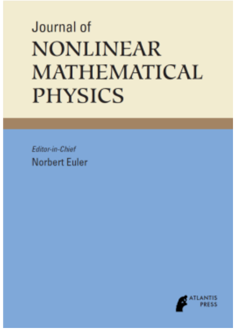

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RELATING THE BOTTOM PRESSURE AND THE SURFACE ELEVATION IN THE WATER WAVE PROBLEM

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An overview is presented of recent progress on the relation between the pressure at the bottom of the flat water bed and the elevation of the free water boundary within the context of the one-dimensional, irrotational water wave problem. We present five different approaches to this problem. All are compared to (1) numerical data for Stokes waves, one-dimensional traveling wave solutions of the full irrotational water wave problem, and (2) experimental data for high-amplitude waves in a long, narrow (i.e. one-dimensional) wave tank.

Keywords: Water waves; pressure; review.

Mathematics Subject Classification 2000: 35Q31, 76B15, 76B07

1. Introduction

The dynamics of an irrotational, inviscid free-surface water wave is accurately modeled by the Euler equations. In the case of a one-dimensional surface, these equations are

$$\phi_{xx} + \phi_{zz} = 0, \quad (x, z) \in D, \tag{1.1a}$$

$$\phi_z = 0, \quad z = -h, \tag{1.1b}$$

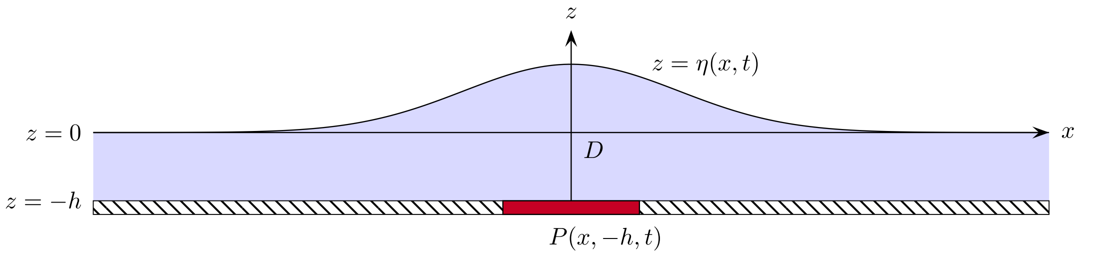

Fig. 1. The fluid domain D for the water wave problem. An idealized pressure sensor is indicated at the bottom. In all models discussed the pressure measurement is assumed to be a point measurement.

$$\eta_t + \eta_x \phi_x = \phi_z, \quad z = \eta(x,t), \tag{1.1c}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0, \qquad z = \eta(x,t), \tag{1.1d}$$

where x and z are the horizontal and vertical coordinate, respectively, see Fig. 1; $z = \eta(x,t)$ is the free top boundary and $\phi(x,z,t)$ is the velocity potential. Indices are used to denote partial derivatives. Further, g is the acceleration due to gravity and h is the average depth of the fluid.

A quantity of significant physical importance is the pressure $P(x,z,t)$ in the fluid. This quantity does not appear in the Euler equations. Instead, it is obtained from the Bernoulli equation

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + gz + \frac{P(x,z,t)}{\rho} = 0, \quad -h \le z \le \eta(x,t), \tag{1.2}$$

which allows one to find the pressure once (1.1a)–(1.1d) is solved.

However, from a physical perspective one may wish to ask a different question: is it possible to recover the surface elevation $\eta(x,t)$ from knowledge of the pressure, especially when measured at the bottom $z = -h$? This problem is of paramount importance for experimentalists and field practitioners, for whom direct measurement of the surface elevation is difficult. Instead, the surface elevation is often inferred from measurements of the bottom pressure, see e.g. [2, 4, 18, 19, 24, 25]. For instance, the predictions made by the Pacific Tsunami Warning Center [21] are determined this way. The pressure is not only of physical importance: it is relevant for the mathematical study of the Euler equations. Its qualitative properties play a central role in understanding properties of irrotational traveling water waves (so-called Stokes waves) such as in showing that their free surface is the graph of a function [23, 26], and in discerning the patterns of particle paths beneath them [7, 8, 10].

These considerations prompt us to investigate (1.1a)–(1.1d) and (1.2) in a different way, where the bottom pressure is considered as input, and the goal is to recover the surface quantities $\phi(x,\eta(x,t),t)$ and, especially, $\eta(x,t)$. In addition, we impose that the solutions of (1.1a)–(1.1d) and (1.2) are stationary in a frame of reference moving with constant velocity (in other words, we restrict ourselves to Stokes wave solutions) and one-dimensional. When comparing with experimental data in Sec. 4 this can no longer be justified, and we explore the extent to which the different methods for surface reconstruction still provide accurate results.

2. Relating the Bottom Pressure and the Surface Elevation

The problem of relating the water surface elevation to measurements at the flat fluid bed has received much attention recently. In this section we review five relationships between the bottom pressure and the surface elevation. Three of these are the result of recent work. In the following sections we compare these relationships against numerical and experimental data.

- The oldest and perhaps most commonly used reconstruction formula is that obtaining using a *hydrostatic approximation* (HA) (Archimedes' relation) [5, 13]:

$$\eta(x,t) = \frac{P(x,-h,t)}{\rho g} - h. \tag{2.1}$$

  As mentioned in the introduction, the HA is used, for instance, in open-ocean buoys employed for tsunami detection, see [21]. Despite the limitations of a HA, its application in a shallow-water regime (e.g. tsunamis) are remarkably accurate. A comparison with field data is not included below but one easily verifies, using pressure data available from [21], for instance, that the four formulations below are consistent with the HA in that they give $\eta = 0$ in that limit.
- The *transfer function* (TF) approach is obtained by linearizing the equations of motion around quiescent water, and using Fourier transforms to solve the resulting linear constant coefficient equations, similar to how one derives the dispersion relation for the water wave problem. This results in a linear relationship between the Fourier transforms $\mathcal{F}$ of the dynamical part of the pressure and the elevation of the surface [5, 13, 16, 17]:

$$\mathcal{F}\{\eta(x,t)\}(k) = \cosh(kh)\mathcal{F}\{p(x,t)/g\}(k). \tag{2.2}$$

  Here $p(x,t) = (P(x,-h,t) - \rho g h)/\rho$ is the dynamic (or non-static) part of the pressure $P(x,z,t)$ evaluated at the bottom of the fluid $z = -h$, scaled by the fluid density ρ. In this relationship, η and p are regarded as functions of the spatial coordinate x, with parametric dependence on time t. Since the formula is the result of linearizing (1.1a)–(1.1d) and (1.2), one can only expect good agreement for waves of small amplitude. In particular, the reconstruction of waves that do not have a linear analogue (like solitary waves, which also have a broad spectrum) cannot be assumed to be accurate using (2.2). In contrast to the other relations presented here, (2.2) is not restricted to Stokes waves. In measurements, time series are more common than spatial series. The formula (2.2) is easily modified to accommodate this, using temporal instead of spatial Fourier transforms. This results in extra factors of the wave speed $c(k)$, not necessarily constant.

Both approaches above rely on linear theory. Until the work of Escher and Schlurmann [16], no consideration was given to the importance of nonlinear effects in the pressure-surface reconstruction problem. Such effects are expected to be especially significant for high-amplitude shallow-water waves or for waves in the surf zone (see [3, 4, 24], for instance). Since nonlinear effects are not captured by the linear TF (2.2), different modifications of (2.2) have been proposed, see [18, 19, 24] or [3, 20]. See Bishop and Donelan [4] for a comparison of these modified TF approaches.

Since 2008, much progress has been made in understanding the qualitative and quantitative properties of the pressure in the water wave problem, accounting for all nonlinear effects. Constantin and Strauss [12] examine different properties and relations between the pressure and the surface elevation qualitatively. They do not present a reconstruction method to accurately determine one function in terms of the other, but the mathematical properties they uncover are revisited and verified in an experimental setting by Constantin, Escher and Hsu [11].

- The first nonlinear reconstruction method was introduced in [14, 22] and is based on a fully nonlinear nonlocal implicit relationship between the surface elevation and bottom pressure. The *nonlocal relationship* is obtained from (1.1a)–(1.1d) and (1.2) with a traveling wave assumption but without any other approximation. In order to recover the surface elevation from the pressure measurements one solves

$$\sqrt{\frac{c^2-2g\eta}{1+\eta_\xi^2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\xi}\cosh(k(\eta+h))\mathcal{F}\{\sqrt{c^2-2p}\}(k)\ dk, \tag{2.3}$$

  for $\eta(\xi) = \eta(x-ct)$ given $p(\xi) = p(x-ct)$. Here c is the speed of the traveling wave. Using an implicit function theorem argument, given c and p, it is shown in [22] that (2.3) has a solution, at least for waves of sufficiently small amplitude. Using numerical and experimental comparisons, excellent agreement between $\eta(\xi)$ and the true surface elevation was obtained. These comparisons were carried out well beyond the small amplitude regime, and with waves that were not necessarily traveling. The insensitivity of the results obtained with regard to the parameter c seems to be one of the advantages of this relationship. This is explored further below.

  As given above, (2.3) applies to solitary waves. An equivalent relation for periodic solutions is easily derived, see [22]. It is this periodic equivalent that is used below when comparing with numerical or experimental data.
- As demonstrated in [22], the nonlocal relationship (2.3) may be used to derive several asymptotic relationships. One such approximation is the *renormalized transfer function* (RTF) given by

$$\eta = \frac{\mathcal{F}^{-1}\{\hat{p}(k)\cosh(kh)\}/g}{1-\mathcal{F}^{-1}\{\hat{p}(k)\,k\sinh(kh)\}/g}, \tag{2.4}$$

  where $\hat{p}(k) = \mathcal{F}\{p(\xi)\}(k)$. This relation may be viewed as an improvement to the traditional TF (2.2) and is valid for small amplitude waves. As noted in [22] (or, see below), this approximation is a good compromise between accuracy and computational efficiency, even for waves of large amplitude. Unlike the fully nonlinear model (2.3), the RTF does not require knowledge of the wave speed c explicitly, although the traveling wave assumption is made in its derivation.
- Recently, Constantin [9] derived a fully nonlinear relation that allows for the recovery of a traveling solitary wave profile from measurements of the bottom pressure. In contrast to the methods mentioned above, the recovery formula holds for solitary wave profiles

only. An analogue for periodic waves is not known. It is given in parametric form as

$$\xi(q) = q + \int_{-\infty}^{q} \mathcal{F}^{-1}\left\{\cosh(kh)\mathcal{F}\left\{\frac{c}{\sqrt{c^2-2p}} - 1\right\}(k)\right\}(s)ds,$$
$$\eta(q) = \mathcal{F}^{-1}\left\{\frac{\sinh(kh)}{k}\mathcal{F}\left\{\frac{c}{\sqrt{c^2-2p}} - 1\right\}(k)\right\}(q). \tag{2.5}$$

We refer to this relation as the *explicit solitary wave reconstruction* (ESWR). Formula (2.5) is obtained without approximation from (1.1a)–(1.1d) and (1.2) with the imposition of a traveling wave assumption, posed on the infinite line. It is amazing that the relationship between pressure and surface elevation is explicit: from the knowledge of c and p, the graph of $\eta(\xi)$ is obtained immediately.

3. Numerical Comparisons of the Different Approaches

In this section, we compare the reconstructed surface using the various relationships discussed in Sec. 2 using numerical data. We investigate the accuracy of the reconstruction and the sensitive dependence, if any, on the parameter c of the different reconstructions.

3.1. *Reconstruction using numerical data*

The five reconstruction methods introduced above are compared with periodic Stokes wave data generated from the Euler equations (1.1a)–(1.1d), using the methods discussed in [15]. This comparison is an extension of that in [22], where the ESWR was not included. Without loss of generality, we assume the waves have period 2π. It is well known that once the period and the depth h are fixed, there exists a single branch of Stokes wave solutions bifurcating away from the trivial, quiescent solution (see [15] and many of the references therein). We start with a solution on this bifurcation curve, i.e. we prescribe c and a 2π-periodic Stokes wave. Using (2.3), we solve for the pressure at the bottom by equating the Fourier coefficients on both its left- and right-hand side (for $k = -N \dots N$, N a chosen cut-off value). Using the truncated algebraic system of equations, we solve the linear system for the Fourier series representation of the term $\sqrt{c^2 - 2p}$. This allows for the direct solution for $p(\xi)$ in terms of the Stokes wave data set $\mathcal{S} = (\eta_{\text{true}}(\xi), c_{\text{true}})$.

Having obtained the pressure underneath a Stokes wave, we use the various relations presented in Sec. 2 to reconstruct the surface elevation. As a validation, we expect that the reconstruction (2.3) using the nonlocal relation (NR) should be accurate to machine precision since we are using a nonlinear solver to invert the transformation that generated the pressure data.

Using the non-dimensional parameters $h = 0.1$, $g = 1$, $\rho = 1$ and $L = 2\pi$, the reconstruction using the different relations is shown in Fig. 2. The top panel shows the reconstruction of a solution of small amplitude. It appears that all reconstruction methods are in good agreement. This is somewhat surprising for the ESWR using (2.5), which is valid only for solitary wave solutions. The small-amplitude solution is not near the solitary wave regime. In order to compare (2.5) with periodic data, its range of x values is simply truncated. Increasing the amplitude ($= (\eta_{\max} - \eta_{\min})/2$) of the true solution $\eta_{\text{true}}(\xi)$ shows differences between the reconstructions, even though the amplitude is less than 35% of the limiting