, \ldots, N. \end{aligned} \tag{3.21}$$

Substitution of (3.20) into (3.18), we arrive at

$$\begin{aligned} &\partial_y s_{jN+1} + \rho_1 s_{jN+1} + i\rho_0 = 0, \quad \partial_y s_{N+1j} + \rho_1 s_{N+1j} - i\rho_0 = 0, \\ &\partial_y w_{jN+1} + (\rho_1 + 2i\rho_2 s_{jN+1}^2 + 2\rho_3 s_{jN+1}^3)w_{jN+1} = 0, \\ &\partial_y w_{N+1j} + (\rho_1 - 2i\rho_2 s_{N+1j}^2 + 2\rho_3 s_{N+1j}^3)w_{N+1j} = 0, \end{aligned} \tag{3.22}$$

from which, we obtain

$$\begin{aligned} s_{jN+1} &= e^{-\int \rho_1 dy}\left(c_0 - i\int \rho_0 e^{\int \rho_1 dy} dy\right), \\ s_{N+1j} &= e^{-\int \rho_1 dy}\left(c_0 + i\int \rho_0 e^{\int \rho_1 dy} dy\right), \\ w_{jN+1} &= c_{jN+1}e^{-\int(\rho_1 + 2i\rho_2 s_{jN+1}^2 + 2\rho_3 s_{jN+1}^3)dy}, \\ w_{N+1j} &= c_{N+1j}e^{\int(-\rho_1 + 2i\rho_2 s_{jN+1}^2 - 2\rho_3 s_{jN+1}^3)dy}, \end{aligned}$$

with c_0, c_{jN+1} and c_{N+1j} being arbitrary constants.

We suppose that $s_{jN+1} = s_{1N+1}$, $s_{N+1j} = s_{N+11}$, $s_{N+11}^* = s_{1N+1}$ and $F(x,z,y) = f(x,y)g(z,y)$, where $f(x,y)$ and $g(z,y)$ are given by

$$f(x,y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & w_{1N+1}e^{s_{1N+1}x} \\ 0 & 0 & \cdots & 0 & w_{2N+1}e^{s_{1N+1}x} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & w_{NN+1}e^{s_{1N+1}x} \\ w_{N+11}e^{s_{N+11}x} & w_{N+12}e^{s_{N+11}x} & \cdots & w_{N+1N}e^{s_{N+11}x} & 0 \end{pmatrix},$$

$$g(z,y) = \begin{pmatrix} e^{s_{N+11}z} & 0 & \cdots & 0 & 0 \\ 0 & e^{s_{N+11}z} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{s_{N+11}z} & 0 \\ 0 & 0 & \cdots & 0 & e^{s_{1N+1}z} \end{pmatrix}.$$

Using (2.16) ($N = 1$), it yields that

$$L = \begin{pmatrix} 1 & \cdots & 0 & -\frac{w_{1N+1}e^{(s_{1N+1}+s_{N+11})x}}{s_{1N+1}+s_{N+11}} \\ 0 & \cdots & 0 & -\frac{w_{2N+1}e^{(s_{1N+1}+s_{N+11})x}}{s_{1N+1}+s_{N+11}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -\frac{w_{NN+1}e^{(s_{1N+1}+s_{N+11})x}}{s_{1N+1}+s_{N+11}} \\ -\frac{w_{N+11}e^{(s_{1N+1}+s_{N+11})x}}{s_{1N+1}+s_{N+11}} & \cdots & -\frac{w_{N+1N}e^{(s_{1N+1}+s_{N+11})x}}{s_{1N+1}+s_{N+11}} & 1 \end{pmatrix}.$$

From (2.20), we have

$$\widehat{K} = -\frac{1}{|L|} \times \begin{pmatrix} w_{1N+1}L^*_{(1N+1)}e^{(s_{1N+1}+s_{N+11})x} & \cdots & w_{1N+1}e^{2s_{1N+1}x} \\ w_{2N+1}L^*_{(1N+1)}e^{(s_{1N+1}+s_{N+11})x} & \cdots & w_{2N+1}e^{2s_{1N+1}x} \\ \vdots & \ddots & \vdots \\ w_{NN+1}L^*_{(1N+1)}e^{(s_{1N+1}+s_{N+11})x} & \cdots & w_{NN+1}e^{2s_{1N+1}x} \\ \sum_{j=1}^{N} w_{N+1j}L^*_{(1j)}e^{2s_{N+11}x} & \cdots & \sum_{j=1}^{N} w_{N+1j}L^*_{(N+1j)}e^{(s_{1N+1}+s_{N+11})x} \end{pmatrix}.$$

Thus, the one-soliton solutions of (3.16) are given by

$$q_j = -\frac{2}{|L|}w_{jN+1}e^{2s_{1N+1}x}, \tag{3.23}$$

where $|L| = 1 - \frac{\sum_{j=1}^{N} w_{jN+1}w_{N+1j}}{(s_{1N+1}+s_{N+11})^2}e^{2(s_{1N+1}+s_{N+11})x}$, $\sum_{l=1}^{N} w_{N+1l}L^*_{(jl)} = w_{N+1j}$ and $w_{jN+1} = \epsilon w^*_{N+1j}$ are used.

According to the form of $|L|$, it is easy to see that the one-soliton solutions of (3.16) is dark soliton solution for ($\epsilon = -1$) and bright soliton solution for ($\epsilon = -1$). Similarly, through tedious calculations, we can drive N-soliton solutions of (3.16) with help of mathematics.

3.3. *Integrable variable-coefficient N-coupled mKdV-type equations and high-order NLS equations*

Soliton solutions to the mkdv equation is important in inhomogeneous plasmas, which describe produced filamentation by the ponderomotive force between the dispersive effects and the nonlinear perturbation.

We consider operators $\mathbf{M_1}$ and $\mathbf{M_2}$ defined by

$$\mathbf{M_1} = \alpha\partial_x + a_0(y)I, \tag{3.24}$$

$$\mathbf{M_2} = I\partial_y + I\rho_3(y)\partial_x^3 + Ix\rho_1(y)\partial_x, \tag{3.25}$$

where

$$\alpha = i\begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix}_{(2N+1)\times(2N+1)}, \quad I = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{(2N+1)\times(2N+1)}. \tag{3.26}$$

Substitution of (3.23) and (3.24) into (2.6) and (2.7) yields that

$$a_0 = c_0 e^{-\int \rho_1 dy}, \tag{3.27}$$

where c_0 is an arbitrary constant.

Similarly, we have

$$\mathbf{D_1} = i\begin{pmatrix} 0 & 0 & \cdots & 0 & q_1 \\ 0 & 0 & \cdots & 0 & q_1^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_N \\ 0 & 0 & \cdots & 0 & q_N^* \\ q_1^* & q_1 & \cdots & q_N & 0 \end{pmatrix},$$

where $\hat{k}^{(2k-1,2N+1)} = \frac{q_k}{2}$, $\hat{k}^{(2k,2N+1)} = \frac{q_k^*}{2}$, $\hat{k}^{(2N+1,2n-1)} = -\frac{q_n^*}{2}$, $\hat{k}^{(2N+1,2n)} = -\frac{q_n}{2}$, $\widehat{k}_x^{(2k-1,2n-1)} = \frac{q_k q_n^*}{2}$, $\widehat{k}_x^{(2k-1,2n)} = \frac{q_k q_n}{2}$, $\widehat{k}_x^{(2k,2n-1)} = \frac{q_k^* q_n^*}{2}$, $\widehat{k}_x^{(2k,2n)} = \frac{q_k^* q_n}{2}$, $\widehat{k}_x^{(2N+1,2N+1)} = \sum_{s=1}^{N} |q_s|^2$, $(n, k = 1, \ldots, N)$.

With the aid of (2.12)–(2.14), we have $C_2 = 3\rho_3(y)\widehat{K}_x$. $C_1 = (C_1^{(l,j)})_{(2N+1)\times(2N+1)}$ is determined by (3.27) and (3.28).

Substitution of (3.23) and (3.24) into (2.13) and (2.14) produces that

$$\alpha C_1 - C_1\alpha + \alpha c_{2x} + 3\rho_3\alpha\widehat{K}_{xx} - 3\rho_3\mathbf{D}_{1xx} + 3\rho_3(\mathbf{D}_1\widehat{K}_x - \widehat{K}_x\mathbf{D}_1) = 0, \tag{3.28}$$

$$\alpha C_{1x} - \mathbf{D}_{1y} - x\rho_1\mathbf{D}_{1x} - \rho_3\mathbf{D}_{1xxx} + \mathbf{D}_1C_1 - C_1\mathbf{D}_1 - C_2\mathbf{D}_{1x} = \phi_1\mathbf{D}_1. \tag{3.29}$$

From which, we obtain

$$C_1^{(2n-1,2N+1)} = \frac{3\rho_3}{4}q_{nxx}, \quad C_1^{(2n,2N+1)} = \frac{3\rho_3}{4}q_{nxx}^*, \quad C_1^{(2N+1,2n-1)} = -\frac{3\rho_3}{4}q_{nxx}^*,$$

$$C_1^{(2N+1,2n)} = -\frac{3\rho_3}{4}q_{nxx}, \quad C_1^{(2k-1,2n-1)} = \frac{3\rho_3}{4}(q_kq_n^*)_x, \quad C_1^{(2k-1,2n)} = \frac{3\rho_3}{4}(q_kq_n)_x,$$

$$C_1^{(2k,2n-1)} = \frac{3\rho_3}{4}(q_k^*q_n^*)_x, \quad C_1^{(2k,2n)} = \frac{3\rho_3}{4}(q_k^*q_n)_x, \quad C_1^{(2N+1,2N+1)} = \frac{3\rho_3}{2}\sum_{s=1}^{N}|q_s|_x^2.$$

Further, the integrable variable-coefficient N-coupled mKdV type equations are derived:

$$q_{ky} + \rho_1(xq_k)_x + \frac{\rho_3}{4}\left[q_{kxxx} + 3q_k\sum_{s=1}^{N}|q_s|_x^2 + 6q_{kx}\sum_{s=1}^{N}|q_s|^2\right] = 0. \tag{3.30}$$

Under the transformation

$$q_k(x,y) = u_k(X,Y)\exp\frac{-i}{6}\left(X - \frac{Y}{18}\right), \quad y = Y, \quad x = X - \frac{Y}{12}.$$

Equations (3.29) are reduced to the integrable variable-coefficient N-coupled high-order NLS equations:

$$u_{kY} + \left(1 - \frac{\rho_3}{4}\right)\frac{u_{kX}}{12} + \left(\rho_1 - \frac{i}{216} + \frac{3\rho_3}{4}\right)u_k + \rho_1\left(X - \frac{Y}{12}\right)\left(u_{kX} - \frac{i}{6}u_k\right)$$
$$+ \frac{\rho_3}{4}\left[u_{kXXX} + 3u_k\sum_{s=1}^{N}|u_s|_X^2 + 6u_{kX}\sum_{s=1}^{N}|u_s|^2\right] - \frac{i\rho_3}{8}u_{kXX} - \frac{i\rho_3}{4}u_k\sum_{s=1}^{N}|u_s|^2 = 0. \tag{3.31}$$

Then Eqs. (3.29) have Lax pair $\mathbf{N_1}$ and $\mathbf{N_2}$, defined by

$$\mathbf{N_1} = \mathbf{M_1} + \mathbf{D_1}, \quad \mathbf{N_2} = \mathbf{M_2} + \mathbf{D_2}, \quad \mathbf{D_2} = C_1 + C_2\partial_x,$$

where $\mathbf{D_1}$ is the same as before, and

$$C_1 = \frac{3\rho_3}{4}\begin{pmatrix} |q_1|_x^2 & (q_1)_x^2 & \cdots & (q_1q_N^*)_x & (q_1q_N)_x & q_{1xx} \\ (q_1^*)_x^2 & |q_1|_x^2 & \cdots & (q_1^*q_N^*)_x & (q_1^*q_N)_x & q_{1xx}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (q_Nq_1^*)_x & (q_Nq_1)_x & \cdots & |q_N|_x^2 & (q_N)_x^2 & q_{Nxx} \\ (q_N^*q_1^*)_x & (q_N^*q_1)_x & \cdots & (q_N^*)_x^2 & |q_N|_x^2 & q_{Nxx}^* \\ -q_{1xx}^* & -q_{1xx} & \cdots & -q_{Nxx}^* & -q_{Nxx} & 2\sum_{l=1}^{N}|q_l|_x^2 \end{pmatrix}$$

and

$$C_2 = \frac{3\rho_3}{2}\begin{pmatrix} |q_1|^2 & (q_1)^2 & \cdots & q_1q_N^* & q_1q_N & q_{1x} \\ (q_1^*)^2 & |q_1|^2 & \cdots & q_1^*q_N^* & q_1^*q_N & q_{1x}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ q_Nq_1^* & q_Nq_1 & \cdots & |q_N|^2 & (q_N)^2 & q_{Nx} \\ q_N^*q_1^* & q_N^*q_1 & \cdots & (q_N^*)^2 & |q_N|^2 & q_{Nx}^* \\ -q_{1x}^* & -q_{1x} & \cdots & -q_{Nx}^* & -q_{Nx} & 2\sum_{l=1}^{N}|q_l|^2 \end{pmatrix}.$$

Next, we shall formulate one-soliton solutions of (3.29).

Similarly, we obtain the evolution equations for F

$$\alpha F_x + F_z\alpha = 0, \tag{3.32}$$

$$F_y + \rho_1(xF_x + zF_z) + \rho_1 F + \rho_3(F_{xxx} + F_{zzz}) = 0. \tag{3.33}$$

From (3.32), we have

$$\begin{aligned}
&F^{(2k-12N+1)} = w_{2k-12N+1}e^{s_{2k-12N+1}(x+z)},\\
&F^{(2k2N+1)} = w_{2k2N+1}e^{s_{2k2N+1}(x+z)},\\
&F^{(2N+12k-1)} = w_{2N+12k-1}e^{s_{2N+12k-1}(x+z)},\\
&F^{(2N+12k)} = w_{2N+12k}e^{s_{2N+12k}(x+z)},\\
&F^{(2k-12n-1)} = F^{(2k-12n)} = F^{(2k2n-1)} = F^{(2k2n)} = F^{(2N+12N+1)} = 0.
\end{aligned} \tag{3.34}$$

Substitution of (3.33) into (3.32) yields that

$$\begin{aligned}
&\partial_y s_{2k-12N+1} + \rho_1 s_{2k-12N+1} = 0, \quad \partial_y s_{2k2N+1} + \rho_1 s_{2k2N+1} = 0,\\
&\partial_y s_{2N+12k-1} + \rho_1 s_{2N+12k-1} = 0, \quad \partial_y s_{2N+12k} + \rho_1 s_{2N+12k} = 0,\\
&\partial_y w_{2k-12N+1} + (\rho_1 + 2\rho_3 s^3_{2k-12N+1})w_{2k-12N+1} = 0,\\
&\partial_y w_{2k2N+1} + (\rho_1 + 2\rho_3 s^3_{2k2N+1})w_{2k2N+1} = 0,\\
&\partial_y w_{2N+12k-1} + (\rho_1 + 2\rho_3 s^3_{2N+12k-1})w_{2N+12k-1} = 0,\\
&\partial_y w_{2N+12k} + (\rho_1 + 2\rho_3 s^3_{2N+12k})w_{2N+12k} = 0.
\end{aligned} \tag{3.35}$$

Further, we suppose that

$$\begin{aligned}
&s_{2k-12N+1} = s_{2k2N+1} = s_{2N+12k-1} = s_{2N+12k} = c_0 e^{-\int \rho_1 dy},\\
&w_{2k-12N+1} = c_{2k-12N+1}e^{-\int(\rho_1+2\rho_3 s^3_{2k-12N+1})dy}, \quad w_{2k2N+1} = c_{2k2N+1}e^{-\int(\rho_1+2\rho_3 s^3_{2k2N+1})dy},\\
&w_{2N+12k-1} = c_{2N+12k-1}e^{-\int(\rho_1+2\rho_3 s^3_{2k2N+1})dy}, \quad w_{2N+12k} = c_{2N+12k}e^{-\int(\rho_1+2\rho_3 s^3_{2N+12k})dy},
\end{aligned}$$

where $c_{2k-12N+1}$, c_{2k2N+1}, $c_{2N+12k-1}$ and c_{2N+12k} are arbitrary constants, c_0 is an arbitrary negative constant.

In the same way, we assume that $F(x,z,y) = f(x,y)g(z,y)$,

$$f(x,y) = \begin{pmatrix}
0 & \cdots & 0 & w_{12N+1}e^{s_{12N+1}x}\\
0 & \cdots & 0 & w_{22N+1}e^{s_{12N+1}x}\\
\vdots & \ddots & \vdots & \vdots\\
0 & \cdots & 0 & w_{2N2N+1}e^{s_{12N+1}x}\\
w_{2N+11}e^{s_{12N+1}x} & \cdots & w_{2N+12N}e^{s_{12N+1}x} & 0
\end{pmatrix},$$

$$g(z,y) = \begin{pmatrix}
e^{s_{12N+1}z} & 0 & \cdots & 0 & 0\\
0 & e^{s_{12N+1}z} & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & e^{s_{12N+1}z} & 0\\
0 & 0 & \cdots & 0 & e^{s_{12N+1}z}
\end{pmatrix}.$$