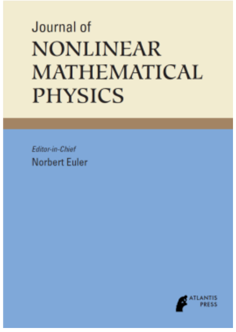

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Article

World Scientific
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Journal of Nonlinear Mathematical Physics, Vol. 19, No. 4 (2012) 1250031 (15 pages)

DOI: 10.1142/S1402925112500313

EXACT SOLUTIONS TO LATTICE BOUSSINESQ-TYPE EQUATIONS

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Received 23 May 2012
Accepted 11 July 2012
Published 31 December 2012

In this paper several kinds of exact solutions to lattice Boussinesq-type equations are constructed by means of generalized Cauchy matrix approach, including soliton solutions and mixed solutions. The introduction of the general condition equation set yields that all solutions contain two kinds of plane-wave factors.

Keywords: Lattice Boussinesq-type equations; generalized Cauchy matrix approach; exact solutions.

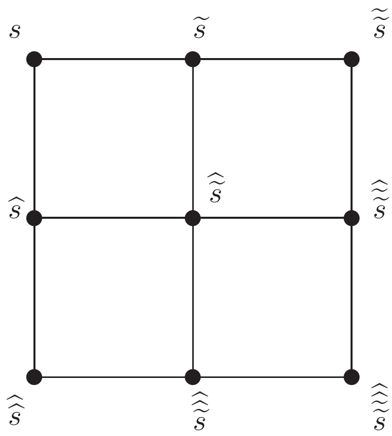
Mathematics Subject Classification 2000: 39A14

1. Introduction

The first example of the lattice Boussinesq (BSQ) equation is given in [1], being a straight dimensional reduction of the Hirota bilinear KP equation [5]. In [11], Nijhoff *et al.* presented a lattice BSQ equation on two-dimensional lattice (see also [6, 8, 14]):

$$\begin{aligned}\frac{p^3-q^3}{p-q+s_{n+1,m+1}-s_{n+2,m}} &- \frac{p^3-q^3}{p-q+s_{n,m+2}-s_{n+1,m+1}} \\ &= (p-q+s_{n+1,m+2}-s_{n+2,m+1})(2p+q+s_{n,m+1}-s_{n+2,m+2}) \\ &\quad -(p-q+s_{n,m+1}-s_{n+1,m})(2p+q+s_{n,m}-s_{n+2,m+1}),\end{aligned} \tag{1.1}$$

which appeared as the first higher-rank case of the so-called "lattice Gel'fand–Dikii (GD)" hierarchy, whose bottom member is the lattice potential KdV equation. Equation (1.1) is defined on 9-point stencil as follows:



The notation employed in (1.1) is illustrated in this figure: $s = s_{n,m}$ denotes the dependent variable of the lattice points labeled by $(n, m) \in \mathbb{Z}^2$; p and q are continuous lattice parameters associated with the grid size in the directions of the lattice given by the independent variables n and m, respectively. For the sake of clarity we prefer to use notations with elementary lattice shifts denoted by

$$s = s_{n,m} \ \mapsto\ \widetilde{s} = s_{n+1,m}, \quad s = s_{n,m} \ \mapsto\ \widehat{s} = s_{n,m+1},$$

in terms of which we also have

$$\widehat{\widetilde{s}} = s_{n+1,m+1}, \quad \widehat{\widetilde{\widetilde{s}}} = s_{n+2,m+1}, \quad \widehat{\widehat{\widetilde{s}}} = s_{n+1,m+2}, \quad \widehat{\widehat{\widetilde{\widetilde{s}}}} = s_{n+2,m+2}.$$

Together with the lattice BSQ equation, also the lattice modified BSQ (MBSQ) equation [8, 11, 14], the lattice Schwarzian BSQ (SBSQ) equation [7, 14] and lattice BSQ analogue of the $(\mathrm{Q}3)_0$ [9, 19] have been presented in the form of 9-point equations and defined on the above stencil.

Besides the scalar representations, the lattice BSQ-type equations can also be expressed as three-component forms defined on an elementary square [13, 14], possessing three-dimensional consistency property. Recently, by considering lattice equations defined on edges of the elementary square with some constraints, Hietarinta gave a classification of the BSQ-like multi-component lattice equations via the property of multidimensional consistency [2], where four BSQ-like three-component lattice equations and two BSQ-like two-component lattice equations were obtained. The four BSQ-like three-component lattice equations (after some point transformations on the dependent variables) are related to more general types of dispersion relations, which were studied systematically in [19] by using the direct linearization (DL) method. Up to now, various methods have been developed to construct exact solutions to the lattice BSQ-type equations, such as DL method [11, 13, 14, 19], Hirota bilinear method [3, 4], generalized Cauchy matrix method [20].

In recent years, the study of various kinds of exact solutions to the integrable partial difference equations (P△Es) becomes a hot topic and some significant progress has been made. With the help of the property of multidimensional consistency, generalized solutions to H1 equation [17] and rational solutions to H3 equation and Q1 equation [12] were carried out by using Hirota bilinear method. Meanwhile, Cauchy matrix approach [10] was also extended to study the exact solutions for ABS lattice equations [18] and lattice BSQ-type equations [20]. In [20], Zhang *et al.* derived soliton solutions, Jordan block solutions and mixed solutions for the lattice BSQ-type equations by introducing the following condition equations set (CES)

$$-\omega MK + KM = r\,{}^t c, \tag{1.2a}$$

$$(pI-K)\widetilde{r} = (pI-\omega K)r, \quad (qI-K)\widehat{r} = (qI-\omega K)r, \tag{1.2b}$$

$$(pI-K)\widetilde{M} = (pI-\omega K)M, \quad (qI-K)\widehat{M} = (qI-\omega K)M, \tag{1.2c}$$

where parameter $\omega \neq 1$ is a cubic root of unity. Since just parameter ω was involved in the CES (1.2), only one plane-wave factor was contained in these solutions.

Motivated by the earlier works [3, 19], in this paper, we will start by introducing a more general CES which is comprised of ω and ω^2 to establish exact solutions for the lattice BSQ-type equations by means of generalized Cauchy matrix approach. As an upshot of the extension for the CES, two kinds of plane-wave factors will be involved in these solutions. The outline of this paper is as follows. In Sec. 2, the more general CES will be set up, from which the dynamical properties of $\boldsymbol{M}$ will be obtained. To establish the lattice BSQ-type equations, we will introduce several objects and consider their relationships. In Sec. 3, a deformed CES and corresponding objects will be introduced by taking the canonical form $\boldsymbol{\Gamma}$ of matrix $\boldsymbol{K}$ in the original CES. Finally, due to the forms of $\boldsymbol{\Gamma}$, we will get several kinds of solutions besides soliton solutions.

2. Generalized Cauchy Matrix Approach of Lattice BSQ-type Equations

2.1. *CES and recurrence structure*

To begin, we consider the following CES

$$\boldsymbol{K}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{K}' = \boldsymbol{r}\,{}^t\boldsymbol{s}, \tag{2.1a}$$

$$\widetilde{\boldsymbol{r}} = (p\boldsymbol{I}+\boldsymbol{K})\boldsymbol{r}, \quad \widehat{\boldsymbol{r}} = (q\boldsymbol{I}+\boldsymbol{K})\boldsymbol{r}, \tag{2.1b}$$

$${}^t\widetilde{\boldsymbol{s}} = {}^t\boldsymbol{s}(p\boldsymbol{I}-\boldsymbol{K}')^{-1}, \quad {}^t\widehat{\boldsymbol{s}} = {}^t\boldsymbol{s}(q\boldsymbol{I}-\boldsymbol{K}')^{-1}, \tag{2.1c}$$

$$\widetilde{\boldsymbol{M}}(p\boldsymbol{I}-\boldsymbol{K}') = (p\boldsymbol{I}+\boldsymbol{K})\boldsymbol{M}, \quad \widehat{\boldsymbol{M}}(q\boldsymbol{I}-\boldsymbol{K}') = (q\boldsymbol{I}+\boldsymbol{K})\boldsymbol{M}, \tag{2.1d}$$

where $p,\ q \in \mathbb{C}$ are lattice parameters; $\boldsymbol{I} = \mathrm{diag}(\boldsymbol{I}_{N_1}, \boldsymbol{I}_{N_2})$ is the $(N_1+N_2)\times(N_1+N_2)$ unit matrix; $\boldsymbol{K} = \mathrm{diag}(\boldsymbol{K}_1, \boldsymbol{K}_2)$, $\boldsymbol{K}' = \mathrm{diag}(-\omega\boldsymbol{K}_1, -\omega^2\boldsymbol{K}_2)$ with $\omega^2+\omega+1=0$; $\boldsymbol{K}_1$ is a $N_1\times N_1$ complex matrix, whose two arbitrary eigenvalues $k_i^{(1)},\ k_j^{(1)}$ satisfy $k_i^{(1)} - \omega k_j^{(1)} \neq 0$ $(i,j=1,2,\ldots,N_1)$, and a is a fixed constant chosen such that $\det(a\boldsymbol{I}_{N_1}+\boldsymbol{K}_1)\neq 0$; $\boldsymbol{K}_2$ is a $N_2\times N_2$ complex matrix, whose two arbitrary eigenvalues $k_i^{(2)},\ k_j^{(2)}$ satisfy $k_i^{(2)} - \omega^2 k_j^{(2)} \neq 0$ $(i,j=1,2,\ldots,N_2)$, and b is a fixed constant chosen such that $\det(b\boldsymbol{I}_{N_2}+\boldsymbol{K}_2)\neq 0$; here we also assume $k_i^{(1)} - \omega^2 k_j^{(2)} \neq 0$, $(i=1,2,\ldots,N_1,\ j=1,2,\ldots,N_2)$; $\boldsymbol{M}, \boldsymbol{r}$ and ${}^t\boldsymbol{s}$ are, respectively, undetermined matrix, column vector and row vector, which are dependent on discrete variables n and m. Here and hereafter ${}^t\boldsymbol{s}$ does not mean transpose of $\boldsymbol{s}$ but just a notation, transpose is represented by $\boldsymbol{s}^T$.

Obviously, the tilde-equation in (2.1d) can be rewritten as

$$\boldsymbol{K}\widetilde{\boldsymbol{M}}+\widetilde{\boldsymbol{M}}\boldsymbol{K}'=(p\boldsymbol{I}+\boldsymbol{K})(\widetilde{\boldsymbol{M}}-\boldsymbol{M}). \tag{2.2}$$

Taking $\widetilde{\ }$-shift of (2.1a) and making use of the equations (2.1b) and (2.2), we get

$$\widetilde{\boldsymbol{M}}-\boldsymbol{M}=\boldsymbol{r}\,{}^{t}\widetilde{\boldsymbol{s}}. \tag{2.3}$$

Replacing $\widetilde{\ }$-shift by $\widehat{\ }$-shift in (2.3), we obtain

$$\widehat{\boldsymbol{M}}-\boldsymbol{M}=\boldsymbol{r}\,{}^{t}\widehat{\boldsymbol{s}}. \tag{2.4}$$

Equations (2.3) and (2.4) can be viewed as the dynamical properties of matrix $\boldsymbol{M}$ w.r.t. the discrete variables n and m.

The introduction of the following quantities involving the matrix $\boldsymbol{M}$ leads to exact solutions for the lattice BSQ-type equations:

$$\boldsymbol{u}^{(i)}(b)=(\boldsymbol{I}+\boldsymbol{M})^{-1}(b\boldsymbol{I}+\boldsymbol{K})^{i}\boldsymbol{r}, \tag{2.5a}$$

$${}^{t}\boldsymbol{u}^{(j)}(a)={}^{t}\boldsymbol{s}(a\boldsymbol{I}+\boldsymbol{K}')^{j}(\boldsymbol{I}+\boldsymbol{M})^{-1}, \tag{2.5b}$$

$$S^{(i,j)}(a,b)={}^{t}\boldsymbol{s}(a\boldsymbol{I}+\boldsymbol{K}')^{j}(\boldsymbol{I}+\boldsymbol{M})^{-1}(b\boldsymbol{I}+\boldsymbol{K})^{i}\boldsymbol{r} \tag{2.5c}$$

for $i,\ j\in\mathbb{Z}$. Then the latter objects can also be written as

$$S^{(i,j)}(a,b)={}^{t}\boldsymbol{s}(a\boldsymbol{I}+\boldsymbol{K}')^{j}\boldsymbol{u}^{(i)}(b)={}^{t}\boldsymbol{u}^{(j)}(a)(b\boldsymbol{I}+\boldsymbol{K})^{i}\boldsymbol{r}, \tag{2.6}$$

which are not symmetric w.r.t. the interchange of the pairs $(i,\ b)$ and $(j,\ a)$, i.e. $S^{(i,j)}(a,b)\neq S^{(j,i)}(b,a)$ (cf. [20]).

Taking $\widetilde{\ }$-shift and $\underset{\sim}{\ }$-shift (backward direction) of (2.5a), respectively, and using (2.3) and (2.4), we obtain

$$\widetilde{\boldsymbol{u}}^{(i)}(b)=(p-b)\boldsymbol{u}^{(i)}(b)+\boldsymbol{u}^{(i+1)}(b)-\widetilde{S}^{(i,0)}(a,b)\boldsymbol{u}^{(0)}(b), \tag{2.7a}$$

$$\begin{aligned}\left[\prod_{h=1}^{3}(\omega^{h}p\boldsymbol{I}+\boldsymbol{K})\boldsymbol{u}^{(i)}(b)\right]_{\sim}&=\left[\prod_{h=1}^{2}(p+\omega^{h}(E_1-b))\widetilde{\boldsymbol{u}}^{(i)}(b)\right]_{\sim}\\&\quad-\sum_{l=1}^{2}\omega^{l}\left[\prod_{h=2}^{l}(p+\omega^{h-1}(a-E_2))S^{(i,0)}(a,b)\right]_{\sim}\\&\quad\cdot\left[\prod_{h=l+1}^{2}(p+\omega^{h}(E_1-b))\widetilde{\boldsymbol{u}}^{(0)}(b)\right]_{\sim},\end{aligned} \tag{2.7b}$$

where operators E_1,E_2 are defined by their actions on the indices i and j as: $E_1\boldsymbol{u}^{(i)}(b)=\boldsymbol{u}^{(i+1)}(b)$, $E_2S^{(i,j)}(a,b)=S^{(i,j+1)}(a,b)$. Replacing p by q and $\widetilde{\ }$-shift by $\widehat{\ }$-shift, we derive

$$\widehat{\boldsymbol{u}}^{(i)}(b)=(q-b)\boldsymbol{u}^{(i)}(b)+\boldsymbol{u}^{(i+1)}(b)-\widehat{S}^{(i,0)}(a,b)\boldsymbol{u}^{(0)}(b), \tag{2.7c}$$

$$\begin{aligned}\left[\prod_{h=1}^{3}(\omega^h q\boldsymbol{I}+\boldsymbol{K})\boldsymbol{u}^{(i)}(b)\right] &= \left[\prod_{h=1}^{2}(q+\omega^h(E_1-b))\widehat{\boldsymbol{u}}^{(i)}(b)\right] \\ &\quad -\sum_{l=1}^{2}\omega^l\left[\prod_{h=2}^{l}(q+\omega^{h-1}(a-E_2))S^{(i,0)}(a,b)\right] \\ &\quad \cdot\left[\prod_{h=l+1}^{2}(q+\omega^h(E_1-b))\widehat{\boldsymbol{u}}^{(0)}(b)\right]. \end{aligned}\tag{2.7d}$$

These equations constitute the dynamical properties of $\boldsymbol{u}^{(i)}(b)$ w.r.t. discrete variables n and m. In quite a similar fashion, we obtain a system of recurrence relations for the objects ${}^t\boldsymbol{u}^{(j)}(a)$:

$${}^t\boldsymbol{u}^{(j)}(a)=(p-a){}^t\widetilde{\boldsymbol{u}}^{(j)}(a)+{}^t\widetilde{\boldsymbol{u}}^{(j+1)}(a)+S^{(0,j)}(a,b){}^t\widetilde{\boldsymbol{u}}^{(0)}(a),\tag{2.8a}$$

$$\begin{aligned}\left[{}^t\widetilde{\boldsymbol{u}}^{(j)}(a)\prod_{h=1}^{3}(\omega^h p\boldsymbol{I}-\boldsymbol{K}')\right] &= \left[\prod_{h=1}^{2}(p-\omega^h(E_3-a)){}^t\boldsymbol{u}^{(j)}(a)\right] \\ &\quad +\sum_{l=1}^{2}\omega^l\left[\prod_{h=l+1}^{2}(p-\omega^h(E_3-a)){}^t\boldsymbol{u}^{(0)}(a)\right] \\ &\quad \cdot\left[\prod_{h=2}^{l}(p-\omega^{h-1}(b-E_4))\widetilde{S}^{(0,j)}(a,b)\right], \end{aligned}\tag{2.8b}$$

$${}^t\boldsymbol{u}^{(j)}(a)=(q-a){}^t\widehat{\boldsymbol{u}}^{(j)}(a)+{}^t\widehat{\boldsymbol{u}}^{(j+1)}(a)+S^{(0,j)}(a,b){}^t\widehat{\boldsymbol{u}}^{(0)}(a),\tag{2.8c}$$

$$\begin{aligned}\left[{}^t\widehat{\boldsymbol{u}}^{(j)}(a)\prod_{h=1}^{3}(\omega^h q\boldsymbol{I}-\boldsymbol{K}')\right] &= \left[\prod_{h=1}^{2}(q-\omega^h(E_3-a)){}^t\boldsymbol{u}^{(j)}(a)\right] \\ &\quad +\sum_{l=1}^{2}\omega^l\left[\prod_{h=l+1}^{2}(q-\omega^h(E_3-a)){}^t\boldsymbol{u}^{(0)}(a)\right] \\ &\quad \cdot\left[\prod_{h=2}^{l}(q-\omega^{h-1}(b-E_4))\widehat{S}^{(0,j)}(a,b)\right], \end{aligned}\tag{2.8d}$$

where operators E_3, E_4 satisfy $E_3{}^t\boldsymbol{u}^{(j)}(a)={}^t\boldsymbol{u}^{(j+1)}(a)$, $E_4S^{(i,j)}(a,b)=S^{(i+1,j)}(a,b)$.

Furthermore, multiplying (2.7c) from the left by the row vector ${}^t\boldsymbol{s}(a\boldsymbol{I}+\boldsymbol{K}')^j$, we have the recurrence relations for $S^{(i,j)}(a,b)$:

$$\begin{aligned}(p+a)\widetilde{S}^{(i,j)}(a,b)-\widetilde{S}^{(i,j+1)}(a,b)&=(p-b)S^{(i,j)}(a,b)+S^{(i+1,j)}(a,b)\\&\quad-\widetilde{S}^{(i,0)}(a,b)S^{(0,j)}(a,b),\end{aligned}\tag{2.9a}$$

$$\begin{aligned}\left[\prod_{h=1}^{2}(p+\omega^h(a-E_2))S^{(i,j)}(a,b)\right]&=\left[\prod_{h=1}^{2}(p+\omega^h(E_4-b))\widetilde{S}^{(i,j)}(a,b)\right]\\&\quad-\sum_{l=1}^{2}\omega^l\left[\prod_{h=2}^{l}(p+\omega^{h-1}(a-E_2))S^{(i,0)}(a,b)\right]\\&\quad\cdot\left[\prod_{h=l+1}^{2}(p+\omega^h(E_4-b))\widetilde{S}^{(0,j)}(a,b)\right],\end{aligned}\tag{2.9b}$$

$$\begin{aligned}(q+a)\widehat{S}^{(i,j)}(a,b)-\widehat{S}^{(i,j+1)}(a,b)&=(q-b)S^{(i,j)}(a,b)+S^{(i+1,j)}(a,b)\\&\quad-\widehat{S}^{(i,0)}(a,b)S^{(0,j)}(a,b),\end{aligned}\tag{2.9c}$$

$$\begin{aligned}\left[\prod_{h=1}^{2}(q+\omega^h(a-E_2))S^{(i,j)}(a,b)\right]&=\left[\prod_{h=1}^{2}(q+\omega^h(E_4-b))\widehat{S}^{(i,j)}(a,b)\right]\\&\quad-\sum_{l=1}^{2}\omega^l\left[\prod_{h=2}^{l}(q+\omega^{h-1}(a-E_2))S^{(i,0)}(a,b)\right]\\&\quad\cdot\left[\prod_{h=l+1}^{2}(q+\omega^h(E_4-b))\widehat{S}^{(0,j)}(a,b)\right],\end{aligned}\tag{2.9d}$$

which can also be obtained from the dynamic system (2.8) by multiplying the column vector $(b\boldsymbol{I}+\boldsymbol{K})^i\boldsymbol{r}$ from the right side.

In order to construct closed-form equations from the relations (2.9), we introduce the objects:

$$v_a:=S^{(0,-1)}(a,0)-1,\quad w_b:=S^{(-1,0)}(0,b)-1,\tag{2.10a}$$

$$s_a:=S^{(1,-1)}(a,0)-a,\quad t_b:=S^{(-1,1)}(0,b)-b,\tag{2.10b}$$

$$r_a:=S^{(2,-1)}(a,0)-a^2,\quad z_b:=S^{(-1,2)}(0,b)-b^2,\tag{2.10c}$$

and

$$s_{a,b}=S^{(-1,-1)}(a,b).\tag{2.11}$$

For convenience, we denote $S^{(i,j)}(0,0) = S^{(i,j)}$ and $S^{(0,0)} = s$. Then for $S^{(i,j)}$ and the objects defined in (2.10), (2.11), we have the following relations

$$p\widetilde{S}^{(i,j)} - \widetilde{S}^{(i,j+1)} = pS^{(i,j)} + S^{(i+1,j)} - \widetilde{S}^{(i,0)}S^{(0,j)}, \tag{2.12a}$$

$$\begin{aligned} p^2S^{(i,j)} + pS^{(i,j+1)} + S^{(i,j+2)} &= p^2\widetilde{S}^{(i,j)} - p\widetilde{S}^{(i+1,j)} + \widetilde{S}^{(i+2,j)} \\ &\quad + pS^{(i,0)}\widetilde{S}^{(0,j)} - S^{(i,0)}\widetilde{S}^{(1,j)} + S^{(i,1)}\widetilde{S}^{(0,j)}, \end{aligned} \tag{2.12b}$$

as well as

$$1 - (p+a)\widetilde{s}_{a,b} + (p-b)s_{a,b} = v_a\widetilde{w}_b, \tag{2.13a}$$

$$(p-a+b) + p_a^{(1)}s_{a,b} - p_b^{(2)}\widetilde{s}_{a,b} = \widetilde{v}_a t_b - \widetilde{s}_a w_b + p\widetilde{v}_a w_b, \tag{2.13b}$$

and

$$s_a = (p+a)\widetilde{v}_a - (p-\widetilde{s}\,)v_a, \tag{2.14a}$$

$$\widetilde{t}_b = (p+s)\widetilde{w}_b - (p-b)w_b, \tag{2.14b}$$

and

$$r_a = -ps_a + (p+a)\widetilde{s}_a + v_a\widetilde{S}^{(1,0)}, \tag{2.15a}$$

$$\widetilde{z}_b = p\widetilde{t}_b - (p-b)t_b + \widetilde{w}_b S^{(0,1)}, \tag{2.15b}$$

and

$$\widetilde{r}_a = p_a^{(1)}v_a + (p+s)\widetilde{s}_a - (p(p+s) + S^{(0,1)})\widetilde{v}_a, \tag{2.16a}$$

$$z_b = p_b^{(2)}\widetilde{w}_b - (p-\widetilde{s}\,)t_b - (p(p-\widetilde{s}\,) + \widetilde{S}^{(1,0)})w_b, \tag{2.16b}$$

where $p_a^{(1)}$ and $p_b^{(2)}$ are defined as

$$\begin{aligned} &p_a^{(1)} = \frac{G(p,a)}{p+a}, \quad q_a^{(1)} = \frac{G(q,a)}{q+a}, \quad p_b^{(2)} = \frac{G(p,-b)}{p-b}, \quad q_b^{(2)} = \frac{G(q,-b)}{q-b}, \\ &G(a,b) = a^3 + b^3. \end{aligned} \tag{2.17}$$

All relations (2.12)–(2.16) also hold for their hat-q counterparts obtained by replacing the $\widetilde{\ }$-shift by the $\widehat{\ }$-shift whilst replacing the parameter p by q.

From Eqs. (2.12a), (2.14a), (2.16a) and their hat-q counterparts, a set of further relations can be derived

$$\left(p+q+s-\frac{\widehat{\widetilde{s}}_a}{\widehat{\widetilde{v}}_a}\right)(p-q+\widehat{s}-\widetilde{s}\,) = p_a^{(1)}\frac{\widehat{v}_a}{\widehat{\widetilde{v}}_a} - q_a^{(1)}\frac{\widetilde{v}_a}{\widehat{\widetilde{v}}_a}, \tag{2.18a}$$

$$p-q+\widehat{s}-\widetilde{s} = (p+a)\frac{\widetilde{v}_a}{v_a} - (q+a)\frac{\widehat{v}_a}{v_a}. \tag{2.18b}$$

Similarly, the utilization of the relations (2.12a), (2.14b), (2.16b) and their hat-q counterparts yields

$$\left(p+q-\widehat{\widetilde{s}}+\frac{t_b}{w_b}\right)(p-q+\widehat{s}-\widetilde{s}\,) = p_b^{(2)}\frac{\widetilde{w}_b}{w_b} - q_b^{(2)}\frac{\widehat{w}_b}{w_b}, \tag{2.19a}$$

$$p-q+\widehat{s}-\widetilde{s}=(p-b)\frac{\widehat{w}_b}{\widehat{\widetilde{w}}_b}-(q-b)\frac{\widetilde{w}_b}{\widehat{\widetilde{w}}_b}. \tag{2.19b}$$

2.2. *Lattice BSQ-type equations*

From the relations (2.12)–(2.19) one can construct the lattice BSQ-type equations (see also [13, 14, 19, 20]). To get the lattice BSQ equation we take $i = j = 0$ in (2.12) and its hat-q counterpart and get

$$p\widetilde{s}-\widetilde{S}^{(0,1)}=ps+S^{(1,0)}-s\widetilde{s}, \quad q\widehat{s}-\widehat{S}^{(0,1)}=qs+S^{(1,0)}-s\widehat{s}, \tag{2.20a}$$

$$p^2 s+pS^{(0,1)}+S^{(0,2)}=p^2\widetilde{s}-p\widetilde{S}^{(1,0)}+\widetilde{S}^{(2,0)}+ps\widetilde{s}-s\widetilde{S}^{(1,0)}+S^{(0,1)}\widetilde{s}, \tag{2.20b}$$

$$q^2 s+qS^{(0,1)}+S^{(0,2)}=q^2\widehat{s}-q\widehat{S}^{(1,0)}+\widehat{S}^{(2,0)}+qs\widehat{s}-s\widehat{S}^{(1,0)}+S^{(0,1)}\widehat{s}. \tag{2.20c}$$

Let us focus on (2.20b) and (2.20c). By subtraction one can delete $S^{(0,2)}$, and to delete $S^{(2,0)}$ from the remains, one can make use of (2.9a) where we take $i = 1,\ j = 0$. Then, after some algebra we can reach to

$$\widehat{\widetilde{S}}^{(1,0)}+S^{(0,1)}=pq-(p+q-\widehat{\widetilde{s}})(p+q+s)+\frac{G(p,-q)}{p-q+\widehat{s}-\widetilde{s}}. \tag{2.21}$$

This equation together with (2.20a) composes of the three-component lattice BSQ equation.

Making use of relations (2.13)–(2.19), one can give rise to the three-component lattice MBSQ/SBSQ equation (cf. [19]). In fact, from (2.13b)$^\wedge$ deleting $\widehat{\widetilde{s}}_a$ by using (2.18a) and $\widehat{t}_b$ by using the hat-q version of (2.14b) and also making use of (2.18b), one has

$$(q-b)\widehat{\widetilde{v}}_a w_b=\frac{p_a^{(1)}\widehat{v}_a-q_a^{(1)}\widetilde{v}_a}{(p+a)\widetilde{v}_a-(q+a)\widehat{v}_a}v_a\widehat{w}_b-p_a^{(1)}\widehat{s}_{a,b}+p_b^{(2)}\widehat{\widetilde{s}}_{a,b}+\frac{p_a^{(1)}-p_b^{(2)}}{a+b}. \tag{2.22}$$

This equation can also be rewritten in a compact form

$$\widehat{\widetilde{v}}_a w_b=v_a\frac{\frac{p_a^{(1)}}{p-b}\widehat{v}_a\widetilde{w}_b-\frac{q_a^{(1)}}{q-b}\widetilde{v}_a\widehat{w}_b}{(p+a)\widetilde{v}_a-(q+a)\widehat{v}_a}-\frac{G(a,b)}{(p-b)(q-b)}\left(\widehat{\widetilde{s}}_{a,b}-\frac{1}{a+b}\right). \tag{2.23}$$

The system composed of (2.23), (2.13a) and its hat-q version can be viewed as a modified version of the three-component lattice MBSQ/SBSQ equation (cf. [2, 19]).

The elimination of t_b from (2.13b) by using (2.19a) and $\widetilde{s}_a$ by using the hat-q version of (2.14a) and the usage of (2.18b) and (2.13a) deduce an alternative form of (2.23) (cf. [19]). This alternative form together with (2.13a) and its q-hat counterpart composes another form of the modified version of three-component lattice MBSQ/SBSQ equation.

The limit of the modified version of three-component lattice MBSQ/SBSQ equation as $a \to 0$, $b \to 0$ leads to the usual three-component lattice MBSQ/SBSQ equation, i.e.

$$1-p\widetilde{s}_{0,0}+ps_{0,0}=v_0\widetilde{w}_0, \quad 1-q\widehat{s}_{0,0}+qs_{0,0}=v_0\widehat{w}_0, \tag{2.24a}$$

$$\widehat{\widetilde{v}}_0 w_0=v_0\frac{p\widehat{v}_0\widetilde{w}_0-q\widetilde{v}_0\widehat{w}_0}{p\widetilde{v}_0-q\widehat{v}_0}. \tag{2.24b}$$

From the above three-component BSQ-type systems, one-component lattice BSQ equation (s), lattice MBSQ equation (v_0 or w_0), lattice SBSQ equation ($s_{0,0}$) [7, 8, 11, 14] and BSQ-type NQC equation ($s_{a,b}$) [19] can be obtained by removing the other two variables.

3. Explicit Solutions of CES

3.1. *Simplification of the CES*

Analogous to the earlier analysis [18, 20], we just need to discuss general solutions for the CES (2.1) according to the coefficient matrix $\boldsymbol{K}$ which is in canonical forms. We replace $\boldsymbol{K}_1$ by matrix $\boldsymbol{\Gamma}_1$ which is similar to $\boldsymbol{K}_1$, i.e. $\boldsymbol{K}_1 = \boldsymbol{T}_1^{-1}\boldsymbol{\Gamma}_1\boldsymbol{T}_1$, and $\boldsymbol{K}_2$ by matrix $\boldsymbol{\Gamma}_2$ which is similar to $\boldsymbol{K}_2$, i.e. $\boldsymbol{K}_2 = \boldsymbol{T}_2^{-1}\boldsymbol{\Gamma}_2\boldsymbol{T}_2$. Then the CES (2.1) becomes

$$\boldsymbol{\Gamma}\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{\Gamma}' = \boldsymbol{r}_1{}^t\boldsymbol{s}_1, \tag{3.1a}$$

$$\widetilde{\boldsymbol{r}}_1 = (p\boldsymbol{I}+\boldsymbol{\Gamma})\boldsymbol{r}_1, \quad \widehat{\boldsymbol{r}}_1 = (q\boldsymbol{I}+\boldsymbol{\Gamma})\boldsymbol{r}_1, \tag{3.1b}$$

$${}^t\widetilde{\boldsymbol{s}}_1 = {}^t\boldsymbol{s}_1(p\boldsymbol{I}-\boldsymbol{\Gamma}')^{-1}, \quad {}^t\widehat{\boldsymbol{s}}_1 = {}^t\boldsymbol{s}_1(q\boldsymbol{I}-\boldsymbol{\Gamma}')^{-1}, \tag{3.1c}$$

$$\widetilde{\boldsymbol{M}}_1(p\boldsymbol{I}-\boldsymbol{\Gamma}') = (p\boldsymbol{I}+\boldsymbol{\Gamma})\boldsymbol{M}_1, \quad \widehat{\boldsymbol{M}}_1(q\boldsymbol{I}-\boldsymbol{\Gamma}') = (q\boldsymbol{I}+\boldsymbol{\Gamma})\boldsymbol{M}_1, \tag{3.1d}$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)$, $\boldsymbol{\Gamma}' = \mathrm{diag}(-\omega\boldsymbol{\Gamma}_1, -\omega^2\boldsymbol{\Gamma}_2)$ and $\boldsymbol{M}_1 = \boldsymbol{T}\boldsymbol{M}\boldsymbol{T}^{-1}$, $\boldsymbol{r}_1 = \boldsymbol{T}\boldsymbol{r}$, ${}^t\boldsymbol{s}_1 = {}^t\boldsymbol{s}\boldsymbol{T}^{-1}$ with $\boldsymbol{T} = \mathrm{diag}(\boldsymbol{T}_1, \boldsymbol{T}_2)$. Now, we turn to the new CES (3.1), from which we can define the following objects:

$$\boldsymbol{u}_1^{(i)}(b) = (\boldsymbol{I}+\boldsymbol{M}_1)^{-1}(b\boldsymbol{I}+\boldsymbol{\Gamma})^i\boldsymbol{r}_1, \tag{3.2a}$$

$${}^t\boldsymbol{u}_1^{(j)}(a) = {}^t\boldsymbol{s}_1(a\boldsymbol{I}+\boldsymbol{\Gamma}')^j(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}, \tag{3.2b}$$

$$S_1^{(i,j)}(a,b) = {}^t\boldsymbol{s}_1(a\boldsymbol{I}+\boldsymbol{\Gamma}')^j(\boldsymbol{I}+\boldsymbol{M}_1)^{-1}(b\boldsymbol{I}+\boldsymbol{\Gamma})^i\boldsymbol{r}_1. \tag{3.2c}$$

In fact, (3.2) and (2.5) are related by

$$\boldsymbol{u}_1^{(i)}(b) = \boldsymbol{T}\boldsymbol{u}^{(i)}(b), \quad {}^t\boldsymbol{u}_1^{(j)}(a) = {}^t\boldsymbol{u}^{(j)}(a)\boldsymbol{T}^{-1}, \quad S_1^{(i,j)}(a,b) = S^{(i,j)}(a,b). \tag{3.3}$$

Obviously $S_1^{(i,j)}(a,b) = {}^t\boldsymbol{s}_1(a\boldsymbol{I}+\boldsymbol{\Gamma}')^j\boldsymbol{u}_1^{(i)}(b) = {}^t\boldsymbol{u}_1^{(j)}(a)(b\boldsymbol{I}+\boldsymbol{\Gamma})^i\boldsymbol{r}_1$. Equation (3.3) implies that K and its canonical from Γ lead to the same $S^{(i,j)}(a,b)$ for the lattice BSQ-type equations.

Summarizing the above results, we have the following statement.

Proposition 3.1. *Suppose that* $\boldsymbol{\Gamma} = \boldsymbol{T}\boldsymbol{K}\boldsymbol{T}^{-1}$, $\boldsymbol{\Gamma}' = \boldsymbol{T}\boldsymbol{K}'\boldsymbol{T}^{-1}$. *Then* $\boldsymbol{M} = \boldsymbol{T}^{-1}\boldsymbol{M}_1\boldsymbol{T}$, $\boldsymbol{r} = \boldsymbol{T}^{-1}\boldsymbol{r}_1$ *and* ${}^t\boldsymbol{s} = {}^t\boldsymbol{s}_1\boldsymbol{T}$ *provide general solutions to the CES* (2.1), *where* $\boldsymbol{M}_1$, $\boldsymbol{r}_1$ *and* ${}^t\boldsymbol{s}_1$ *are the general solutions to the deformed CES* (3.1).

3.2. *Solutions relate to* $\boldsymbol{\Gamma}$

Since equation (3.1d) can be deduced from (3.1a) and (3.1b) (cf. [18, 20]), we just give explicit expressions of $\boldsymbol{r}_1$, ${}^t\boldsymbol{s}_1$ and $\boldsymbol{M}_1$ in (3.1a)–(3.1c), where $\boldsymbol{\Gamma}$ takes different canonical forms of the $(N_1+N_2)\times(N_1+N_2)$ constant matrix $\boldsymbol{K}$, which corresponds to $\boldsymbol{K}$ having different kinds of eigenvalues. Generally speaking, $\mathbf{\Gamma}_i,\ (i=1,2)$ are of forms

$$\mathbf{\Gamma}_i=\mathrm{diag}\big(\mathbf{\Gamma}_{i,[D,h_i^{(1)}]},\ \mathbf{\Gamma}_{i,[J;h_i^{(2)}]},\ldots,\mathbf{\Gamma}_{i,[J;h_i^{(k)}]}\big),\quad \sum_{j=1}^{k}h_i^{(j)}=N_i,\quad (i=1,2),\tag{3.4}$$

where $\mathbf{\Gamma}_{i,[D,h_i^{(1)}]}$ $(i=1,2)$ represent $h_i^{(1)}\times h_i^{(1)}$ diagonal matrices; $\mathbf{\Gamma}_{i,[J;h_i^{(j)}]}$ $(i=1,2,\ j=2,\ldots,k)$ are $h_i^{(j)}\times h_i^{(j)}$ Jordan-blocks. With the above form of $\mathbf{\Gamma}$ one can derive general mixed solutions to the lattice BSQ-type equations (see also [18]).

In the following we only discuss three forms of the canonical matrix $\mathbf{\Gamma}$:

	Case 1	Case 2	Case 3
$\mathbf{\Gamma}_1$	diagonal	diagonal	Jordan block
$\mathbf{\Gamma}_2$	diagonal	Jordan block	Jordan block

The case 1 leads to the soliton solutions and the latter two cases yield mixed solutions.

Case 1.

$$\mathbf{\Gamma}=\mathrm{diag}(\mathbf{\Gamma}_{1,[D;N_1]},\ \mathbf{\Gamma}_{2,[D;N_2]}),\quad \mathbf{\Gamma}'=\mathrm{diag}(-\omega\mathbf{\Gamma}_{1,[D;N_1]},-\omega^2\mathbf{\Gamma}_{2,[D;N_2]}),\tag{3.5a}$$

$$\mathbf{\Gamma}_{i,[D;N_i]}=\mathrm{diag}(k_{i,1},k_{i,2},\ldots,k_{i,N_i}),\quad (i=1,2).\tag{3.5b}$$

In this case, $\boldsymbol{r}_1$ in (3.1b) is given by

$$\boldsymbol{r}_1=\mathrm{diag}(\boldsymbol{\rho}_{1,[D;N_1]},\ \boldsymbol{\rho}_{2,[D;N_2]})\cdot(\boldsymbol{I}_{N_1}^{(1)^T},\boldsymbol{I}_{N_2}^{(1)^T})^T,\quad \boldsymbol{I}_{N_i}^{(1)}=(1,1,\ldots,1)_{N_i}^T,\tag{3.6a}$$

$$\boldsymbol{\rho}_{i,[D;N_i]}=\mathrm{diag}(\rho_{i,1},\rho_{i,2},\ldots,\rho_{i,N_i}),\quad \rho_{i,l}=(p+k_{i,l})^n(q+k_{i,l})^m\rho_{i,l}^0,\quad (i=1,2),\tag{3.6b}$$

where $\{\rho_{i,l}^0\}$ are complex constants. From (3.1c) we deduce

$${}^t\boldsymbol{s}_1=(\boldsymbol{I}_{N_1}^{(1)^T},\boldsymbol{I}_{N_2}^{(1)^T})\cdot\mathrm{diag}(\boldsymbol{S}_{1,[D;N_1]},\ \boldsymbol{S}_{2,[D;N_2]}),\tag{3.7a}$$

$$\boldsymbol{S}_{j,[D;N_j]}=\mathrm{diag}(\sigma_{j,1},\sigma_{j,2},\ldots,\sigma_{j,N_j}),\quad \sigma_{j,h}=(p+\omega^jk_{j,h})^{-n}(q+\omega^jk_{j,h})^{-m}\sigma_{j,h}^0,\ (j=1,2),\tag{3.7b}$$

where $\{\sigma_{j,h}^0\}$ are complex constants. Substituting (3.6) and (3.7) into (3.1a), then we get

$$\boldsymbol{M}_1=\mathrm{diag}(\boldsymbol{\rho}_{1,[D;N_1]},\ \boldsymbol{\rho}_{2,[D;N_2]})\cdot\begin{pmatrix}\boldsymbol{G}_{[D,D]}^{(1,1)} & \boldsymbol{G}_{[D,D]}^{(1,2)}\\ \boldsymbol{G}_{[D,D]}^{(2,1)} & \boldsymbol{G}_{[D,D]}^{(2,2)}\end{pmatrix}\cdot\mathrm{diag}(\boldsymbol{S}_{1,[D;N_1]},\ \boldsymbol{S}_{2,[D;N_2]}),\tag{3.8}$$

where

$$\boldsymbol{G}_{[D,D]}^{(i,j)}=\left(\frac{1}{k_{i,l}-\omega^jk_{j,h}}\right)_{l,h},\quad (i,j=1,2).\tag{3.9}$$

Obviously, $\{\boldsymbol{G}_{[D,D]}^{(i,j)}\}$ are Cauchy matrices. In this case, the corresponding solutions $S^{(i,j)}(a,b)$ generate soliton solutions to the lattice BSQ-type equations in light of $S^{(i,j)}=S^{(i,j)}(0,0)$, (2.10) and (2.11).

In particular, when $N_1 = N_2 = 1$ and $k_{1,1} = k_{2,1} = k$, we obtain the 1-soliton solution for the three-component lattice BSQ equation

$$s = \frac{k(\omega^2 - 1)(\varrho_1 + \varrho_2)}{k(\omega^2 - 1) + \omega^2 \varrho_1 - \varrho_2}, \tag{3.10a}$$

$$S^{(1,0)} = ks, \quad S^{(0,1)} = \frac{k^2(1-\omega)(\varrho_1 + \omega\varrho_2)}{k(\omega^2 - 1) + \omega^2 \varrho_1 - \varrho_2}, \tag{3.10b}$$

where $\varrho_i = (\frac{p+k}{p+\omega^i k})^n (\frac{q+k}{q+\omega^i k})^m \varrho_i^0$ with $\varrho_i^0 = \frac{\rho^0}{\sigma_i^0}$, $i = 1, 2$ are two discrete plane-wave factors. From (2.10) and (2.11), 1-soliton solution for the modified version of three-component lattice MBSQ/SBSQ equation can be described as

$$v_a = \frac{-k(\omega^2-1) + \dfrac{(a-k)}{(k^2+ak+a^2)}[\omega(k - a\omega)\varrho_1 + (a - k\omega)\varrho_2]}{k(\omega^2 - 1) + \omega^2 \varrho_1 - \varrho_2}, \tag{3.11a}$$

$$w_b = \frac{s}{b+k} - 1, \quad s_{a,b} = \frac{1}{b+k}(v_a + 1), \tag{3.11b}$$

where s is given by (3.10a). Whilst the limit of (3.11) as $a \to 0$ and $b \to 0$ gives 1-soliton solution of equation (2.24)

$$v_0 = \frac{-k(\omega^2-1) - \omega(\varrho_1 - \varrho_2)}{k(\omega^2 - 1) + \omega^2 \varrho_1 - \varrho_2}, \quad w_0 = \frac{s}{k} - 1, \quad s_{0,0} = \frac{1}{k}(v_0 + 1). \tag{3.12}$$

Case 2.

$$\boldsymbol{\Gamma} = \mathrm{diag}(\boldsymbol{\Gamma}_{[D;N_1]},\ \boldsymbol{\Gamma}_{[J;N_2]}(k)), \quad \boldsymbol{\Gamma}' = \mathrm{diag}(-\omega\boldsymbol{\Gamma}_{[D;N_1]},\ -\omega^2\boldsymbol{\Gamma}_{[J;N_2]}(k)), \tag{3.13}$$

where $\boldsymbol{\Gamma}_{[D;N_1]} = \boldsymbol{\Gamma}_{1,[D;N_1]}$ satisfies (3.5) and

$$\boldsymbol{\Gamma}_{[J;N_2]}(k) = \begin{pmatrix} k & 0 & 0 & \cdots & 0 & 0 \\ 1 & k & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & k \end{pmatrix}_{N_2 \times N_2}, \tag{3.14}$$

where $k \neq 0$ is a complex constant.

In this case, $\boldsymbol{r}_1$ in (3.1b) is of form

$$\boldsymbol{r}_1 = \mathrm{diag}(\boldsymbol{\rho}_{[D;N_1]},\ \boldsymbol{\rho}_{[J;N_2]}(k)) \cdot (\boldsymbol{I}_{N_1}^{(1)^T}, \boldsymbol{I}_{N_2}^{(2)^T})^T, \quad \boldsymbol{I}_{N_2}^{(2)} = (1, 0, \ldots, 0)_{N_2}^T, \tag{3.15}$$

where $\boldsymbol{\rho}_{[D;N_1]} = \boldsymbol{\rho}_{1,[D;N_1]}$, $\boldsymbol{I}_{N_1}^{(1)}$ are given by (3.6) and

$$\boldsymbol{\rho}_{[J;N_2]}(k) = \begin{pmatrix} \rho & 0 & 0 & \cdots & 0 \\ \rho^{(1)} & \rho & 0 & \cdots & 0 \\ \dfrac{\rho^{(2)}}{2!} & \rho^{(1)} & \rho & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\rho^{(N_2-1)}}{(N_2-1)!} & \dfrac{\rho^{(N_2-2)}}{(N_2-2)!} & \dfrac{\rho^{(N_2-3)}}{(N_2-3)!} & \cdots & \rho \end{pmatrix}_{N_2 \times N_2}, \quad \rho^{(j)} = \partial_k^j \rho, \tag{3.16}$$

with $\rho = (p+k)^n(q+k)^m\rho^0$. From (3.1c) we also have

$$ {}^t\boldsymbol{s}_1 = (\boldsymbol{I}_{N_1}^{(1)^T}, \boldsymbol{I}_{N_2}^{(2)^T}) \cdot \mathrm{diag}(\boldsymbol{S}_{[D;N_1]},\ \boldsymbol{S}_{[J;N_2]}(k)), \tag{3.17} $$

where $\boldsymbol{S}_{[D;N_1]} = \boldsymbol{S}_{1,[D;N_1]}$ is given by (3.7) and

$$ \boldsymbol{S}_{[J;N_2]}(k) = \begin{pmatrix} \frac{\sigma^{(N_2-1)}}{(N_2-1)!} & \cdots & \frac{\sigma^{(2)}}{2!} & \sigma^{(1)} & \sigma \\ \frac{\sigma^{(N_2-2)}}{(N_2-2)!} & \cdots & \sigma^{(1)} & \sigma & 0 \\ \frac{\sigma^{(N_2-3)}}{(N_2-3)!} & \cdots & \sigma & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma & \cdots & 0 & 0 & 0 \end{pmatrix}_{N_2\times N_2}, \quad \sigma^{(j)} = \partial_k^j\sigma, \tag{3.18} $$

with $\sigma = (p+\omega^2 k)^{-n}(q+\omega^2 k)^{-m}\sigma^0$. Set

$$ \boldsymbol{M}_1 = \mathrm{diag}(\boldsymbol{\rho}_{[D;N_1]},\ \boldsymbol{\rho}_{[J;N_2]}(k)) \cdot \begin{pmatrix} \boldsymbol{G}_{[D,D]}^{(1,1)} & \boldsymbol{G}_{[D,J]}^{(1,2)} \\ \boldsymbol{G}_{[D,J]}^{(2,1)} & \boldsymbol{G}_{[J,J]}^{(2,2)} \end{pmatrix} \cdot \mathrm{diag}(\boldsymbol{S}_{[D;N_1]},\ \boldsymbol{S}_{[J;N_2]}(k)). \tag{3.19} $$

Then (3.1a) becomes

$$ \boldsymbol{\Gamma}_{[D;N_1]}\boldsymbol{G}_{[D,D]}^{(1,1)} - \omega\boldsymbol{G}_{[D,D]}^{(1,1)}\boldsymbol{\Gamma}_{[D;N_1]} = \boldsymbol{I}_{N_1}^{(1)}\boldsymbol{I}_{N_1}^{(1)^T}, \tag{3.20a} $$

$$ \boldsymbol{\Gamma}_{[D;N_1]}\boldsymbol{G}_{[D,J]}^{(1,2)} - \omega^2\boldsymbol{G}_{[D,J]}^{(1,2)}\boldsymbol{\Gamma}_{[J;N_2]}^T(k) = \boldsymbol{I}_{N_1}^{(1)}\boldsymbol{I}_{N_2}^{(2)^T}, \tag{3.20b} $$

$$ \boldsymbol{\Gamma}_{[J;N_2]}(k)\boldsymbol{G}_{[D,J]}^{(2,1)} - \omega\boldsymbol{G}_{[D,J]}^{(2,1)}\boldsymbol{\Gamma}_{[D;N_1]} = \boldsymbol{I}_{N_2}^{(2)}\boldsymbol{I}_{N_1}^{(1)^T}, \tag{3.20c} $$

$$ \boldsymbol{\Gamma}_{[J;N_2]}(k)\boldsymbol{G}_{[J,J]}^{(2,2)} - \omega^2\boldsymbol{G}_{[J,J]}^{(2,2)}\boldsymbol{\Gamma}_{[J;N_2]}^T(k) = \boldsymbol{I}_{N_2}^{(2)}\boldsymbol{I}_{N_2}^{(2)^T}, \tag{3.20d} $$

where $\boldsymbol{G}_{[D,D]}^{(1,1)}$ is given by (3.9). $\boldsymbol{G}_{[D,J],j}^{(1,2)}$ and $\boldsymbol{G}_{[J,J],j}^{(2,2)}$ denote the jth row vectors of $\boldsymbol{G}_{[D,J]}^{(1,2)}$ and $\boldsymbol{G}_{[J,J]}^{(2,2)}$, respectively. From (3.20b), we obtain the following system for $\{\boldsymbol{G}_{[D,J],j}^{(1,2)}\}$

$$ k_{1,j}\boldsymbol{G}_{[D,J],j}^{(1,2)} - \omega^2\boldsymbol{G}_{[D,J],j}^{(1,2)}\boldsymbol{\Gamma}_{[J;N_2]}^T(k) = \boldsymbol{I}_{N_2}^{(2)^T}, \quad j = 1, 2, \ldots, N_1, \tag{3.21} $$

which yield

$$ \boldsymbol{G}_{[D,J],j}^{(1,2)} = \boldsymbol{I}_{N_2}^{(2)^T}(k_{1,j}\boldsymbol{I}_{N_2} - \omega^2\boldsymbol{\Gamma}_{[J;N_2]}^T(k))^{-1}, \quad j = 1, 2, \ldots, N_1. \tag{3.22} $$

From (3.20d) we arrive at a system for $\{\boldsymbol{G}_{[J,J],j}^{(2,2)}\}$

$$ k\boldsymbol{G}_{[J,J],1}^{(2,2)} - \omega^2\boldsymbol{G}_{[J,J],1}^{(2,2)}\boldsymbol{\Gamma}_{[J;N_2]}^T(k) = \boldsymbol{I}_{N_2}^{(2)^T}, \tag{3.23a} $$

$$ \boldsymbol{G}_{[J,J],j-1}^{(2,2)} + k\boldsymbol{G}_{[J,J],j}^{(2,2)} - \omega^2\boldsymbol{G}_{[J,J],j}^{(2,2)}\boldsymbol{\Gamma}_{[J;N_2]}^T(k) = \boldsymbol{0}_{N_2}^T, \tag{3.23b} $$

where $\mathbf{0}_{N_2}$ is the N_2-order zero column vector. From (3.23) we easily get

$$\boldsymbol{G}^{(2,2)}_{[J,J],j} = (-1)^{j-1}\boldsymbol{I}^{(2)^T}_{N_2}(k\boldsymbol{I}_{N_2} - \omega^2\boldsymbol{\Gamma}^T_{[J;N_2]}(k))^{-j}, \quad j = 1,2,\ldots,N_2. \tag{3.24}$$

Comparing (3.20b) with (3.20c), we have $\boldsymbol{G}^{(2,1)}_{[D,J]} = -\omega^2\boldsymbol{G}^{(1,2)^T}_{[D,J]}$.

Define

$$\begin{aligned} \boldsymbol{r}^{[D,J]} &= \mathrm{diag}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{r}_1, \quad {}^t\boldsymbol{s}^{[D,J]} = {}^t\boldsymbol{s}_1\mathrm{diag}(\boldsymbol{A},\boldsymbol{B}),\\ \boldsymbol{M}^{[D,J]} &= \mathrm{diag}(\boldsymbol{A},\boldsymbol{B})\boldsymbol{M}_1\mathrm{diag}(\boldsymbol{A},\boldsymbol{B}), \end{aligned} \tag{3.25}$$

where $\boldsymbol{A}$ is an arbitrary $N_1\times N_1$ constant diagonal matrix and $\boldsymbol{B}$ is an arbitrary $N_2\times N_2$ constant lower triangular Toeplitz matrix (commute with Jordan block (cf. [15, 16])). It is readily to see that $\boldsymbol{\rho}_{[J;N_2]}(k)$ and $\bar{\boldsymbol{I}}\boldsymbol{S}_{[J;N_2]}(k)$ are lower triangular Toeplitz matrices, where $\bar{\boldsymbol{I}}$ is the $N_2\times N_2$ anti-diagonal unit matrix. Similar to [15], one can prove that (3.25) provides general solutions to the CES (3.1) when $\boldsymbol{\Gamma}$ is (3.13) and (3.14).

Case 3.

$$\boldsymbol{\Gamma} = \mathrm{diag}(\boldsymbol{\Gamma}_{[J;N_1]}(\kappa),\ \boldsymbol{\Gamma}_{[J;N_2]}(k)), \quad \boldsymbol{\Gamma}' = \mathrm{diag}(-\omega\boldsymbol{\Gamma}_{[J;N_1]}(\kappa), -\omega^2\boldsymbol{\Gamma}_{[J;N_2]}(k)), \tag{3.26}$$

where nonzero constants κ and k satisfy $\kappa-\omega^2k\neq 0$.

For this case $\boldsymbol{r}_1$ and ${}^t\boldsymbol{s}_1$ are given by

$$\boldsymbol{r}_1 = \mathrm{diag}(\boldsymbol{\rho}_{[J;N_1]}(\kappa),\ \boldsymbol{\rho}_{[J;N_2]}(k))\cdot(\boldsymbol{I}^{(2)^T}_{N_1}, \boldsymbol{I}^{(2)^T}_{N_2})^T, \tag{3.27a}$$

$${}^t\boldsymbol{s}_1 = (\boldsymbol{I}^{(2)^T}_{N_1}, \boldsymbol{I}^{(2)^T}_{N_2})\cdot\mathrm{diag}(\bar{\boldsymbol{S}}_{[J;N_1]}(\kappa),\ \boldsymbol{S}_{[J;N_2]}(k)), \tag{3.27b}$$

where $\boldsymbol{\rho}_{[J;\cdot](\cdot)}$, $\boldsymbol{S}_{[J;N_2]}(k)$ are, respectively, defined by (3.16) and (3.18). Here $\bar{\boldsymbol{S}}_{[J;N_1]}(\kappa)$ is of form

$$\bar{\boldsymbol{S}}_{[J;N_1]}(\kappa) = \begin{pmatrix} \dfrac{\theta^{(N_1-1)}}{(N_1-1)!} & \cdots & \dfrac{\theta^{(2)}}{2!} & \theta^{(1)} & \theta \\ \dfrac{\theta^{(N_1-2)}}{(N_1-2)!} & \cdots & \theta^{(1)} & \theta & 0 \\ \dfrac{\theta^{(N_1-3)}}{(N_1-3)!} & \cdots & \theta & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta & \cdots & 0 & 0 & 0 \end{pmatrix}_{N_1\times N_1}, \quad \theta^{(j)} = \partial^j_\kappa\theta, \tag{3.28}$$

with $\theta = (p+\omega\kappa)^{-n}(q+\omega\kappa)^{-m}\theta^0$, where θ^0 is a constant. Let

$$\boldsymbol{M}_1 = \mathrm{diag}(\boldsymbol{\rho}_{[J;N_1]}(\kappa),\ \boldsymbol{\rho}_{[J;N_2]}(k))\cdot\begin{pmatrix} \boldsymbol{G}^{(1,1)}_{[J,J]} & \boldsymbol{G}^{(1,2)}_{[J,J]} \\ \boldsymbol{G}^{(2,1)}_{[J,J]} & \boldsymbol{G}^{(2,2)}_{[J,J]} \end{pmatrix}\cdot\mathrm{diag}(\bar{\boldsymbol{S}}_{[J;N_1]}(\kappa),\ \boldsymbol{S}_{[J;N_2]}(k)). \tag{3.29}$$

Plugging (3.29) into (3.1a) leads to

$$\boldsymbol{\Gamma}_{[J;N_1]}(\kappa)\boldsymbol{G}^{(1,1)}_{[J,J]} - \omega\boldsymbol{G}^{(1,1)}_{[J,J]}\boldsymbol{\Gamma}^T_{[J;N_1]}(\kappa) = \boldsymbol{I}^{(2)}_{N_1}\boldsymbol{I}^{(2)^T}_{N_1}, \tag{3.30a}$$

$$\boldsymbol{\Gamma}_{[J;N_1]}(\kappa)\boldsymbol{G}^{(1,2)}_{[J,J]} - \omega^2\boldsymbol{G}^{(1,2)}_{[J,J]}\boldsymbol{\Gamma}^T_{[J;N_2]}(k) = \boldsymbol{I}^{(2)}_{N_1}\boldsymbol{I}^{(2)^T}_{N_2}, \tag{3.30b}$$

$$\boldsymbol{\Gamma}_{[J;N_2]}(k)\boldsymbol{G}^{(2,1)}_{[J,J]} - \omega\boldsymbol{G}^{(2,1)}_{[J,J]}\boldsymbol{\Gamma}^T_{[J;N_1]}(\kappa) = \boldsymbol{I}^{(2)}_{N_2}\boldsymbol{I}^{(2)^T}_{N_1}, \tag{3.30c}$$

$$\boldsymbol{\Gamma}_{[J;N_2]}(k)\boldsymbol{G}^{(2,2)}_{[J,J]} - \omega^2\boldsymbol{G}^{(2,2)}_{[J,J]}\boldsymbol{\Gamma}^T_{[J;N_2]}(k) = \boldsymbol{I}^{(2)}_{N_2}\boldsymbol{I}^{(2)^T}_{N_2}, \tag{3.30d}$$

where $\boldsymbol{G}^{(2,2)}_{[J,J]}$ is given by (3.24). The jth row vectors of $\boldsymbol{G}^{(1,1)}_{[J,J]}$, $\boldsymbol{G}^{(1,2)}_{[J,J]}$ and $\boldsymbol{G}^{(2,1)}_{[J,J]}$ can be described as

$$\boldsymbol{G}^{(1,1)}_{[J,J],j} = (-1)^{j-1}\boldsymbol{I}^{(2)^T}_{N_1}(\kappa\boldsymbol{I}_{N_1} - \omega\boldsymbol{\Gamma}^T_{[J;N_1]}(\kappa))^{-j}, \quad j = 1,2,\ldots,N_1, \tag{3.31a}$$

$$\boldsymbol{G}^{(1,2)}_{[J,J],j} = (-1)^{j-1}\boldsymbol{I}^{(2)^T}_{N_2}(\kappa\boldsymbol{I}_{N_2} - \omega^2\boldsymbol{\Gamma}^T_{[J;N_2]}(k))^{-j}, \quad j = 1,2,\ldots,N_1, \tag{3.31b}$$

$$\boldsymbol{G}^{(2,1)}_{[J,J],j} = (-1)^{j-1}\boldsymbol{I}^{(2)^T}_{N_1}(k\boldsymbol{I}_{N_1} - \omega\boldsymbol{\Gamma}^T_{[J;N_1]}(\kappa))^{-j}, \quad j = 1,2,\ldots,N_2. \tag{3.31c}$$

The comparison of (3.30b) and (3.30c) gives $\boldsymbol{G}^{(1,2)}_{[J,J]} = -\omega\boldsymbol{G}^{(2,1)^T}_{[J,J]}$. Hence, when $\boldsymbol{\Gamma}$ takes (3.26), the general solutions of the CES (3.1) can be given out by defining

$$\begin{aligned}\boldsymbol{r}^{[J,J]} &= \mathrm{diag}(\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}})\boldsymbol{r}_1, \quad {}^t\boldsymbol{s}^{[J,J]} = {}^t\boldsymbol{s}_1\mathrm{diag}(\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}}),\\ \boldsymbol{M}^{[J,J]} &= \mathrm{diag}(\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}})\boldsymbol{M}_1\mathrm{diag}(\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}}),\end{aligned} \tag{3.32}$$

where $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are arbitrary $N_1 \times N_1$ and $N_2 \times N_2$ constant lower triangular Toeplitz matrices, respectively.

4. Conclusion

The study of the discrete versions of soliton systems, i.e. systems given by integrable partial difference equations, has attracted lots of attentions in recent years. Up to now, many discrete integrable systems as well as their corresponding semi-discrete systems have been proposed, such as ABS lattices, lattice BSQ-type equation, lattice GD hierarchy and lattice KP-type equations. Subsequently, several constructive approaches have also been developed to construct exact solutions for these systems. Cauchy matrix approach, which is deeply connected to the DL method, was first used by Nijhoff *et al.* to study the soliton solutions for ABS lattices [10] and generalized by Zhang *et al.* [18, 20] to construct more kinds of exact solutions for ABS lattices and lattice BSQ-type equations. In present contribution, we extended the Cauchy matrix approach to derive exact solutions for the lattice BSQ-type equations. Comparing with the previous work [20], here a more general CES (2.1) was introduced. Consequently, all the solutions listed in this paper contain two kinds of plane-wave factors. Our treatments in this paper can also be applied to the extended lattice BSQ systems [19] and lattice GD hierarchy [11]. For the semi-discrete BSQ-type systems, their exact solutions can be derived through continuum limits of the discrete plane-wave factors, i.e. replacing the discrete plane-wave factors by exponential functions.

Acknowledgments

The authors are very grateful to the referees for the invaluable and expert comments. This project was supported by the National Natural Science Foundation of China under Grant (No. 11071157), Shanghai Leading Academic Discipline Project (No. J50101), SRF of the DPHE of China (No. 20113108110002) and Postgraduate Innovation Foundation of Shanghai University (No. SHUCX111027).

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