# On the Application of a Generalized Version of the Dressing Method to the Integration of Variable Coefficient $\boldsymbol{N}$-Coupled Nonlinear Schrödinger Equation 

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# ON THE APPLICATION OF A GENERALIZED VERSION OF THE DRESSING METHOD TO THE INTEGRATION OF VARIABLE COEFFICIENT $N$-COUPLED NONLINEAR SCHRÖDINGER EQUATION 

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$N$-coupled nonlinear Schrödinger (NLS) equations have been proposed to describe $N$-pulse simultaneous propagation in optical fibers. When the fiber is nonuniform, $N$-coupled variable-coefficient NLS equations can arise. In this paper, a family of $N$-coupled integrable variable-coefficient NLS equations are studied by using a generalized version of the dressing method. We first extend the dressing method to the versions with $(N+1) \times(N+1)$ operators and $(2 N+1) \times(2 N+1)$ operators. Then, we obtain three types of $N$-coupled variable-coefficient equations ( $N$-coupled NLS equations, $N$-coupled Hirota equations and $N$-coupled high-order NLS equations). Then, the compatibility conditions are given, which insure that these equations are integrable. Finally, the explicit solutions of the new integrable equations are obtained.

Keywords: Variable-coefficient; the generalized dressing method; integrability.
Mathematics Subject Classification 2000: 22E46, 53C35, 57S20

## 1. Introduction

The dressing method is a power tool for studying integrable nonlinear evolution equations. This method was first presented by Zakharov and Shabat [15] and used to solve some
nonlinear evolution equations. Subsequently many authors developed the dressing method and obtained explicit solutions of a number of nonlinear evolution equations $[3,5,14,16]$. In [4, 9], Dai and Jeffrey extended the dressing method to a generalized version for solving nonlinear evolution equations with variable-coefficients, in which a key is that variablecoefficient dressing operators are transformed to different variable-coefficient ones.

As the technology of optical fibers for long distance communication and signal processing has rapidly developed, coupled nonlinear Schrödinger (NLS) equations have become a hot research topic. Hasegawa and Tappert [7] theoretically proved that the principle of soliton in optical fiber is based on the exact balance between group velocity dispersion (GVD) and self-phase modulation (SPM). Mollenauer [12] verified the principle with the aid of experiments. $N$-coupled NLS equations (homogeneous and inhomogeneous) model $N$-pulse simultaneous propagation. Nakkeeran in [13] studied a family of $N$-coupled NLS by using Bäcklund transformation method. In [8], Hioe discussed the solitary wave solutions for $N$-coupled NLS equations. Actually, some NLS equations with constant coefficients are so idealized that they fail to model some complex physics situations, thus a class of variable-coefficient Higher-order NLS (HNLS) equations have arisen in physics fibers. Kodama and Hasegawa first presented the integrable variable-coefficient HNLS equation [10]

$$
\begin{align*}
& i u_{z}+a(z) u_{t t}+b(z) u|u|^{2}+\left(c(z, t)+i c_{1}\right) u \\
& \quad+i h_{1}(z, t) u_{t}+p u\left(|u|^{2}\right)_{t}+i l(z)\left(u|u|^{2}\right)_{t}+i b(z) u_{t t t}=0 \tag{1.1}
\end{align*}
$$

In the sequel, many investigators carried out research in this topic from different aspects $[1$, $6,11]$.

In this paper, we first extend the generalized dressing method to $(N+1) \times(N+1)$ matrix operators, from which we propose the integrable variable-coefficient $N$-coupled cylindrical NLS equations:

$$
\begin{align*}
Q_{j t} & +\frac{i \rho_{2}}{2} Q_{j x x}+\left(\rho_{1}+i x^{2} \rho_{1}+a_{0} \rho_{2}-\frac{i \rho_{2}}{2}\right) Q_{j}+x\left(\rho_{1}-\rho_{2}\right) Q_{j x} \\
& -i \epsilon \rho_{2} Q_{j} \sum_{l=1}^{N}\left|Q_{l}\right|^{2}=0, \quad j=1, \ldots, N \tag{1.2}
\end{align*}
$$

which govern simulatance propagation of fields in our homogeneous fiber medium, with the effect of GVD, the inverse velocity and self-phase modulation, where, $\rho_{1}, \rho_{2}$ and $a_{0}$ are arbitrary functions of $t$, the coefficients of $Q_{j}, Q_{j} \sum_{l=1}^{N}\left|Q_{l}\right|^{2}, Q_{j x x}, Q_{j x}$, are related to gain (loss), phase modulation, GVD and the inverse velocity.

And the integrable variable-coefficient $N$-coupled Hirota equations:

$$
\begin{align*}
q_{j, y} & +\frac{i \rho_{2}}{2} q_{j, x x}-i \epsilon \rho_{2} \sum_{n=1}^{N}\left|q_{n}\right|^{2} q_{j}+2 i x \rho_{0} q_{j}+\rho_{1}\left(x q_{j}\right)_{x} \\
& +\frac{\rho_{3}}{4}\left[q_{j, x x x}-3 \epsilon\left(\sum_{n=1}^{N}\left|q_{n}\right|^{2} q_{j, x}+\sum_{n=1}^{N} q_{n}^{*} q_{n, x} q_{j}\right)\right]=0, \tag{1.3}
\end{align*}
$$

with $\rho_{1}, \rho_{2}$ being arbitrary functions of $y$.

In addition, we apply the generalization to the $(2 N+1) \times(2 N+1)$ matrix operators, we derive the integrable variable-coefficient $N$-coupled high-order NLS equations:

$$
\begin{align*}
u_{k Y} & +\left(1-\frac{\rho_{3}}{4}\right) \frac{u_{k X}}{12}+\left(\rho_{1}-\frac{i}{216}+\frac{3 \rho_{3}}{4}\right) u_{k}+\rho_{1}\left(X-\frac{Y}{12}\right)\left(u_{k X}-\frac{i}{6} u_{k}\right) \\
& +\frac{\rho_{3}}{4}\left[u_{k X X X}+3 u_{k} \sum_{s=1}^{N}\left|u_{s}\right|_{X}^{2}+6 u_{k X} \sum_{s=1}^{N}\left|u_{s}\right|^{2}\right] \\
& -\frac{i \rho_{3}}{8} u_{k X X}-\frac{i \rho_{3}}{4} u_{k} \sum_{s=1}^{N}\left|u_{s}\right|^{2}=0 \tag{1.4}
\end{align*}
$$

where $\rho_{1}, \rho_{3}$ are arbitrary functions of $Y$.
This paper is organized as follows. In Sec. 2, we briefly describe the generalized dressing method and its properties. In Sec. 3, as an application, this generalization is applied to a family $N$-coupled NLS equations. Their solutions and Lax pairs are also given. In Sec. 4, we give some simplest cases for reductions and discuss their solutions form.

## 2. A Generalized Dressing Method

First, we summarize the outline of the generalized dressing method. To this end, we consider three integral operators $\mathbf{F}(x, z, y), \mathbf{K}_{+}(x, z, y), \mathbf{K}_{-}(x, z, y)$ defined by

$$
\begin{align*}
\mathbf{K}_{+}(x, z, y) \psi(x) & \equiv \int_{x}^{\infty} K_{+}(x, z, y) \psi(z) d z \\
\mathbf{K}_{-}(x, z, y) \psi(x) & \equiv \int_{-\infty}^{x} K_{-}(x, z, y) \psi(z) d z  \tag{2.1}\\
\mathbf{F}(x, z, y) \psi(x) & \equiv \int_{-\infty}^{\infty} F(x, z, y) \psi(z) d z
\end{align*}
$$

where $\mathbf{F}(x, z, y), \mathbf{K}_{+}(x, z, y), \mathbf{K}_{-}(x, z, y)$ are $n \times n$ matrices, $\psi(x)$ is any $n \times 1$ matrix. $\mathbf{K}_{+}(x, z, y)$ and $\mathbf{K}_{-}(x, z, y)$ are the Volterra operators, so that $K_{+}(x, z, y)=0$ for $z<x$ and $K_{-}(x, z, y)=0$ for $z>x$. We assume that $\left(\mathbf{I}+\mathbf{K}_{+}\right)^{-\mathbf{1}}$ exists and $\mathbf{F}$ admits the triangular factorization

$$
\begin{equation*}
\mathbf{I}+\mathbf{F}=\left(\mathbf{I}+\mathbf{K}_{+}\right)^{-\mathbf{1}}\left(\mathbf{I}+\mathbf{K}_{-}\right), \tag{2.2}
\end{equation*}
$$

where $\mathbf{I}$ is the identity operator. From (2.2), a direct calculation shows that $\mathbf{F}$ and $\mathbf{K}_{+}$ satisfy the Gel'fand-Levitan-Marchenko (GLM) equation [15]

$$
\begin{equation*}
K_{+}(x, z, y)+F(x, z, y)+\int_{x}^{\infty} K_{+}(x, s, y) F(s, z, y) d s=0, \quad z>x \tag{2.3}
\end{equation*}
$$

Similarly, we have

$$
F(x, z, y)-K_{-}(x, z, y)+\int_{x}^{\infty} K_{+}(x, s, y) F(s, z, y) d s=0, \quad z<x
$$

Here it is supposed that

$$
\sup \int_{x_{0}}^{+\infty}\left|K_{ \pm}(x, z, y)\right| \psi(z) d z<+\infty, \quad \sup \int_{x_{0}}^{+\infty}|F(x, z, y)| \psi(z) d z<+\infty, \quad x_{0}>-\infty
$$

We now introduce two differential operators $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{m}}$ defined by

$$
\begin{gather*}
\mathbf{M}_{1}=\alpha \partial_{x}+A\left(x, t_{m}\right) \\
\mathbf{M}_{m}=\beta \partial_{t_{m}}+\mathbf{L}\left(x, t_{m}\right), \quad \mathbf{L}\left(x, t_{m}\right)=\sum_{j=0}^{m} b_{j} \frac{\partial^{j}}{\partial_{x}^{j}}, \tag{2.4}
\end{gather*}
$$

where $\alpha, \beta, b_{j}$ are matrix functions of their arguments. Suppose that the operator $\mathbf{F}$ commutes with $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{m}}$, that is

$$
\begin{equation*}
\left[\mathbf{M}_{1}, \mathbf{F}\right]=\mathbf{M}_{1} \mathbf{F}-\mathbf{F M}_{1}=0, \quad\left[\mathbf{M}_{\mathbf{m}}, \mathbf{F}\right]=\mathbf{M}_{\mathrm{m}} \mathbf{F}-\mathbf{F M}_{\mathbf{m}}=0 \tag{2.5}
\end{equation*}
$$

Equation (2.5) together with (2.4) implies the following equations

$$
\begin{gather*}
\alpha \mathbf{F}_{x}+\mathbf{F}_{z} \alpha+A\left(x, t_{m}\right) \mathbf{F}-\mathbf{F} A\left(z, t_{m}\right)=0,  \tag{2.6}\\
\beta \mathbf{F}_{t_{m}}+\mathbf{L F}-\mathbf{F L}^{+}\left(z, t_{m}\right)=0, \tag{2.7}
\end{gather*}
$$

where

$$
\mathbf{F} \mathbf{L}^{+}\left(z, t_{m}\right)=\sum_{j=0}^{m}(-1)^{j} \frac{\partial^{j}}{\partial_{z}^{j}}\left(\mathbf{F} b_{j}\left(z, t_{m}\right)\right) .
$$

Now we "dress" the two differential operators $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{m}}$ to obtain the dressed operators $\mathbf{N}_{1}$ and $\mathbf{N}_{\mathbf{m}}$. The dressing procedure is accomplished through the relations

$$
\begin{array}{r}
\mathbf{N}_{\mathbf{1}}\left(\mathbf{I}+\mathbf{K}_{+}\right)-\left(\mathbf{I}+\mathbf{K}_{+}\right) \mathbf{M}_{\mathbf{1}}=0 \\
\mathbf{N}_{\mathbf{m}}\left(\mathbf{I}+\mathbf{K}_{+}\right)-\left(\mathbf{I}+\mathbf{K}_{+}\right) \mathbf{M}_{\mathbf{m}}=0 \tag{2.8}
\end{array}
$$

where $\mathbf{N}_{1}$ and $\mathbf{N}_{\mathbf{m}}$ can be ensured to be simple differential operators by the above equations.
A key of the generalized dressing method is to let the differential operators $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{m}}$ satisfy the relation

$$
\begin{equation*}
\left[\mathbf{M}_{\mathbf{1}}, \mathbf{M}_{\mathbf{m}}\right]=\phi_{1} \mathbf{M}_{\mathbf{1}}+\phi_{2} \mathbf{M}_{\mathbf{m}} \tag{2.9}
\end{equation*}
$$

where $\phi_{1}$ and $\phi_{2}$ are arbitrary functions of their arguments. According to [4, 9], the corresponding dressing operators obey the equation

$$
\begin{equation*}
\left[\mathbf{N}_{\mathbf{1}}, \mathbf{N}_{\mathbf{m}}\right]=\phi_{1} \mathbf{N}_{\mathbf{1}}+\phi_{2} \mathbf{N}_{\mathbf{m}} \tag{2.10}
\end{equation*}
$$

We consider the case of $m=3$ and denote $t_{3}=y, \mathbf{N}_{\mathbf{1}}=\mathbf{M}_{\mathbf{1}}+\mathbf{D}_{\mathbf{1}}, \mathbf{N}_{\mathbf{2}}=\mathbf{M}_{\mathbf{3}}+\mathbf{D}_{\mathbf{2}}$. For convenience, we denote $\hat{K}=\left.K_{+}(x, z)\right|_{z=x}$. From the first expression of (2.8), we obtain

$$
\begin{gather*}
\mathbf{D}_{1}=\alpha \widehat{K}-\widehat{K} \alpha  \tag{2.11}\\
\alpha K_{x}+A(x, y) K+\mathbf{D}_{1} K+K_{z} \alpha-K A(z, y)=0
\end{gather*}
$$

Using the second expression of (2.8), we have $\mathbf{D}_{\mathbf{2}}=C_{1}+C_{2} \partial_{x}$, where

$$
\begin{gather*}
C_{2}-b_{2} \widehat{K}+\widehat{K} b_{2}-2 b_{3} \widehat{K}_{x}-\left.b_{3} K_{x}\right|_{z=x}-\left.\left(K b_{3}(z)\right)_{z}\right|_{z=x}=0, \\
\widehat{K} b_{3}-b_{3} \widehat{K}=0 \tag{2.12}
\end{gather*}
$$

and $C_{1}$ is determined from (2.10) by the following two equations:

$$
\begin{align*}
& \alpha C_{1 x}+A C_{1}-C_{1} A-C_{2} A_{x}+\mathbf{D}_{1} b_{0}-b_{0} \mathbf{D}_{\mathbf{1}}-\mathbf{D}_{1, y}-b_{1} \mathbf{D}_{1 x}-b_{2} \mathbf{D}_{1 x x} \\
& \quad-b_{3} \mathbf{D}_{1 x x x}+\mathbf{D}_{1} C_{1}-C_{1} \mathbf{D}_{1}-C_{2} \mathbf{D}_{1 x}=\phi_{1} \mathbf{D}_{\mathbf{1}}  \tag{2.13}\\
& \alpha C_{1} \\
& \quad-C_{1} \alpha+\alpha C_{2 x}+A C_{2}-C_{2} A+\mathbf{D}_{\mathbf{1}} b_{1}-b_{1} \mathbf{D}_{1}  \tag{2.14}\\
& \quad-2 b_{2} \mathbf{D}_{1 x}-3 b_{3} \mathbf{D}_{1 x x}+\mathbf{D}_{1} C_{2}-C_{2} \mathbf{D}_{1}=0
\end{align*}
$$

Further, from (2.9), we derive $\phi_{2}=0$ and

$$
\begin{gather*}
\alpha b_{0}-b_{0} \alpha+A b_{1}-b_{1} A+\alpha b_{1 x}-2 b_{2} A_{x}-3 b_{3} A_{x x}=\phi_{1} \alpha \\
\alpha b_{0 x}-\beta A_{y}-b_{0} A+A b_{0}-b_{1} A_{0 x}-b_{2} A_{x x}-b_{3} A_{x x x}=\phi_{1} A . \tag{2.15}
\end{gather*}
$$

Actually, we obtain nonlinear evolution equations from (2.13). In what follows, we give the solution formula of the obtained equations.

Assume that (2.6) and (2.7) have solutions in the form of separation of variables

$$
\begin{equation*}
F(x, z, y)=\sum_{j=1}^{N} f_{j}(x, y) g_{j}(z, y) \tag{2.16}
\end{equation*}
$$

where $f_{j}(x, y), g_{j}(z, y)$ are some $n \times n$ matrices. Moreover, we suppose that

$$
\begin{equation*}
K(x, z, y)=\sum_{j=1}^{N} k_{j}(x, y) g_{j}(z, y) . \tag{2.17}
\end{equation*}
$$

Substituting (2.16) and (2.17) into the GLM equation (2.3) yields that

$$
\begin{equation*}
\widehat{K}=\sum_{j=1}^{N} k_{j}(x, y) g_{j}(x, y)=-\left(f_{1}, f_{2}, \ldots, f_{N}\right) L^{-1}\left(g_{1}, g_{2}, \ldots, g_{N}\right)^{T} \tag{2.18}
\end{equation*}
$$

where $L$ is defined by

$$
\begin{equation*}
L_{j l}=\delta_{j l}+\int_{x}^{\infty} g_{j}(s, y) f_{l}(s, y) d s, \quad 1 \leq j, \quad l \leq N \tag{2.19}
\end{equation*}
$$

and $\delta_{j l}$ is the Kronecker's delta.
For the case of $N=1$ in (2.16), we have known that

$$
\begin{equation*}
\widehat{K}=-f_{1}\left[1+\int_{x}^{\infty} g_{1}(s) f_{1}(s) d s\right]^{-1} g_{1} \tag{2.20}
\end{equation*}
$$

In view of (2.20), the one-soliton solutions of the obtained equation are given.

## 3. Applications to the Integrable Variable-Coefficient $N$-Coupled NLS Equations

In this section, we will discuss a family of the integrable variable-coefficient $N$-coupled NLS equations, including integrable variable-coefficient $N$-coupled cylindrical NLS equations, integrable variable-coefficient $N$-coupled Hirota equations, integrable variable-coefficient $N$-coupled mKdV-type equations and high-order NLS equations. Further, some new conclusions of these equations are given. For convenience, the same symbols stand for different means in different sections.

### 3.1. Variable-coefficient $N$-coupled cylindrical NLS equations

Let $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{2}}$ be

$$
\begin{align*}
& \mathbf{M}_{\mathbf{1}}=\alpha \partial_{x}+x a_{0} I  \tag{3.1}\\
& \mathbf{M}_{\mathbf{2}}=I \partial_{t}+\rho_{2} \alpha \partial_{x}^{2}+x \rho_{1} I \partial_{x} \tag{3.2}
\end{align*}
$$

where $\rho_{1}, \rho_{2}$ and $a_{0}$ are functions of $t$, and

$$
\alpha=i\left(\begin{array}{rrrr}
1 & \cdots & 0 & 0  \tag{3.3}\\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & -1
\end{array}\right)_{(N+1) \times(N+1)} \quad I=\left(\begin{array}{cccc}
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1
\end{array}\right)_{(N+1) \times(N+1)} .
$$

Substituting (3.1) and (3.2) into (2.15), we have

$$
\rho_{1}-2 \rho_{2} a_{0}=\phi_{1}, \quad a_{0, t}+2 \rho_{1} a_{0}-2 \rho_{2} a_{0}^{2}=0 .
$$

From which, it yields that $a_{0}=\frac{1}{e^{2 \int \rho_{1} d t}\left(c_{0}-\int 2 \rho_{2} e^{-2 \int \rho_{1} d t} d t\right)}$, where $c_{0}$ is an integration constant.

Similarly, we derive from (2.11)

$$
\mathbf{D}_{\mathbf{1}}=i\left(\begin{array}{cccc}
0 & \cdots & 0 & q_{1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & q_{N} \\
-\epsilon q_{1}^{*} & \cdots & -\epsilon q_{N}^{*} & 0
\end{array}\right)
$$

with $\hat{k}^{(j N+1)}=\frac{q_{j}}{2}, \hat{k}^{(N+1 j)}=\epsilon \frac{q_{j}^{*}}{2}, \widehat{k}_{x}^{(l j)}=-\epsilon \frac{q_{l} q_{j}^{*}}{2}, \widehat{k}_{x}^{(N+1 N+1)}=-\frac{\epsilon}{2} \sum_{k=1}^{N}\left|q_{k}\right|^{2}, \epsilon^{2}=1$, $(l, j=1, \ldots, N)$. It yields from (2.12) that $C_{2}=\rho_{2} \mathbf{D}_{\mathbf{1}}$. For the sake of simplicity, we denote $C_{1}=\left(C_{1}^{(k s)}\right)_{(N+1) \times(N+1)}, C_{2}=\left(C_{2}^{(k s)}\right)_{(N+1) \times(N+1)},(k, s=1, \ldots, N+1)$.

From (2.13) and (2.14), we obtain

$$
\begin{gather*}
C_{1}^{(j l)}=-i \frac{\epsilon \rho_{2}}{2} q_{j} q_{l}^{*}, \quad C_{1}^{(j N+1)}=i \frac{\rho_{2}}{2} q_{j x}, \\
C_{1}^{(N+1 j)}=-i \frac{\epsilon \rho_{2}}{2} q_{j x}^{*}, \quad C_{1}^{(N+1 N+1)}=i \frac{\epsilon \rho_{2}}{2} \sum_{l=1}^{N}\left|q_{l}\right|^{2} \tag{3.4}
\end{gather*}
$$

and the integrable variable-coefficient $N$-coupled cylindrical NLS equations:

$$
\begin{equation*}
q_{j t}+\rho_{1}\left(x q_{j}\right)_{x}+\rho_{2} a_{0} q_{j}+i \frac{\rho_{2}}{2} q_{j x x}-i \epsilon \rho_{2} q_{j} \sum_{l=1}^{N}\left|q_{l}\right|^{2}=0, \quad j=1, \ldots, N . \tag{3.5}
\end{equation*}
$$

Under the transformation $q_{j}=Q_{j} e^{\frac{i}{2} x^{2}}$, the above equations are reduced to different integrable variable-coefficient $N$-coupled cylindrical NLS equations:

$$
\begin{align*}
Q_{j t} & +\frac{i \rho_{2}}{2} Q_{j x x}+\left(\rho_{1}+i x^{2} \rho_{1}+a_{0} \rho_{2}-\frac{i \rho_{2}}{2}\right) Q_{j}+x\left(\rho_{1}-\rho_{2}\right) Q_{j x} \\
& -i \epsilon \rho_{2} Q_{j} \sum_{l=1}^{N}\left|Q_{l}\right|^{2}=0, \quad j=1, \ldots, N \tag{3.6}
\end{align*}
$$

Equation (3.5) have Lax pair $\mathbf{N}_{\mathbf{1}}$ and $\mathbf{N}_{\mathbf{2}}$ defined by

$$
\mathbf{N}_{1}=\mathbf{M}_{1}+\mathbf{D}_{1}, \quad \mathbf{N}_{\mathbf{2}}=\mathbf{M}_{\mathbf{2}}+\mathbf{D}_{\mathbf{2}}, \quad \mathbf{D}_{\mathbf{2}}=C_{1}+C_{2} \partial_{x}
$$

where

$$
C_{1}=-\frac{i \rho_{2}}{2}\left(\begin{array}{cccc}
\epsilon\left|q_{1}\right|^{2} & \cdots & \epsilon q_{1} q_{N}^{*} & -q_{1 x} \\
\epsilon q_{2} q_{1}^{*} & \cdots & \epsilon q_{2} q_{N}^{*} & -q_{2 x} \\
\vdots & \ddots & \vdots & \vdots \\
\epsilon q_{N} q_{1}^{*} & \cdots & \epsilon\left|q_{N}\right|^{2} & -q_{N x} \\
\epsilon q_{1 x}^{*} & \cdots & \epsilon q_{N x}^{*} & -\epsilon \sum_{l=1}^{N}\left|q_{l}\right|^{2}
\end{array}\right), \quad C_{2}=i \rho_{2}\left(\begin{array}{cccc}
0 & \cdots & 0 & q_{1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & q_{N} \\
-\epsilon q_{1}^{*} & \cdots & -\epsilon q_{N}^{*} & 0
\end{array}\right) .
$$

In what follows, we will derive the one-soliton solution of (3.5).
From (2.6) and (2.7), we have the equations for $F$

$$
\begin{align*}
\alpha F_{x}+F_{z} \alpha+(x-z) a_{0} F & =0,  \tag{3.7}\\
F_{t}+\rho_{1}(t)\left(x F_{x}+z F_{z}\right)+\rho_{1}(t) F+\rho_{2}(t)\left(\alpha F_{x x}-F_{z z} \alpha\right) & =0 . \tag{3.8}
\end{align*}
$$

In view of (3.7), we set $F$ as follows

$$
\begin{array}{ll}
F^{(j N+1)}=w_{j N+1} e^{i \frac{a_{0}}{2}\left(x^{2}+z^{2}\right)+\mu_{j N+1}(x+z)}, & F^{(j l)}=0, \\
F^{(N+1 j)}=w_{N+1 j} e^{-i \frac{a_{0}}{2}\left(x^{2}+z^{2}\right)+\mu_{N+1 j}(x+z)}, & F^{(N+1 N+1)}=0, \tag{3.9}
\end{array}
$$

with $F^{(j l)}$ being elements of $F$.
Substitution of (3.9) into (3.8), we derive

$$
\begin{align*}
& \partial_{t} \mu_{j N+1}+\left[\rho_{1}-2 a_{0} \rho_{2}\right] \mu_{j N+1}=0, \\
& \partial_{t} \mu_{N+1 j}+\left[\rho_{1}-2 a_{0} \rho_{2}\right] \mu_{N+1 j}=0, \\
& \partial_{t} w_{j N+1}+\left[\rho_{1}-2 a_{0} \rho_{2}+2 i \rho_{2} \mu_{j N+1}^{2}\right] w_{j N+1}=0,  \tag{3.10}\\
& \partial_{t} w_{N+1 j}+\left[\rho_{1}-2 a_{0} \rho_{2}-2 i \rho_{2} \mu_{N+1 j}^{2}\right] w_{N+1 j}=0 .
\end{align*}
$$

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We assume that $\mu_{j N+1}=\mu_{1 N+1}, \mu_{N+1 j}=\mu_{N+11}$ and $F=f(x, t) g(z, t)$, with

$$
\begin{aligned}
& f(x, t)=\left(\begin{array}{ccccc}
0 & \cdots & 0 & w_{1 N+1} e^{i \frac{a_{0} x^{2}}{2}+\mu_{1 N+1} x} \\
\vdots & \ddots & & \vdots & \vdots \\
0 & \cdots & & 0 & w_{N N+1} e^{i \frac{a_{0} x^{2}}{2}+\mu_{1 N+1} x} \\
w_{N+11} e^{-i \frac{a_{0} x^{2}}{2}+\mu_{N+11} x} & \cdots & w_{N+1 N} e^{-i \frac{a_{0} x^{2}}{2}+\mu_{N+11} x} & 0
\end{array}\right), \\
& g(z, t)=\left(\begin{array}{ccccc}
e^{-i \frac{a_{0} z^{2}}{2}+\mu_{N+11} z} & 0 & \cdots & 0 & 0 \\
0 & e^{-i \frac{a_{0} z^{2}}{2}+\mu_{N+11} z} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & e^{-i \frac{a_{0} z^{2}}{2}+\mu_{N+11} z} & 0 \\
0 & 0 & \cdots & 0 & e^{i \frac{a_{0} z^{2}}{2}+\mu_{1 N+1} z}
\end{array}\right)
\end{aligned}
$$

Let $K=k(x, t) g(z, t)$. We take $N=1$ in (2.16). From (2.19), we obtain

$$
L=\left(\begin{array}{cccc}
1 & \cdots & 0 & -\frac{w_{1 N+1} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x}}{\mu_{1 N+1}+\mu_{N+11}} \\
0 & \cdots & 0 & -\frac{w_{2 N+1} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x}}{\mu_{1 N+1}+\mu_{N+1,1}} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & -\frac{w_{N N+1} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x}}{\mu_{1 N+1}+\mu_{N+11}} \\
-\frac{w_{N+11} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x}}{\mu_{1 N+1}+\mu_{N+11}} & \cdots & -\frac{w_{N+1 N e}}{\mu_{1 N+1}+\mu_{N+11}} & 1
\end{array}\right)
$$

where $|L|=1-\sum_{j=1}^{N} \frac{w_{j N+1} w_{N+1 j}}{\left(\mu_{N+11}+\mu_{1 N+1}\right)^{2}} e^{2\left(\mu_{N+11}+\mu_{1 N+1}\right) x}, L_{(N+1 N+1)}^{*}=1, \quad L=$ $\left(L_{(l k)}\right)_{(N+1) \times(N+1)},|L|$ is determinant of $L, L_{(l k)}^{*}$ is algebraic cofactor of $L$.

From (2.20), we can derive

$$
\begin{aligned}
\widehat{K}= & -\frac{1}{|L|} \\
& \times\left(\begin{array}{ccc}
w_{1 N+1} L_{(1 N+1)}^{*} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x} & \ldots & w_{1 N+1} e^{2 \mu_{1 N+1} x+i a_{0} x^{2}} \\
w_{2 N+1} L_{(1 N+1)}^{*} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x} & \ldots & w_{2 N+1} e^{2 \mu_{1 N+1} x+i a_{0} x^{2}} \\
\vdots & \ddots & \vdots \\
w_{N N+1} L_{(1 N+1)}^{*} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x} & \ldots & w_{N N+1} e^{2 \mu_{1 N+1} x+i a_{0} x^{2}} \\
\sum_{k=1}^{N} w_{N+1 k} L_{(1 k)}^{*} e^{2 \mu_{N+11} x-i a_{0} x^{2}} & \ldots & \sum_{k=1}^{N} w_{N+1 k} L_{(N+1 k)}^{*} e^{\left(\mu_{1 N+1}+\mu_{N+11}\right) x}
\end{array}\right) .
\end{aligned}
$$

From which, it yields the one-soliton solution of (3.5):

$$
q_{j}=-\frac{2}{|L|} w_{j N+1} e^{i a_{0} x^{2}+2 \mu_{1 N+1} x}, \quad q_{j}^{*}=-\frac{2}{|L|} w_{N+1 j} e^{-i a_{0} x^{2}+2 \mu_{N+11} x}
$$

where

$$
\begin{gathered}
\sum_{k=1}^{N} w_{N+1 k} L_{(j k)}^{*}=w_{N+1 j}, \quad w_{j N+1}=\epsilon w_{N+1 j}^{*}, \quad \mu_{N+11}=\mu_{1 N+1}^{*}, \\
L_{(j k)}^{*}=w_{N+1 j} w_{k N+1}, \quad L_{(j j)}^{*}=1-\sum_{k=1, k \neq j}^{N} w_{N+1 k} w_{k N+1},
\end{gathered}
$$

are used.
Further, we obtain the one-soliton solution of (3.6):

$$
Q_{j}=-\frac{2 q_{j}}{|L|} w_{j N+1} e^{i\left(a_{0}-\frac{1}{2}\right) x^{2}+2 \mu_{1 N+1} x} .
$$

According to the form $|L|$, we can find that $\frac{1}{|L|}$ has not singular solution for $(\epsilon=-1)$ and has singular solution for $(\epsilon=1)$. Therefore, the one-soliton solution of (3.5) for has singular solution for $(\epsilon=1)$. Theoretically, we can give $N$-soliton solution of (3.5) and (3.6). However, it is necessary to complex calculations with the aid of mathematics.

### 3.2. Variable-coefficient $N$-coupled Hirota equations

We set $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{3}}$ to be

$$
\begin{align*}
& \mathbf{M}_{\mathbf{1}}=\alpha \partial_{x}+a_{0}(y) I  \tag{3.11}\\
& \mathbf{M}_{\boldsymbol{3}}=I \partial_{y}+I \rho_{3}(y) \partial_{x}^{3}+\rho_{2}(y) \alpha \partial_{x}^{2}+I x \rho_{1}(y) \partial_{x}+x \rho_{0}(y) \alpha \tag{3.12}
\end{align*}
$$

with $\alpha$ and $I$ are given in (3.3).
Substituting (3.11) and (3.12) into (2.15), which gives that

$$
\rho_{1}=\phi_{1}, \quad a_{0, y}+\rho_{1} a_{0}+\rho_{0}=0,
$$

from which, we have $a_{0}=e^{-\int \rho_{1} d y}\left(c-\int \rho_{0} e^{\int \rho_{1} d y} d y\right)$, with $c$ being an arbitrary constant.
In the same way, we obtain

$$
\mathbf{D}_{1}=i\left(\begin{array}{cccc}
0 & \cdots & 0 & q_{1} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & q_{N} \\
-\epsilon q_{1}^{*} & \cdots & -\epsilon q_{N}^{*} & 0
\end{array}\right)
$$

where, $\hat{k}^{(j N+1)}=\frac{q_{j}}{2}, \hat{k}^{(N+1 j)}=\frac{\epsilon q_{j}^{*}}{2}, \hat{k}_{x}^{(N+1 N+1)}=-\epsilon \sum_{j=1}^{N} \frac{\left|q_{j}\right|^{2}}{2}, \hat{k}_{x}^{(l j)}=-\epsilon \frac{q_{l} q_{j}^{*}}{2}, l, j=$ $1, \ldots, N$.

Using (2.12)-(2.14), we have $C_{2}=\rho_{2} \mathbf{D}_{1}+3 \rho_{3} \hat{K}_{x} . C_{1}$ is to be determined by (3.13) and (3.14).

Substitution of (3.11) and (3.12) into (2.13), we have

$$
\begin{equation*}
\alpha C_{1}-C_{1} \alpha-\rho_{2} \alpha \mathbf{D}_{1, x}+3 \rho_{3} \alpha \widehat{K}_{x x}-3 \rho_{3} \mathbf{D}_{1, x x}+3 \rho_{3}\left(\mathbf{D}_{1} \widehat{K}_{x}-\widehat{K}_{x} \mathbf{D}_{\mathbf{1}}\right)=0 \tag{3.13}
\end{equation*}
$$

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$$
\begin{align*}
& \alpha C_{1, x}+x \rho_{0}\left(\mathbf{D}_{\mathbf{1}} \alpha-\alpha \mathbf{D}_{\mathbf{1}}\right)-x \rho_{1} \mathbf{D}_{1, x}-\rho_{2} \mathbf{D}_{1, x x}-\rho_{3} \mathbf{D}_{1, x x x} \\
& \quad+\mathbf{D}_{\mathbf{1}} C_{1}-C_{1} \mathbf{D}_{\mathbf{1}}-C_{2} \mathbf{D}_{1, x}-\mathbf{D}_{1, y}=\phi_{1} \mathbf{D}_{\mathbf{1}} \tag{3.14}
\end{align*}
$$

From which, we derive

$$
\begin{gather*}
C_{1}^{(l j)}=-\frac{3 \epsilon \rho_{3}}{4}\left(q_{l} q_{j}^{*}\right)_{x}-\frac{i \epsilon \rho_{2}}{2} q_{l} q_{j}^{*}, \quad C_{1}^{(N+1 N+1)}=-\frac{3 \epsilon \rho_{3}}{4} \sum_{j=1}^{N}\left(\left|q_{j}\right|^{2}\right)_{x}+\frac{i \epsilon \rho_{2}}{2} \sum_{j=1}^{N}\left|q_{j}\right|^{2}, \\
C_{1}^{(j N+1)}=\frac{i \rho_{2}}{2} q_{j x}+\frac{3 \rho_{3}}{4} q_{j, x x}, \quad C_{1}^{(N+1 j)}=-\epsilon \frac{i \rho_{2}}{2} q_{j x}^{*}+\frac{3 \epsilon \rho_{3}}{4} q_{j, x x}^{*} \tag{3.15}
\end{gather*}
$$

and the integrable variable-coefficient $N$-coupled Hirota equations:

$$
\begin{align*}
q_{j, y} & +\frac{i \rho_{2}}{2} q_{j, x x}-i \epsilon \rho_{2} \sum_{n=1}^{N}\left|q_{n}\right|^{2} q_{j}+2 i x \rho_{0} q_{j}+\rho_{1}\left(x q_{j}\right)_{x} \\
& +\frac{\rho_{3}}{4}\left[q_{j, x x x}-3 \epsilon\left(\sum_{n=1}^{N}\left|q_{n}\right|^{2} q_{j, x}+\sum_{n=1}^{N} q_{n}^{*} q_{n, x} q_{j}\right)\right]=0 . \tag{3.16}
\end{align*}
$$

The nonlinear wave propagation of simultaneous fields in an optical fiber with core medium for not homogeneous, with the effects, various GVD, SPM, higher-order dispersion (HOD) and Kerr dispersion is by governed by the integrable variable-coefficients $N$-coupled by (3.16).

The Lax pair is $\mathbf{N}_{\mathbf{1}}$ and $\mathbf{N}_{\mathbf{2}}$, given by

$$
\mathbf{N}_{\mathbf{1}}=\mathbf{M}_{1}+\mathbf{D}_{\mathbf{1}}, \quad \mathbf{N}_{\mathbf{2}}=\mathbf{M}_{\mathbf{2}}+\mathbf{D}_{\mathbf{2}}, \quad \mathbf{D}_{\mathbf{2}}=C_{1}+C_{2} \partial_{x}
$$

with

$$
\begin{aligned}
& C_{1}=\left(\begin{array}{ccc}
-\frac{3 \epsilon \rho_{3}}{4}\left|q_{1}\right|_{x}^{2}-\frac{i \epsilon \rho_{2}}{2}\left|q_{1}\right|^{2} & \cdots & \frac{3 \rho_{3}}{4} q_{1 x x}+\frac{i \rho_{2}}{2} q_{1 x} \\
-\frac{3 \epsilon \rho_{3}}{4}\left(q_{2} q_{1}^{*}\right)_{x}-\frac{i \epsilon \rho_{2}}{2} q_{2} q_{1}^{*} & \cdots & \frac{3 \rho_{3}}{4} q_{2 x x}+\frac{i \rho_{2}}{2} q_{2 x} \\
\vdots & & \ddots \\
\vdots \\
-\frac{3 \epsilon \rho_{3}}{4}\left(q_{N} q_{1}^{*}\right)_{x}-\frac{i \epsilon \rho_{2}}{2} q_{N} q_{1}^{*} & \cdots & \vdots \\
\frac{3 \epsilon \rho_{3}}{4} q_{1 x x}^{*}-\frac{i \epsilon \rho_{2}}{2} q_{1}^{*} & \cdots & -\frac{3 \epsilon \rho_{3}}{4} \sum_{j=1}^{N}\left|q_{j}\right|_{x}^{2}+\frac{i \epsilon \rho_{2}}{2} \sum_{j=1}^{N}\left|q_{j}\right|^{2}
\end{array}\right) \\
& C_{2}=\left(\begin{array}{ccc}
-\frac{3 \epsilon \rho_{3}}{2}\left|q_{1}\right|^{2} & \ldots & \frac{3 \rho_{3}}{2} q_{1 x}+i \rho_{2} q_{1} \\
-\frac{3 \epsilon \rho_{3}}{3} q_{2} q_{1}^{*} & \ldots & \frac{3 \rho_{3}}{2} q_{2 x}+i \rho_{2} q_{2} \\
\vdots & \ddots & \vdots \\
-\frac{3 \epsilon \rho_{3}}{2} q_{N} q_{1}^{*} & \cdots & \frac{3 \rho_{3}}{2} q_{N x}+i \rho_{2} q_{N} \\
-\frac{3 \epsilon \rho_{3}}{2} q_{1 x}^{*}-i \epsilon \rho_{2} q_{1}^{*} & \cdots & -\frac{i 3 \epsilon \rho_{3}}{2} \sum_{j=1}^{N}\left|q_{j}\right|^{2}
\end{array}\right)
\end{aligned}
$$

In the sequel, we shall discuss the solution of (3.16).

From (2.6) and (2.7), we have the equations for $F$

$$
\begin{gather*}
\alpha F_{x}+F_{z} \alpha=0  \tag{3.17}\\
F_{y}+\rho_{0}(x \alpha F-z F \alpha)+\rho_{1}\left(x F_{x}+z F_{z}\right)+\rho_{1} F  \tag{3.18}\\
+\rho_{2}\left(\alpha F_{x x}-F_{z z} \alpha\right)+\rho_{3}\left(F_{x x x}+F_{z z z}\right)=0
\end{gather*}
$$

From (3.17), it is easy to derive

$$
\begin{align*}
& F_{x}^{(j N+1)}-F_{z}^{(j N+1)}=0, \quad F_{x}^{(N+1 j)}-F_{z}^{(N+1 j)}=0,  \tag{3.19}\\
& F_{x}^{(l j)}+F_{z}^{(l j)}=0, \quad F_{x}^{(N+1 N+1)}+F_{z}^{(N+1 N+1)}=0 . \tag{3.20}
\end{align*}
$$

Let

$$
\begin{gather*}
F^{(j N+1)}=w_{j N+1} e^{s_{j N+1}(x+z)}, \quad F^{(N+1 j)}=w_{N+1 j} e^{s_{N+1 j}(x+z)}, \\
F^{(l j)}=0, \quad F^{(N+1 N+1)}=0, \quad l, j=1, \ldots, N . \tag{3.21}
\end{gather*}
$$

Substitution of (3.20) into (3.18), we arrive at

$$
\begin{align*}
& \partial_{y} s_{j N+1}+\rho_{1} s_{j N+1}+i \rho_{0}=0, \quad \partial_{y} s_{N+1 j}+\rho_{1} s_{N+1 j}-i \rho_{0}=0, \\
& \partial_{y} w_{j N+1}+\left(\rho_{1}+2 i \rho_{2} s_{j N+1}^{2}+2 \rho_{3} s_{j N+1}^{3}\right) w_{j N+1}=0,  \tag{3.22}\\
& \partial_{y} w_{N+1 j}+\left(\rho_{1}-2 i \rho_{2} s_{N+1 j}^{2}+2 \rho_{3} s_{N+1 j}^{3}\right) w_{N+1 j}=0,
\end{align*}
$$

from which, we obtain

$$
\begin{aligned}
& s_{j N+1}=e^{-\int \rho_{1} d y}\left(c_{0}-i \int \rho_{0} e^{\int \rho_{1} d y} d y\right), \\
& s_{N+1 j}=e^{-\int \rho_{1} d y}\left(c_{0}+i \int \rho_{0} e^{\int \rho_{1} d y} d y\right), \\
& w_{j N+1}=c_{j N+1} e^{-\int\left(\rho_{1}+2 i \rho_{2} s_{j N+1}^{2}+2 \rho_{3} s_{j N+1}^{3}\right) d y}, \\
& w_{N+1 j}=c_{N+1 j} e^{\int\left(-\rho_{1}+2 i \rho_{2} s_{j N+1}^{2}-2 \rho_{3} s_{j N+1}^{3}\right) d y},
\end{aligned}
$$

with $c_{0}, c_{j N+1}$ and $c_{N+1 j}$ being arbitrary constants.
We suppose that $s_{j N+1}=s_{1 N+1}, s_{N+1 j}=s_{N+11}, s_{N+11}^{*}=s_{1 N+1}$ and $F(x, z, y)=$ $f(x, y) g(z, y)$, where $f(x, y)$ and $g(z, y)$ are given by

$$
f(x, y)=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & w_{1 N+1} e^{s_{1 N+1} x} \\
0 & 0 & \cdots & 0 & w_{2 N+1} e^{s_{1 N+1} x} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & w_{N N+1} e^{s_{1 N+1} x} \\
w_{N+11} e^{s_{N+11} x} & w_{N+12} e^{s_{N+11} x} & \cdots & w_{N+1 N} e^{s_{N+11} x} & 0
\end{array}\right)
$$

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$$
g(z, y)=\left(\begin{array}{ccccc}
e^{s_{N+11} z} & 0 & \cdots & 0 & 0 \\
0 & e^{s_{N+11} z} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & e^{s_{N+11} z} & 0 \\
0 & 0 & \cdots & 0 & e^{s_{1 N+1} z}
\end{array}\right)
$$

Using (2.16) $(N=1)$, it yields that

$$
L=\left(\begin{array}{cccc}
1 & \cdots & 0 & -\frac{w_{1 N+1} e^{\left(s_{1 N+1}+s_{N+11}\right) x}}{s_{1 N+1}+s_{N+11}} \\
0 & \cdots & 0 & -\frac{w_{2 N+1} e^{\left(s_{1 N+1}+s_{N+11}\right) x}}{s_{1 N+1}+s_{N+11}} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & -\frac{w_{N N+1} e^{\left(s_{1 N+1}+s_{N+11}\right) x}}{s_{1 N+1}+s_{N+11}} \\
-\frac{w_{N+11 e^{\left(s_{1 N+1}+s_{N+11}\right) x}}^{s_{1 N+1}+s_{N+11}}}{} & \cdots & -\frac{w_{N+1 N} e^{\left(s_{1 N+1}+s_{N+11}\right) x}}{s_{1 N+1}+s_{N+11}} & 1
\end{array}\right)
$$

From (2.20), we have

$$
\begin{aligned}
\widehat{K}= & -\frac{1}{|L|} \\
& \times\left(\begin{array}{ccc}
w_{1 N+1} L_{(1 N+1)}^{*} e^{\left(s_{1 N+1}+s_{N+11}\right) x} & \ldots & w_{1 N+1} e^{2 s_{1 N+1} x} \\
w_{2 N+1} L_{(1 N+1)}^{*} e^{\left(s_{1 N+1}+s_{N+11}\right) x} & \ldots & w_{2 N+1} e^{2 s_{1 N+1} x} \\
\vdots & \ddots & \vdots \\
w_{N N+1} L_{(1 N+1)}^{*} e^{\left(s_{1 N+1}+s_{N+11}\right) x} & \ldots & w_{N N+1} e^{2 s_{1 N+1} x} \\
\sum_{j=1}^{N} w_{N+1 j} L_{(1 j)}^{*} e^{2 s_{N+11} x} & \cdots & \sum_{j=1}^{N} w_{N+1 j} L_{(N+1 j)}^{*} e^{\left(s_{1 N+1}+s_{N+11}\right) x}
\end{array}\right)
\end{aligned}
$$

Thus, the one-soliton solutions of (3.16) are given by

$$
\begin{equation*}
q_{j}=-\frac{2}{|L|} w_{j N+1} e^{2 s_{1 N+1} x}, \tag{3.23}
\end{equation*}
$$

where $|L|=1-\frac{\sum_{j=1}^{N} w_{j N+1} w_{N+1 j}}{\left(s_{1 N+1}+s_{N+11}\right)^{2}} e^{2\left(s_{1 N+1}+s_{N+11}\right) x}, \sum_{l=1}^{N} w_{N+1 l} L_{(j l)}^{*}=w_{N+1 j}$ and $w_{j N+1}=$ $\epsilon w_{N+1 j}^{*}$ are used.

According to the form of $|L|$, it is easy to see that the one-soliton solutions of (3.16) is dark soliton solution for $(\epsilon=-1)$ and bright soliton solution for $(\epsilon=-1)$. Similarly, through tedious calculations, we can drive $N$-soliton solutions of (3.16) with help of mathematics.

### 3.3. Integrable variable-coefficient $N$-coupled $m K d V$-type equations and high-order NLS equations

Soliton solutions to the mkdv equation is important in inhomogeneous plasmas, which describe produced filamentation by the ponderomotive force between the dispersive effects and the nonlinear perturbation.

We consider operators $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{2}}$ defined by

$$
\begin{align*}
& \mathbf{M}_{\mathbf{1}}=\alpha \partial_{x}+a_{0}(y) I,  \tag{3.24}\\
& \mathbf{M}_{\mathbf{2}}=I \partial_{y}+I \rho_{3}(y) \partial_{x}^{3}+\operatorname{Ix} \rho_{1}(y) \partial_{x}, \tag{3.25}
\end{align*}
$$

where

$$
\alpha=i\left(\begin{array}{cccc}
1 & \cdots & 0 & 0  \tag{3.26}\\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & -1
\end{array}\right)_{(2 N+1) \times(2 N+1)} \quad I=\left(\begin{array}{cccc}
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 1
\end{array}\right)_{(2 N+1) \times(2 N+1)} .
$$

Substitution of (3.23) and (3.24) into (2.6) and (2.7) yields that

$$
\begin{equation*}
a_{0}=c_{0} e^{-\int \rho_{1} d y} \tag{3.27}
\end{equation*}
$$

where $c_{0}$ is an arbitrary constant.
Similarly, we have

$$
\mathbf{D}_{\mathbf{1}}=i\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & q_{1} \\
0 & 0 & \cdots & 0 & q_{1}^{*} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & q_{N} \\
0 & 0 & \cdots & 0 & q_{N}^{*} \\
q_{1}^{*} & q_{1} & \cdots & q_{N} & 0
\end{array}\right),
$$

where $\hat{k}^{(2 k-1,2 N+1)}=\frac{q_{k}}{2}, \hat{k}^{(2 k, 2 N+1)}=\frac{q_{k}^{*}}{2}, \hat{k}^{(2 N+1,2 n-1)}=-\frac{q_{n}^{*}}{2}, \hat{k}^{(2 N+1,2 n)}=-\frac{q_{n}}{2}$, $\widehat{k}_{x}^{(2 k-1,2 n-1)}=\frac{q_{k} q_{n}^{*}}{2}, \widehat{k}_{x}^{(2 k-1,2 n)}=\frac{q_{k} q_{n}}{2}, \widehat{k}_{x}^{(2 k, 2 n-1)}=\frac{q_{k}^{*} q_{n}^{*}}{2}, \widehat{k}_{x}^{(2 k, 2 n)}=\frac{q_{k}^{*} q_{n}}{2}, \widehat{k}_{x}^{(2 N+1,2 N+1)}=$ $\sum_{s=1}^{N}\left|q_{s}\right|^{2},(n, k=1, \ldots, N)$.

With the aid of $(2.12)-(2.14)$, we have $C_{2}=3 \rho_{3}(y) \widehat{K}_{x} . C_{1}=\left(C_{1}^{(l, j)}\right)_{(2 N+1) \times(2 N+1)}$ is determined by (3.27) and (3.28).

Substitution of (3.23) and (3.24) into (2.13) and (2.14) produces that

$$
\begin{align*}
& \alpha C_{1}-C_{1} \alpha+\alpha c_{2 x}+3 \rho_{3} \alpha \widehat{K}_{x x}-3 \rho_{3} \mathbf{D}_{1 x x}+3 \rho_{3}\left(\mathbf{D}_{1} \widehat{K}_{x}-\widehat{K}_{x} \mathbf{D}_{1}\right)=0  \tag{3.28}\\
& \alpha C_{1 x}-\mathbf{D}_{1 y}-x \rho_{1} \mathbf{D}_{1 x}-\rho_{3} \mathbf{D}_{1 x x x}+\mathbf{D}_{1} C_{1}-C_{1} \mathbf{D}_{1}-C_{2} \mathbf{D}_{1 x}=\phi_{1} \mathbf{D}_{1} \tag{3.29}
\end{align*}
$$

From which, we obtain

$$
\begin{aligned}
& C_{1}^{(2 n-1,2 N+1)}=\frac{3 \rho_{3}}{4} q_{n x x}, \quad C_{1}^{(2 n, 2 N+1)}=\frac{3 \rho_{3}}{4} q_{n x x}^{*}, \quad C_{1}^{(2 N+1,2 n-1)}=-\frac{3 \rho_{3}}{4} q_{n x x}^{*}, \\
& C_{1}^{(2 N+1,2 n)}=-\frac{3 \rho_{3}}{4} q_{n x x}, \quad C_{1}^{(2 k-1,2 n-1)}=\frac{3 \rho_{3}}{4}\left(q_{k} q_{n}^{*}\right)_{x}, \quad C_{1}^{(2 k-1,2 n)}=\frac{3 \rho_{3}}{4}\left(q_{k} q_{n}\right)_{x}, \\
& C_{1}^{(2 k, 2 n-1)}=\frac{3 \rho_{3}}{4}\left(q_{k}^{*} q_{n}^{*}\right)_{x}, \quad C_{1}^{(2 k, 2 n)}=\frac{3 \rho_{3}}{4}\left(q_{k}^{*} q_{n}\right)_{x}, \quad C_{1}^{(2 N+1,2 N+1)}=\frac{3 \rho_{3}}{2} \sum_{s=1}^{N}\left|q_{s}\right|_{x}^{2} .
\end{aligned}
$$

Further, the integrable variable-coefficient $N$-coupled mKdV type equations are derived:

$$
\begin{equation*}
q_{k y}+\rho_{1}\left(x q_{k}\right)_{x}+\frac{\rho_{3}}{4}\left[q_{k x x x}+3 q_{k} \sum_{s=1}^{N}\left|q_{s}\right|_{x}^{2}+6 q_{k x} \sum_{s=1}^{N}\left|q_{s}\right|^{2}\right]=0 . \tag{3.30}
\end{equation*}
$$

Under the transformation

$$
q_{k}(x, y)=u_{k}(X, Y) \exp \frac{-i}{6}\left(X-\frac{Y}{18}\right), \quad y=Y, \quad x=X-\frac{Y}{12} .
$$

Equations (3.29) are reduced to the integrable variable-coefficient $N$-coupled high-order NLS equations:

$$
\begin{align*}
u_{k Y} & +\left(1-\frac{\rho_{3}}{4}\right) \frac{u_{k X}}{12}+\left(\rho_{1}-\frac{i}{216}+\frac{3 \rho_{3}}{4}\right) u_{k}+\rho_{1}\left(X-\frac{Y}{12}\right)\left(u_{k X}-\frac{i}{6} u_{k}\right) \\
& +\frac{\rho_{3}}{4}\left[u_{k X X X}+3 u_{k} \sum_{s=1}^{N}\left|u_{s}\right|_{X}^{2}+6 u_{k X} \sum_{s=1}^{N}\left|u_{s}\right|^{2}\right]-\frac{i \rho_{3}}{8} u_{k X X}-\frac{i \rho_{3}}{4} u_{k} \sum_{s=1}^{N}\left|u_{s}\right|^{2}=0 . \tag{3.31}
\end{align*}
$$

Then Eqs. (3.29) have Lax pair $\mathbf{N}_{\mathbf{1}}$ and $\mathbf{N}_{\mathbf{2}}$, defined by

$$
\mathbf{N}_{1}=\mathbf{M}_{1}+\mathbf{D}_{\mathbf{1}}, \quad \mathbf{N}_{\mathbf{2}}=\mathbf{M}_{\mathbf{2}}+\mathbf{D}_{\mathbf{2}}, \quad \mathbf{D}_{\mathbf{2}}=C_{1}+C_{2} \partial_{x}
$$

where $\mathbf{D}_{\mathbf{1}}$ is the same as before, and

$$
C_{1}=\frac{3 \rho_{3}}{4}\left(\begin{array}{cccccc}
\left|q_{1}\right|_{x}^{2} & \left(q_{1}\right)_{x}^{2} & \cdots & \left(q_{1} q_{N}^{*}\right)_{x} & \left(q_{1} q_{N}\right)_{x} & q_{1 x x} \\
\left(q_{1}^{*}\right)_{x}^{2} & \left|q_{1}\right|_{x}^{2} & \cdots & \left(q_{1}^{*} q_{N}^{*}\right)_{x} & \left(q_{1}^{*} q_{N}\right)_{x} & q_{1 x x}^{*} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\left(q_{N} q_{1}^{*}\right)_{x} & \left(q_{N} q_{1}\right)_{x} & \cdots & \left|q_{N}\right|_{x}^{2} & \left(q_{N}\right)_{x}^{2} & q_{N x x} \\
\left(q_{N}^{*} q_{1}^{*}\right)_{x} & \left(q_{N}^{*} q_{1}\right)_{x} & \cdots & \left(q_{N}^{*}\right)_{x}^{2} & \left|q_{N}\right|_{x}^{2} & q_{N x x}^{*} \\
-q_{1 x x}^{*} & -q_{1 x x} & \cdots & -q_{N x x}^{*} & -q_{N x x} & 2 \sum_{l=1}^{N}\left|q_{l}\right|_{x}^{2}
\end{array}\right)
$$

and

$$
C_{2}=\frac{3 \rho_{3}}{2}\left(\begin{array}{cccccc}
\left|q_{1}\right|^{2} & \left(q_{1}\right)^{2} & \cdots & q_{1} q_{N}^{*} & q_{1} q_{N} & q_{1 x} \\
\left(q_{1}^{*}\right)^{2} & \left|q_{1}\right|^{2} & \cdots & q_{1}^{*} q_{N}^{*} & q_{1}^{*} q_{N} & q_{1 x}^{*} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
q_{N} q_{1}^{*} & q_{N} q_{1} & \cdots & \left|q_{N}\right|^{2} & \left(q_{N}\right)^{2} & q_{N x} \\
q_{N}^{*} q_{1}^{*} & q_{N}^{*} q_{1} & \cdots & \left(q_{N}^{*}\right)^{2} & \left|q_{N}\right|^{2} & q_{N x}^{*} \\
-q_{1 x}^{*} & -q_{1 x} & \cdots & -q_{N x}^{*} & -q_{N x} & 2 \sum_{l=1}^{N}\left|q_{l}\right|^{2}
\end{array}\right)
$$

Next, we shall formulate one-soliton solutions of (3.29).

Similarly, we obtain the evolution equations for $F$

$$
\begin{array}{r}
\alpha F_{x}+F_{z} \alpha=0 \\
F_{y}+\rho_{1}\left(x F_{x}+z F_{z}\right)+\rho_{1} F+\rho_{3}\left(F_{x x x}+F_{z z z}\right)=0 . \tag{3.33}
\end{array}
$$

From (3.32), we have

$$
\begin{align*}
& F^{(2 k-12 N+1)}=w_{2 k-12 N+1} e^{s_{2 k-12 N+1}(x+z)} \\
& F^{(2 k 2 N+1)}=w_{2 k 2 N+1} e^{s_{2 k 2 N+1}(x+z)} \\
& F^{(2 N+12 k-1)}=w_{2 N+12 k-1} e^{s_{2 N+12 k-1}(x+z)},  \tag{3.34}\\
& F^{(2 N+12 k)}=w_{2 N+12 k} e^{s_{2 N+12 k}(x+z)} \\
& F^{(2 k-12 n-1)}=F^{(2 k-12 n)}=F^{(2 k 2 n-1)}=F^{(2 k 2 n)}=F^{(2 N+12 N+1)}=0 .
\end{align*}
$$

Substitution of (3.33) into (3.32) yields that

$$
\begin{align*}
& \partial_{y} s_{2 k-12 N+1}+\rho_{1} s_{2 k-12 N+1}=0, \quad \partial_{y} s_{2 k 2 N+1}+\rho_{1} s_{2 k 2 N+1}=0, \\
& \partial_{y} s_{2 N+12 k-1}+\rho_{1} s_{2 N+12 k-1}=0, \quad \partial_{y} s_{2 N+12 k}+\rho_{1} s_{2 N+12 k}=0, \\
& \partial_{y} w_{2 k-12 N+1}+\left(\rho_{1}+2 \rho_{3} s_{2 k-12 N+1}^{3}\right) w_{2 k-12 N+1}=0,  \tag{3.35}\\
& \partial_{y} w_{2 k 2 N+1}+\left(\rho_{1}+2 \rho_{3} s_{2 k 2 N+1}^{3}\right) w_{2 k 2 N+1}=0, \\
& \partial_{y} w_{2 N+12 k-1}+\left(\rho_{1}+2 \rho_{3} s_{2 N+12 k-1}^{3}\right) w_{2 N+12 k-1}=0, \\
& \partial_{y} w_{2 N+12 k}+\left(\rho_{1}+2 \rho_{3} s_{2 N+12 k}^{3}\right) w_{2 N+12 k}=0 .
\end{align*}
$$

Further, we suppose that

$$
\begin{aligned}
& s_{2 k-12 N+1}=s_{2 k 2 N+1}=s_{2 N+12 k-1}=s_{2 N+12 k}=c_{0} e^{-\int \rho_{1} d y} \\
& w_{2 k-12 N+1}=c_{2 k-12 N+1} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{2 k-12 N+1}^{3}\right) d y}, \quad w_{2 k 2 N+1}=c_{2 k 2 N+1} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{2 k 2 N+1}^{3}\right) d y} \\
& w_{2 N+12 k-1}=c_{2 N+12 k-1} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{2 k 2 N+1}^{3}\right) d y}, \quad w_{2 N+12 k}=c_{2 N+12 k} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{2 N+12 k}^{3}\right) d y}
\end{aligned}
$$

where $c_{2 k-12 N+1}, c_{2 k 2 N+1}, c_{2 N+12 k-1}$ and $c_{2 N+12 k}$ are arbitrary constants, $c_{0}$ is an arbitrary negative constant.

In the same way, we assume that $F(x, z, y)=f(x, y) g(z, y)$,

$$
\begin{aligned}
& f(x, y)=\left(\begin{array}{ccccc}
0 & \cdots & 0 & w_{12 N+1} e^{s_{12 N+1} x} \\
0 & \cdots & 0 & w_{22 N+1} e^{s_{12 N+1} x} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & w_{2 N 2 N+1} e^{s_{12 N+1} x} \\
w_{2 N+11} e^{s_{12 N+1} x} & \cdots & w_{2 N+12 N} e^{s_{12 N+1} x} & 0
\end{array}\right), \\
& g(z, y)=\left(\begin{array}{ccccc}
e^{s_{12 N+1} z} & 0 & \cdots & 0 & 0 \\
0 & e^{s_{12 N+1} z} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & e^{s_{12 N+1} z} & 0 \\
0 & 0 & \cdots & 0 & e^{s_{12 N+1} z}
\end{array}\right)
\end{aligned}
$$

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Using (2.20), we derive

$$
L=\left(\begin{array}{cccc}
1 & \cdots & 0 & -\frac{w_{12 N+1} e^{2 s_{12 N+1} x}}{2 s_{12 N+1}} \\
0 & \cdots & 0 & -\frac{w_{22 N+1} e^{2 s_{12 N+1} x}}{2 s_{12 N+1}} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & -\frac{w_{2 N 2 N+1} e^{2 s_{1 N+1} x}}{2 s_{12 N+1}} \\
-\frac{w_{2 N+111^{2}} s_{12 N+1^{x}}}{2 s_{12 N+1}} & \cdots & -\frac{w_{2 N+12 N} e^{2 s_{12 N+1} x}}{2 s_{12 N+1}} & 1
\end{array}\right) .
$$

Further, it can be shown that

$$
\widehat{K}=-\frac{e^{2 s_{1 N+1} x}}{|L|}\left(\begin{array}{cccc}
w_{12 N+1} L_{(12 N+1)}^{*} & \cdots & w_{12 N+1} L_{(2 N 2 N+1)}^{*} & w_{12 N+1} L_{(2 N+12 N+1)}^{*} \\
w_{22 N+1} L_{(12 N+1)}^{*} & \cdots & w_{22 N+1} L_{(2 N 2 N+1)}^{*} & w_{22 N+1} L_{(2 N+12 N+1)}^{*} \\
\vdots & \cdots & \vdots & \vdots \\
w_{2 N 2 N+1} L_{(12 N+1)}^{*} & \cdots & w_{2 N 2 N+1} L_{(2 N 2 N+1)}^{*} & w_{2 N 2 N+1} L_{(2 N+12 N+1)}^{*} \\
\sum_{j=1}^{2 N} w_{2 N+1 j} L_{(1, j)}^{*} & \cdots & \sum_{j=1}^{2 N} w_{2 N+1 j} L_{(2 N, j)}^{*} & \sum_{j=1}^{2 N} w_{2 N+1 j} L_{(2 N+1 j)}^{*}
\end{array}\right) .
$$

Thus, one-soliton solution of the integrable variable-coefficient $N$-coupled mKdV equation (3.29) is given by

$$
\begin{equation*}
q_{k}=-\frac{2}{|L|} w_{2 k-12 N+1} e^{2 s_{1 N+1} x}, \quad k=1, \ldots, N \tag{3.36}
\end{equation*}
$$

where

$$
\begin{array}{r}
|L|=1-\frac{\sum_{j=1}^{N} w_{2 j 2 N+1} w_{2 N+12 j}+w_{2 j-12 N+1} w_{2 N+12 j-1}}{4 s_{12 N+1}^{2}} e^{4 s_{12 N+1} x} \\
w_{2 k-12 N+1}=c_{2 k-12 N+1} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{2 k-12 N+1}^{3}\right) d y}
\end{array}
$$

In the derivation, $L_{(2 N+12 N+1)}^{*}=1, w_{2 k-12 N+1}=w_{2 k 2 N+1}^{*}, w_{2 N+12 k-1}=w_{2 N+12 k}^{*}$, $w_{2 N+12 k-1}=-w_{2 k 2 N+1}$ are used.

## 4. Reductions and Conclusions

In this section, we will discuss the simplest reductions of one-field case.
Case 1. The Eq. (3.5) $(N=1)$ is reduced to

$$
\begin{equation*}
q_{1}+\rho_{1}\left(x q_{1}\right)_{x}+\rho_{2} a_{0} q_{1}+i \frac{\rho_{2}}{2} q_{1 x x}-i \epsilon \rho_{2} q_{1}\left|q_{1}\right|^{2}=0 \tag{4.1}
\end{equation*}
$$

which has Lax pairs $N_{1}, N_{2}$ defined by

$$
\begin{aligned}
& \mathbf{N}_{\mathbf{1}}=i\left(\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right) \partial_{x}+x a_{0} I+i\left(\begin{array}{cc}
0 & q_{1} \\
-\epsilon q_{1}^{*} & -1
\end{array}\right), \\
& \mathbf{N}_{\mathbf{2}}=I \partial_{t}+\rho_{2} i\left(\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right) \partial_{x}^{2}+x \rho_{1} I \partial_{x}-i \frac{\rho_{2}}{2}\left(\begin{array}{cc}
\epsilon\left|q_{1}\right|^{2} & -q_{1 x} \\
-\epsilon q_{1 x}^{*} & -\epsilon\left|q_{1}\right|^{2}
\end{array}\right)+i \rho_{2}\left(\begin{array}{cc}
0 & q_{1} \\
-\epsilon q_{1}^{*} & 0
\end{array}\right),
\end{aligned}
$$

where $I$ is a unit matrix. Equation (4.1) has one-soliton solution

$$
q_{1}=-\frac{2}{|L|} w_{12} e^{i a_{0} x^{2}+2 \mu_{12} x}
$$

where $|L|=1-\frac{w_{12} w_{21}}{4 \mu_{12}^{2}} e^{4 \mu_{12}^{2} x}, \mu_{12}=\mu_{21}=e^{\int\left(2 \rho_{2}-\rho_{1}\right) d t}, w_{12}=e^{\int\left(2 \rho_{2}-\rho_{1}-2 i \rho_{2} \mu_{12}^{2}\right) d t}, w_{21}=$ $\epsilon e^{\int\left(2 \rho_{2}-\rho_{1}+2 i \rho_{2} \mu_{21}^{2}\right) d t}$.

Specially, $\rho_{1}=0, \rho_{2}=-2 t, a_{0}=\frac{1}{4 t}$, Eq. (4.1) is reduced to the well known cylindrical NLS equation

$$
\begin{equation*}
i u_{t}+u_{x x}+\frac{i}{2 t} u \pm 2 u|u|^{2}=0 \tag{4.2}
\end{equation*}
$$

Case 2. Equation (3.6) (for $N=1$ ) is reduced to the following equation

$$
\begin{equation*}
q_{1 y}+\frac{i \rho_{2}}{2} q_{1 x x}-i \epsilon \rho_{2} q_{1}\left|q_{1}\right|^{2}+2 i x \rho_{0} q_{1}+\rho_{1}\left(x q_{1}\right)_{x}+\frac{\rho_{3}}{4} q_{1, x x x}-\epsilon \frac{3 \rho_{3}}{2}\left|q_{1}\right|^{2} q_{1, x}=0 \tag{4.3}
\end{equation*}
$$

which has lax pairs $N_{1}, N_{2}$ given by

$$
\begin{aligned}
\mathbf{N}_{\mathbf{1}}= & i\left(\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right) \partial_{x}+x a_{0}(y) I+i\left(\begin{array}{cc}
0 & q_{1} \\
-\epsilon q_{1}^{*} & 0
\end{array}\right) \\
\mathbf{N}_{\mathbf{2}}= & I \partial_{y}+I \rho_{3}(y) \partial_{x}^{3}+\rho_{2}(y) i\left(\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right) \partial_{x}^{2}+I x \rho_{1}(y) \partial_{x}+I x \rho_{0}(y) \\
& +\left(\begin{array}{cc}
-\frac{3 \epsilon}{4} \rho_{3}\left|q_{1}\right|_{x}^{2}-\frac{i \epsilon \rho_{2}}{2}\left|q_{1}\right|^{2} & \frac{3 \epsilon}{4} \rho_{3} q_{1 x x}+\frac{i \rho_{2}}{2} q_{1 x} \\
\frac{3 \epsilon}{4} \rho_{3} q_{1 x x}^{*}-\frac{i \epsilon \rho_{2}}{2} q_{1 x}^{*} & -\frac{3 \epsilon}{4} \rho_{3}\left|q_{1}\right|_{x}^{2}+\frac{i \epsilon \rho_{2}}{2}\left|q_{1}\right|^{2}
\end{array}\right) \\
& +\left(\begin{array}{cc}
-i \frac{3 \epsilon \rho_{3}}{2}\left|q_{1}\right|^{2} & \frac{3 \rho_{3}}{2} q_{1 x}+i \rho_{2} q_{1} \\
-\frac{3 \rho_{3}}{2} q_{1 x}^{*}-i \epsilon \rho_{2} q_{1}^{*} & -\frac{3 \epsilon \rho_{3}}{2}\left|q_{1}\right|^{2}
\end{array}\right) \partial_{x},
\end{aligned}
$$

where $I$ is a unit matrix.
The one-soliton solution of (4.3) is

$$
q_{1}=-\frac{2}{|L|} w_{12} e^{2 s_{12} x}
$$

with $|L|=1-\frac{w_{12} w_{21}}{\left(s_{12}+s_{21}\right)^{2}} e^{2\left(s_{12}+s_{21}\right) x}, s_{12}+s_{21}=c e^{-\int \rho_{1} d y}$,
$w_{12}=c_{1} e^{-\int\left(\rho_{1}+2 i \rho_{2} s_{12}^{2}+2 \rho_{3} s_{12}^{3}\right)}, w_{21}=\epsilon c_{1} e^{-\int\left(\rho_{1}-2 i \rho_{2} s_{21}^{2}+2 \rho_{3} s_{21}^{3}\right)}, c_{1}>0$ is a constant. From the form of the solution, it is easy to see that the solution is singular for $(\epsilon=1)$.
Case 3. Equation (3.29) is reduced to the following equation for $(N=1)$

$$
\begin{equation*}
q_{1 y}+\rho_{1}\left(x q_{1}\right)_{x}+\frac{\rho_{3}}{4}\left[q_{1 x x x}+3 q_{1}\left|q_{1}\right|_{x}^{2}+6 q_{1 x}\left|q_{1}\right|^{2}\right]=0 \tag{4.4}
\end{equation*}
$$

Its lax pair is $N_{1}, N_{2}$ defined by

$$
\mathbf{N}_{\mathbf{1}}=i\left(\begin{array}{rrr}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right) \partial_{x}+a_{0}(y) I+i\left(\begin{array}{rrr}
0 & 0 & q_{1} \\
0 & 0 & q_{1}^{*} \\
q_{1}^{*} & q_{1} & 0
\end{array}\right)
$$

$$
\begin{aligned}
\mathbf{N}_{2}= & I \partial_{y}+I \rho_{3}(y) \partial_{x}^{3}+I x \rho_{1}(y) \partial_{x}+\frac{3 \rho_{3}}{4}\left(\begin{array}{ccc}
\left|q_{1}\right|_{x}^{2} & \left(q_{1}\right)_{x}^{2} & q_{1 x x} \\
\left(q_{1}^{*}\right)_{x}^{2} & \left|q_{1}\right|_{x}^{2} & q_{1 x x}^{*} \\
-q_{1 x x}^{*} & -q_{1 x x} & 2\left|q_{1}\right|_{x}^{2}
\end{array}\right) \\
& +\frac{3 \rho_{3}}{2}\left(\begin{array}{ccc}
\left|q_{1}\right|^{2} & \left(q_{1}\right)^{2} & q_{1 x} \\
\left(q_{1}^{*}\right)^{2} & \left|q_{1}\right|^{2} & q_{1 x}^{*} \\
-q_{1 x}^{*} & -q_{1 x} & 2\left|q_{1}\right|^{2}
\end{array}\right) \partial_{x} .
\end{aligned}
$$

Its one-soliton solution is given by

$$
q_{1}=-\frac{2}{|L|} w_{13} e^{2 s_{13} x}
$$

where $|L|=1-\frac{\left|w_{23}\right|^{2}+\left|w_{13}\right|^{2}}{4 s_{13}^{2}} e^{4 s_{13} x}, s_{13}=c_{0} e^{\int \rho_{1} d y}, w_{23}=c_{23} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{13}^{3}\right) d y}, w_{13}=$ $c_{13} e^{-\int\left(\rho_{1}+2 \rho_{3} s_{13}^{3}\right) d y}, c_{0}, c_{13}$ and $c_{23}$ are arbitrary constant, which shows that the solution $q_{1}$ is singular.

We have applied the generalized dressing method to a family $N$-coupled NLS equations. The hierarchy can be reduced to variable-coefficient $N$-coupled cylindrical NLS equations, variable-coefficient $N$-coupled Hirota equations and etc. The derived equations in our paper are all reduced to well-known constant integrable equations for choosing special parameters. Therefore, the obtained new equations have some important physical applications, especially in optical fiber. In [2], Burtsev et al. have investigated the well known cylindrical NLS equation (4.2) with variable spectral parameter. In fact, the equations in our paper can also be derived by the method described in [2]. Essentially, the generalized dressing method described in our paper is applied to variable coefficient spectral parameters. The point is the same to that paper [2]. The different lies in that "spectral parameters" in our paper is hidden and is visible in [2]. The generalized dressing method has the advantages of bypassing the scattering problem entirely, for example, scattering data. Moreover, the method is more intuitive and in keeping with the spirit of the original dressing method [15]. The method can be used to construct a large class of integrable variable coefficient equations in a systematic way, along with their soliton solutions. The solutions can be more general than those obtainable by the inverse scattering technique. The generalized dressing method has extensive applications, however, the method has also limitation. For example, the AblowLadik hierarchy, self-dual network equations and some discrete equations have not been solved by using the method. These problems need further discussion.

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