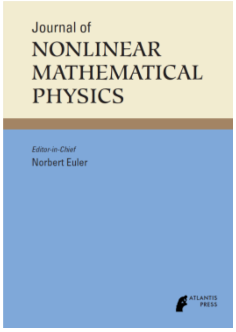

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THE TWO-COMPONENT CAMASSA–HOLM EQUATIONS CH(2,1) AND CH(2,2): FIRST-ORDER INTEGRATING FACTORS AND CONSERVATION LAWS

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Recently, Holm and Ivanov, proposed and studied a class of multi-component generalizations of the Camassa–Holm equations [D. D. Holm and R. I. Ivanov, Multi-component generalizations of the CH equation: geometrical aspects, peakons and numerical examples, *J. Phys A: Math. Theor.* **43** (2010) 492001 (20pp)]. We consider two of those systems, denoted by Holm and Ivanov by CH(2,1) and CH(2,2), and report a class of integrating factors and its corresponding conservation laws for these two systems. In particular, we obtain the complete set of first-order integrating factors for the systems in Cauchy–Kovalevskaya form and evaluate the corresponding sets of conservation laws for CH(2,1) and CH(2,2).

Keywords: Multi-component Camassa–Holm equations; conservation laws; integrating factors.

Mathematics Subject Classification 2000: 35Q35, 76B15

1. Introduction

It is well known that certain conservation laws of shallow water wave equations, such as the Camassa–Holm equation [4] and the Degasperis–Procesi equation [8], are useful to prove blow-up, cf. the papers [5, 15, 18]. Furthermore, conservation laws play a central role in the proof of the global existence (in time) for solutions evolving from certain initial data, cf. the paper [6], and for proving the stability of peakons for both model equations, cf. the papers [7, 12, 13]. In the context of the Camassa–Holm equation they are instrumental in the set-up of a theory of global weak solutions for nonlinear nonlocal conservation laws, cf. the considerations in the papers [2, 3, 10].

In this paper we derive all first-order integrating factors and its corresponding conservation laws for some recently proposed multi-component generalizations of the Camassa–Holm equation [11]. We concentrate on two explicit systems, namely CH(2,1) and CH(2,2), proposed by Holm and Ivanov in [11] (see (1.1a)–(1.1b) and (1.6a)–(1.6b)).

We recently reported in [9] the complete set of first-order integrating factors and conservation laws for a class of Camassa–Holm type equations, which includes the Camassa–Holm equation [4] and the Degasperis–Procesi equation [8]. Our approach applied in this paper is based on the direct method described by Anco and Bluman in their paper [1], which can be applied to derive conservation laws of evolution equations that are in Cauchy–Kovalevskaya form. We also refer the reader to [16, 17] for more details and alternate methods for computing conservation laws for partial differential equations and systems.

Consider the two-component Camassa–Holm equations introduced and denoted by Holm and Ivanov [11] as CH(2,1), which has the following form:

$$\sigma_1 q_t + 2qu_x + uq_x + \sigma\rho\rho_x = 0, \tag{1.1a}$$

$$\rho_t + \rho u_x + u\rho_x = 0, \tag{1.1b}$$

where

$$q = \sigma_1 u - u_{xx} + s \tag{1.2}$$

and s, σ and σ_1 are arbitrary constants. The physically interesting cases are $\sigma = \pm 1$ and $\sigma_1 = 1$ or $\sigma_1 = 0$. By defining the new dependent variables

$$u := U_1, \quad u_x := U_2, \tag{1.3a}$$

$$u_{xx} := U_3, \qquad \rho := U_4 \tag{1.3b}$$

and the change of independent variables,

$$X := t, \quad T := x, \tag{1.4}$$

we can write system (1.1a)–(1.1b) in the following Cauchy–Kovalevskaya form:

$$E_1 := U_{1,T} - U_2 = 0, \tag{1.5a}$$

$$E_2 := U_{2,T} - U_3 = 0, \tag{1.5b}$$

$$\begin{aligned} E_3 := {} & U_{3,T} - \sigma_1^2 U_1^{-1} U_{1,X} + \sigma_1 U_1^{-1} U_{3,X} - 3\sigma_1 U_2 + 2U_1^{-1} U_2 U_3 \\ & + \sigma U_1^{-2} U_4 U_{4,X} + \sigma U_1^{-2} U_2 U_4^2 - 2sU_1^{-1} U_2 = 0, \end{aligned} \tag{1.5c}$$

$$E_4 := U_{4,T} + U_1^{-1} U_{4,X} + U_1^{-1} U_2 U_4 = 0. \tag{1.5d}$$

The second 2-component Camassa–Holm equation that we study in the current paper, denoted by CH(2,2), has the form [11]

$$q_{1,t} + u_0 q_{1,x} + 2q_1 u_{0,x} + u_1 q_{2,x} + 2q_2 u_{1,x} = 0, \tag{1.6a}$$

$$q_{2,t} + u_0 q_{2,x} + 2q_2 u_{0,x} = 0, \tag{1.6b}$$

where

$$q_1 = u_1 - u_{1,xx} + s_1, \tag{1.7a}$$

$$q_2 = u_0 - u_{0,xx} + 3u_1^2 - u_{1x}^2 - 2u_1 u_{1,xx} + 4s_1 u_1 + s_2. \tag{1.7b}$$

Here s_1, s_2 are arbitrary constants. By defining the new dependent variables

$$u_0 := U_1, \quad u_{0,x} := U_2, \quad u_{0,xx} := U_3, \tag{1.8a}$$
$$u_1 := U_4, \quad u_{1,x} := U_5, \quad u_{1,xx} := U_6 \tag{1.8b}$$

and the change of independent variables (1.4), we can present (1.6a)–(1.6b) in the following Cauchy–Kovalevskaya form:

$$E_1 := U_{1,T} - U_2 = 0, \tag{1.9a}$$
$$E_2 := U_{2,T} - U_3 = 0, \tag{1.9b}$$
$$\begin{aligned} E_3 := {} & U_{3,T} + 12U_1^{-1}U_4^3U_5 - 4U_1^{-1}U_4U_{4,X} + 2U_1^{-1}U_5U_{5,X} - 4s_1U_1^{-1}U_{4,X} \\ & + 4U_5U_6 - 4s_1U_5 + 2U_1^{-1}U_2U_3 - 6U_1^{-1}U_2U_4^2 + 2U_1^{-1}U_2U_5^2 - 2s_2U_1^{-1}U_2 \\ & - 4s_1U_1^{-1}U_2U_4 - 12U_1^{-2}U_2U_4^4 + 2U_1^{-1}U_6U_{4,X} - 8U_1^{-1}U_4^2U_5U_6 \\ & + 16s_1U_1^{-1}U_4^2U_5 + 4U_1^{-2}U_4^2U_6U_{4,X} - 8s_1U_1^{-2}U_4^2U_{4,X} + 4U_1^{-2}U_4^2U_2U_3 \\ & + 4U_1^{-2}U_4^2U_2U_5^2 + 8U_1^{-2}U_2U_4^3U_6 - 16s_1U_1^{-2}U_2U_4^3 - 4s_2U_1^{-2}U_2U_4^2 \\ & + 4U_1^{-2}U_4^2U_5U_{5,X} - 4U_1^{-1}U_3U_4U_5 + 4s_2U_1^{-1}U_4U_5 - 12U_1^{-2}U_4^3U_{4,X} \\ & + 2U_1^{-2}U_4^2U_{3,X} + 4U_1^{-2}U_4^3U_{6,X} - 4U_1^{-1}U_4U_5^3 - 2U_1^{-2}U_4^2U_{1,X} \\ & - U_1^{-1}U_{1,X} + U_1^{-1}U_{3,X} - 3U_2 = 0, \end{aligned} \tag{1.9c}$$
$$E_4 := U_{4,T} - U_5 = 0, \tag{1.9d}$$
$$E_5 := U_{5,T} - U_6 = 0, \tag{1.9e}$$
$$\begin{aligned} E_6 := {} & U_{6,T} + 4U_1^{-1}U_4U_5U_6 - 8s_1U_1^{-1}U_4U_5 + 2U_1^{-1}U_5^3 - 3U_5 - U_1^{-1}U_{4,X} \\ & + U_1^{-1}U_{6,X} - 2U_1^{-2}U_4^2U_{6,X} + 6U_1^{-2}U_2U_4^3 - U_1^{-2}U_4U_{3,X} + U_1^{-2}U_4U_{1,X} \\ & + 6U_1^{-2}U_4^2U_{4,X} - 2U_1^{-2}U_4U_6U_{4,X} + 4s_1U_1^{-2}U_4U_{4,X} - 2U_1^{-2}U_2U_3U_4 \\ & - 2U_1^{-2}U_2U_4U_5^2 - 4U_1^{-2}U_2U_4^2U_6 + 8s_1U_1^{-2}U_2U_4^2 + 2s_2U_1^{-2}U_2U_4 \\ & - 2U_1^{-2}U_4U_5U_{5,X} + 2U_1^{-1}U_3U_5 - 2s_2U_1^{-1}U_5 + 2U_1^{-1}U_2U_6 \\ & - 2s_1U_1^{-1}U_2 - 6U_1^{-1}U_4^2U_5 = 0. \end{aligned} \tag{1.9f}$$

The above first-order Cauchy–Kovalevskaya systems can now be investigated for integrating factors to derive conservation laws for the systems; which then leads to conservation laws of the systems CH(1,1) and CH(2,2) in the original variables.

2. General Description

In this section we briefly describe the direct method [1] of integrating factors (or multipliers) for the general first-order Cauchy–Kovalevskaya system of six equations:

$$E_j := U_{j,T} - F_j(U_1, \ldots, U_6, U_{1,X}, \ldots, U_{6,X}) = 0, \quad j = 1, 2, \ldots, 6. \tag{2.1}$$

Every conserved density, Φ^T, and conserved flux, Φ^X, of system (2.1) must satisfy

$$D_T \Phi^T + D_X \Phi^X|_{\vec{E}=\vec{0}} = 0, \tag{2.2}$$

where, in general, both Φ^T and Φ^X are functions of X, T, U_j as well as X-derivatives of U_j. Moreover, every Φ^T requires six integrating factors, $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_6\}$, which are directly related to the conserved density by the relation [1]

$$\Lambda_k = \hat{E}[U_k]\Phi^T, \quad k = 1, 2, \ldots, 6. \tag{2.3}$$

Here $\hat{E}$ is the Euler operator,

$$\hat{E}[U_k] := \frac{\partial}{\partial U_k} - D_T \circ \frac{\partial}{\partial U_{k,T}} + \sum_{j=1}^{q} (-1)^j D_X^j \circ \frac{\partial}{\partial U_{k,jX}}, \tag{2.4}$$

where we use the notation

$$U_{k,jX} := \frac{\partial^j U_k}{\partial X^j}.$$

The conditions on the integrating factors, $\{\Lambda_j\}$, of system (2.1) are

$$\hat{E}[U_k](\Lambda_1 E_1 + \Lambda_2 E_2 + \cdots + \Lambda_6 E_6) = 0, \quad k = 1, 2, \ldots, 6. \tag{2.5}$$

However, since all integrating factors of system (2.1) are adjoint symmetries of the system (2.1), we can calculate $\{\Lambda_j\}$ by the condition

$$\left.\begin{pmatrix} L^*_{E_1}[U_1] & L^*_{E_2}[U_1] & \cdots & L^*_{E_6}[U_1] \\ L^*_{E_1}[U_2] & L^*_{E_2}[U_2] & \cdots & L^*_{E_6}[U_2] \\ \vdots & \vdots & \vdots & \vdots \\ L^*_{E_1}[U_6] & L^*_{E_2}[U_6] & \cdots & L^*_{E_6}[U_6] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_6 \end{pmatrix}\right|_{\vec{E}=\vec{0}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.6}$$

and then require the self-adjointness condition on $\{\Lambda_j\}$ (as integrating factors are variational quantities), namely

$$\begin{aligned} &\begin{pmatrix} L_{\Lambda_1}[U_1] & L_{\Lambda_1}[U_2] & \cdots & L_{\Lambda_1}[U_6] \\ L_{\Lambda_2}[U_1] & L_{\Lambda_2}[U_2] & \cdots & L_{\Lambda_2}[U_6] \\ \vdots & \vdots & \vdots & \vdots \\ L_{\Lambda_6}[U_1] & L_{\Lambda_6}[U_2] & \cdots & L_{\Lambda_6}[U_6] \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_6 \end{pmatrix} \\ &\qquad = \begin{pmatrix} L^*_{\Lambda_1}[U_1] & L^*_{\Lambda_2}[U_1] & \cdots & L^*_{\Lambda_6}[U_1] \\ L^*_{\Lambda_1}[U_2] & L^*_{\Lambda_2}[U_2] & \cdots & L^*_{\Lambda_6}[U_2] \\ \vdots & \vdots & \vdots & \vdots \\ L^*_{\Lambda_1}[U_6] & L^*_{\Lambda_2}[U_6] & \cdots & L^*_{\Lambda_6}[U_6] \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_6 \end{pmatrix}. \end{aligned} \tag{2.7}$$

Here L is the linear operator and L^* its adjoint:

$$L_P[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^{p} \frac{\partial P}{\partial U_{j,iT}} D_T^i + \sum_{k=1}^{q} \frac{\partial P}{\partial U_{j,kX}} D_X^k, \tag{2.8a}$$

$$L_P^*[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^{p} (-1)^i D_T^i \circ \frac{\partial P}{\partial U_{j,iT}} + \sum_{k=1}^{q} (-1)^k D_X^k \circ \frac{\partial P}{\partial U_{j,kX}}. \tag{2.8b}$$

Note that the self-adjointness condition (2.7), is independent of the form of the evolution system (2.1) and only depends on the functional arguments of $\{\Lambda_j\}$ as well as the number of equations in the system.

3. Integrating Factors for System (1.5a)–(1.5d) and Conservation Laws for (1.1a)–(1.1b)

Solving conditions (2.6) and (2.7) for system (1.5a)–(1.5d), the complete set of first-order integrating factors $\{\Lambda_1, \dots, \Lambda_4\}$, of the form

$$\Lambda_j = \Lambda_j(X, T, U_1, \dots, U_4, U_{1,X}, \dots, U_{4,X}), \quad j = 1, 2, \dots, 4$$

for arbitrary σ, $\sigma_1 \neq 0$ and s is as follows:

$$\begin{aligned}\Lambda_1 = {} & \lambda_1 U_4 + 2\lambda_2 \left(s + \frac{3}{2}\sigma_1 U_1 - \frac{1}{2} U_3 \right) \\ & - \lambda_3 (\sigma_1 U_{2,X} - 2sU_1 - \sigma U_4^2 - 3\sigma_1 U_1^2 + 2U_1 U_3)\sigma_1^{-1},\end{aligned} \tag{3.1a}$$

$$\Lambda_2 = -\lambda_2 U_2 + \lambda_3 U_{1,X}, \tag{3.1b}$$

$$\Lambda_3 = -\lambda_2 U_1 - \lambda_3 \sigma_1^{-1} U_1^2, \tag{3.1c}$$

$$\Lambda_4 = \lambda_1 U_1 + \lambda_2 \sigma U_4 + 2\lambda_3 \sigma \sigma_1^{-1} U_1 U_4, \tag{3.1d}$$

where λ_j are arbitrary constants. This leads to the following three sets of conserved density, Φ^t, and conserved flux, Φ^x, for the original system (1.1a)–(1.1b) (separated by means of the arbitrary λ_1, λ_2 and λ_3, respectively):

$$\Phi_1^t = \rho, \tag{3.2a}$$

$$\Phi_1^x = u\rho, \tag{3.2b}$$

$$\Phi_2^t = \sigma_1^2 u - \sigma_1 u_{xx}, \tag{3.3a}$$

$$\Phi_2^x = 2su + \frac{1}{2}\sigma\rho^2 + \frac{3}{2}\sigma_1 u^2 - u u_{xx} - \frac{1}{2}u_x^2, \tag{3.3b}$$

$$\Phi_3^t = \left(-\sigma_1 u_x^2 + \frac{1}{2}\sigma\rho^2 + \frac{1}{2}\sigma_1^2 u^2 - \sigma_1 u u_{xx} + \frac{1}{2}\sigma_1 u_x^2 \right) \sigma_1^{-1}, \tag{3.4a}$$

$$\Phi_3^x = (\sigma_1 u_x u_t + s u_x^2 + \sigma u \rho^2 + \sigma_1 u^3 - u^2 u_{xx})\sigma_1^{-1}. \tag{3.4b}$$

Some special must be considered:

Special Case 1: $\sigma = 0$ with σ_1 arbitrary, but nonzero, and s arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{2U_3 - 3\sigma_1 U_1 - 2s}{(\sigma_1 U_1 - U_3 + s)^{1/2}}, \quad \Lambda_2 = 0, \tag{3.5}$$

$$\Lambda_3 = \frac{U_1}{(\sigma_1 U_1 - U_3 + s)^{1/2}}, \quad \Lambda_4 = 0 \tag{3.6}$$

and the corresponding conserved current for system (1.5a)–(1.5d) is

$$\Phi^t = \sigma_1(\sigma_1 u - u_{xx} + s)^{1/2}, \tag{3.7a}$$

$$\Phi^x = (\sigma_1 u - u_{xx} + s)^{1/2}. \tag{3.7b}$$

Special Case 2: $\sigma = 0$ with $\sigma_1 = 1$ and s arbitrary. The integrating factors are as follows:

$$\Lambda_1 = -\frac{U_1 W^3 H'(W)}{U_4} + 2U_4 H(W), \quad \Lambda_2 = 0, \tag{3.8a}$$

$$\Lambda_3 = \frac{U_1 W^3 H'(W)}{U_4}, \quad \Lambda_4 = 2U_1(WH'(W) + H(W)), \tag{3.8b}$$

where $H(W)$ is an arbitrary differentiable function with

$$W := \frac{U_4}{(U_1 - U_3 + s)^{1/2}}. \tag{3.9}$$

The conserved current for system (1.5a)–(1.5d) is then

$$\Phi^t = H(w)\rho, \tag{3.10a}$$

$$\Phi^x = H(w)u\rho. \tag{3.10b}$$

Here the argument, w, in the arbitrary function H, is

$$w := \frac{\rho}{(u - u_{xx} + s)^{1/2}}. \tag{3.11}$$

Special Case 3: σ arbitrary, but nonzero, with $\sigma_1 = 1$ and s arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{U_3 - 2U_1 - s}{\sigma U_4}, \quad \Lambda_2 = 0, \tag{3.12a}$$

$$\Lambda_3 = \frac{U_1}{\sigma U_4}, \quad \Lambda_4 = \frac{sU_1 - \sigma U_4^2 + U_1^2 - U_1 U_3}{\sigma U_4^2}. \tag{3.12b}$$

The conserved current for system (1.5a)–(1.5d) is then

$$\Phi^t = \frac{u_{xx} - u - s}{\sigma\rho}, \tag{3.13a}$$

$$\Phi^x = \frac{u u_{xx} - u^2 - \sigma\rho^2 - su}{\sigma\rho}. \tag{3.13b}$$

Special Case 4: $\sigma_1 = 0$ with σ and s arbitrary. The integrating factors are as follows:

$$\Lambda_1 = (U_3 - 2s)H(X, W), \quad \Lambda_2 = U_2 H(X, W), \tag{3.14a}$$

$$\Lambda_3 = U_1 H(X, W), \quad \Lambda_4 = -\sigma U_4 H(X, W), \tag{3.14b}$$

where H is an arbitrary differentiable function and

$$W := \frac{1}{2}(-4sU_1 - \sigma U_4^2 + 2U_1U_3 + U_2^2). \tag{3.15}$$

The conserved current for system (1.5a)–(1.5d) is then

$$\Phi^t = 0, \tag{3.16a}$$

$$\Phi^x = wH(t, w), \tag{3.16b}$$

where

$$w := \frac{1}{2}(-4su - \sigma\rho^2 + 2uu_{xx} + u_x^2). \tag{3.17}$$

Special Case 5: $\sigma_1 = 0$ with σ arbitrary, but nonzero, and s arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{2}{\sigma}\left(sU_1 + \frac{\sigma}{2}U_4^2 - U_1U_3\right), \quad \Lambda_2 = 0, \tag{3.18a}$$

$$\Lambda_3 = -\frac{U_1^2}{\sigma}, \quad \Lambda_4 = 2U_1U_4. \tag{3.18b}$$

The conserved current for system (1.5a)–(1.5d) is then

$$\Phi^t = \frac{\rho^2}{2}, \tag{3.19a}$$

$$\Phi^x = \frac{1}{\sigma}(su^2 + \sigma u\rho^2 - u^2u_{xx}). \tag{3.19b}$$

4. Integrating Factors for System (1.9a)–(1.9f) and Conservation Laws for (1.6a)–(1.6b)

Solving conditions (2.6) and (2.7) for system (1.9a)–(1.9f), the complete set of first-order integrating factors $\{\Lambda_1, \ldots, \Lambda_6\}$, of the form

$$\Lambda_j = \Lambda_j(X, T, U_1, \ldots, U_6, U_{1,X}, \ldots, U_{6,X}), \quad j = 1, 2, \ldots, 6$$

are the following:

$$\begin{aligned}\Lambda_1 = {} & \lambda_1(2U_6U_1 + 2U_3U_4 + 2U_4U_5^2 + 4U_4^2U_6 - 6U_4^3 - 2s_1U_1 - 8s_1U_4^2 - 6U_1U_4 \\ & - 2s_2U_4 + U_{5,X}) + \lambda_2(U_3 + 2U_5^2 - 4s_1U_4 - 3U_1 - 2s_2) \\ & + \lambda_3(U_6 - 3U_4 - 2s_1) + \lambda_4(-8s_1^2U_4 - 2s_1s_2 - s_1U_1 + 2s_1U_3 - 22s_1U_4^2 \\ & + 12s_1U_4U_6 + 2s_1U_5^2 - 4s_2U_4 + 2s_2U_6 - 3U_1U_4 + U_1U_6 + 4U_3U_4 - 2U_3U_6 - 12U_4^3 \\ & + 14U_4^2U_6 + 4U_4U_5^2 - 4U_4U_6^2 - 2U_5U_6)Z^{-3/2} \\ & + \frac{\lambda_5}{2}(-8s_1U_4 - 2s_2 - 3U_1 + 2U_3 - 6U_4^2 + 4U_4U_6 + 2U_5^2)Z^{-1/2},\end{aligned} \tag{4.1a}$$

$$\Lambda_2 = -\lambda_1 U_{4,X} + \lambda_2 U_2 + \lambda_3 U_5, \tag{4.1b}$$

$$\begin{aligned}\Lambda_3 &= 2\lambda_1 U_1 U_4 + \lambda_2(U_1 - 2U_4^2) + \lambda_3 U_4 \\ &\quad + \lambda_4(-s_1 U_1 + 8s_1 U_4^2 + 2s_2 U_4 + U_1 U_4 + U_1 U_6 - 2U_3 U_4 + 6U_4^3 \\ &\quad - 4U_4^2 U_6 - 2U_4 U_5^2) Z^{-3/2} + \frac{\lambda_5}{2} U_1 Z^{-1/2},\end{aligned} \tag{4.1c}$$

$$\begin{aligned}\Lambda_4 &= \lambda_1(2U_1 U_5^2 + 2U_1 U_3 + 2U_4 U_{5,X} - 2s_2 U_1 - 3U_1^2 - 18U_1 U_4^2 + U_{2,X} \\ &\quad - 16s_1 U_1 U_4 + 8U_1 U_4 U_6) + \lambda_2(24U_4^3 - 4U_3 U_4 - 4U_4 U_5^2 - 12U_4^2 U_6 - 2U_{5,X} \\ &\quad + 24s_1 U_4^2 - 4s_1 U_1 + 4s_2 U_4) + \lambda_3(U_3 + 4U_4 U_6 - 3U_1 + 2U_5^2 - 12U_4^2 \\ &\quad - 12s_1 U_4 - 2s_2) + 2\lambda_4(2s_1^2 U_1 - 48s_1^2 U_4^2 - 20s_1 s_2 U_4 - 19s_1 U_1 U_4 - 3s_1 U_1 U_6 \\ &\quad + 20s_1 U_3 U_4 - 84s_1 U_4^3 + 48s_1 U_4^2 U_6 + 20s_1 U_4 U_5^2 - 2s_2^2 - 5s_2 U_1 + 4s_2 U_3 - 18s_2 U_4^2 \\ &\quad + 10s_2 U_4 U_6 + 4s_2 U_5^2 - 3U_1^2 + 5U_1 U_3 - 18U_1 U_4^2 + 8U_1 U_4 U_6 + 5U_1 U_5^2 + U_1 U_6^2 - 2U_3^2 \\ &\quad + 18U_3 U_4^2 - 10U_3 U_4 U_6 - 4U_3 U_5^2 - 36U_4^4 + 42U_4^3 U_6 + 18U_4^2 U_5^2 - 12U_4^2 U_6^2 \\ &\quad - 10U_4 U_5^2 U_6 - 2U_5^4) Z^{-3/2} + \lambda_5 U_1(-2s_1 + 3U_4 + U_6) Z^{-1/2},\end{aligned} \tag{4.1d}$$

$$\begin{aligned}\Lambda_5 &= \lambda_1(4U_1 U_4 U_5 - U_{1,X} - 2U_4 U_{4,X}) + \lambda_2(4U_1 U_5 - 4U_4^2 U_5 + 2U_{4,X}) \\ &\quad + \lambda_3(U_2 + 4U_4 U_5) - 2\lambda_4 U_5(s_1 U_1 - 8s_1 U_4^2 - 2s_2 U_4 - U_1 U_4 - U_1 U_6 + 2U_3 U_4 \\ &\quad - 6U_4^3 + 4U_4^2 U_6 + 2U_4 U_5^2) Z^{-3/2} + \lambda_5 U_1 U_5 Z^{-1/2},\end{aligned} \tag{4.1e}$$

$$\begin{aligned}\Lambda_6 &= \lambda_1(U_1^2 + 4U_1 U_4^2) - 4\lambda_2 U_4^3 + \lambda_3(U_1 + 2U_4^2) + 2\lambda_4(3s_1 U_1 U_4 + 8s_1 U_4^3 + s_2 U_1 \\ &\quad + 2s_2 U_4^2 - U_1^2 - U_1 U_3 + 4U_1 U_4^2 - U_1 U_4 U_6 - U_1 U_5^2 - 2U_3 U_4^2 + 6U_4^4 \\ &\quad - 4U_4^3 U_6 - 2U_4^2 U_5^2) Z^{-3/2} + \lambda_5 U_1 U_4 Z^{-1/2},\end{aligned} \tag{4.1f}$$

where

$$Z := s_1 U_4 - s_2 - U_1 + U_3 - 3U_4^2 + 2U_4 U_6 + U_5^2. \tag{4.2}$$

This leads to the following set of three conserved densities and conserved flux for the system (1.6a)–(1.6b):

$$\Phi_1^t = u_1 u_{0,xx} + u_1^2 u_{1,xx} - u_0 u_1 - 2s_1 u_1^2 - 2u_1^3, \tag{4.3a}$$

$$\begin{aligned}\Phi_1^x &= (u_0 + u_1^2) u_{1,xt} + 2u_0 u_1 u_{0,xx} + 2u_0 u_1 u_{1,x}^2 + (4u_0 u_1^2 + u_0^2) u_{1,xx} \\ &\quad - \frac{1}{2} u_0^2 (6u_1 + 2s_1) - u_0(6u_1^3 + 2s_2 u_1 + 8s_1 u_1^2) - u_{0,x} u_{1,t},\end{aligned} \tag{4.3b}$$

$$\Phi_2^t = 2u_1 u_{1,xx} + u_{0,xx} - u_0 - 2u_1^2 + 2u_{1,x}^2 - 4s_1 u_1, \tag{4.4a}$$

$$\begin{aligned}\Phi_2^x &= -2u_1 u_{1,xt} + (u_0 - 2u_1^2) u_{0,xx} - 4u_1^3 u_{1,xx} + \frac{1}{2} u_{0,x}^2 + 2(u_0 - u_1^2) u_{1,x}^2 \\ &\quad - 2(s_2 + 2s_1 u_1) u_0 - \frac{3}{2} u_0^2 + 2u_1^2 (s_2 + 4s_1 u_1 + 3u_1^2),\end{aligned} \tag{4.4b}$$

$$\Phi_3^t = u_{1,xx} - u_1, \tag{4.5a}$$

$$\Phi_3^x = (u_0 + 2u_1^2)u_{1,xx} + u_1 u_{0,xx} + u_{0,x}u_{1,x} + 2u_1 u_{1,x}^2 - (2s_1 + 3u_1)u_0 \\ - 2u_1(s_2 + 3s_1 u_1 + 2u_1^2), \tag{4.5b}$$

$$\Phi_4^t = 2(s_1 - u_{1,xx} + u_1)z^{-1/2}, \tag{4.6a}$$

$$\Phi_4^x = 2(s_1 u_0 + 8s_1 u_1^2 + 2s_2 u_1 + 3u_0 u_1 - u_0 u_{1,xx} - 2u_{0,xx}u_1 + 6u_1^3 \\ - 4u_1^2 u_{1,xx} - 2u_1 u_{1,x}^2)z^{-1/2}, \tag{4.6b}$$

$$\Phi_5^t = z^{1/2}, \tag{4.7a}$$

$$\Phi_5^x = u_0 z^{1/2}, \tag{4.7b}$$

where

$$z := s_1 u_1 - s_2 - u_0 + u_{0,xx} - 3u_1^2 + 2u_1 u_{1,xx} + u_{1,x}^2. \tag{4.8}$$

5. Concluding Remarks

We have derived the complete set of first-order integrating factors for the systems CH(2,1) and CH(2,2) in Cauchy–Kovalevskaya form. The corresponding sets of conservation laws related to these integrating factors have been derived for both these systems. It would certainly be interesting to calculate higher-order integrating factors, although the computations involved for such calculations appear to be rather challenging. We aim to report some results in a future paper.

We expect that the same method that was applied here could also be used to find conservation laws for more general CH-systems proposed in [11, 14].

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