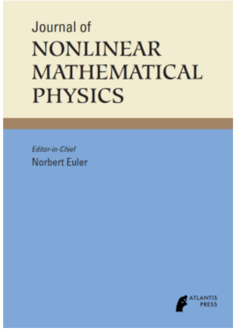

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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Article

Journal of Nonlinear Mathematical Physics, Vol. 19, Suppl. 1 (2012) 1240013 (18 pages)

DOI: 10.1142/S140292511240013X

GEOMETRICAL METHODS FOR EQUATIONS OF HYDRODYNAMICAL TYPE

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Received 23 May 2012
Accepted 22 June 2012
Published 28 November 2012

We describe some recent results for a class of nonlinear hydrodynamical approximation models where the geometric approach gives insight into a variety of aspects. The main contribution concerns analytical results for Euler equations on the diffeomorphism group of the circle for which the inertia operator is a nonlocal operator.

Keywords: Euler equation; diffeomorphism group; fractional Sobolev metrics.

Mathematics Subject Classification 2010: 58D05, 35Q53

1. Introduction

In a seminal paper [1], Arnold pointed out that the Euler equations of hydrodynamics can be viewed geometrically as the geodesic equations on the diffeomorphism group endowed with an invariant metric. This work can be viewed as a generalization of the theory developed by Euler [21] around 1765 which involves the rotation group in $\mathbb{R}^3$ in the description of the motion of a rigid body.

More recently, many approximations models to the governing equations of the classical water-wave problem have been found to arise in a similar way; examples include the equations of Korteweg–de Vries (KdV), Camassa–Holm (CH), Degasperis–Procesi (DP), Hunter–Saxton (HS), and of Constantin–Lax–Majda (CLM), to mention the most prominent ones (see [13, 29, 31] for the hydrodynamical relevance of these equations). This geometric viewpoint is not only aesthetically appealing, but is also useful in the study of well-posedness and stability issues, as well as in gaining insight into blow-up phenomena [6].

There is an extensive activity in deriving approximations models for a variety of specific physical regimes that cannot be summarized here. Nevertheless, it appears that among all these models, some have a particular intricate geometric structure. The KdV equation is a re-expression of the geodesic flow on the Bott–Viraoro group equipped with the L^2-right invariant metric [34]. It has an integrable structure as an infinite bi-Hamiltonian system [37] and its solitary wave solutions are solitons [38]. The CH equation [3] can be recast as the geodesic flow on the diffeomorphism group of the circle $\mathrm{Diff}^\infty(\mathbb{S}^1)$ equipped with the H^1-right-invariant metric [41]. Recently, it has been shown [17] that the DP equation [15] can also be recast as a geodesic equation on $\mathrm{Diff}^\infty(\mathbb{S}^1)$, although in this case the linear connection does not derive from an invariant metric [35]. Both the CH and DP equations are integrable (they have a bi-Hamiltonian structure) and admit peakon solutions [5, 7, 9, 14]. Finally the HS and CLM equations admit geometric interpretations on homogeneous spaces.

The theory of Euler equations on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is the study of *right-invariant Riemannian metrics* on $\mathrm{Diff}^\infty(\mathbb{S}^1)$. Such a metric is defined by an inner product on $\mathrm{Vect}(\mathbb{S}^1) \simeq \mathrm{C}^\infty(\mathbb{S}^1)$, which usually can be written as

$$\langle u, v\rangle = \int_{\mathbb{S}^1} (Au)v dx,$$

where A is a symmetric, positive, linear operator, which is called for historical reasons the *inertia operator* of the corresponding system. This problem has been extensively studied by many authors (see [11, 12, 17, 32, 44] for instance) when the inertia operator A is a *differential operator*. In [19], the CLM equation was considered. It corresponds to the homogeneous $\dot{H}^{1/2}$-right-invariant metric on the homogeneous space $\mathrm{Diff}^\infty(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$. Its inertia operator, which is not *local*, is of the form $\mathcal{H}D$, where $\mathcal{H}$ denotes the Hilbert transform and $D := d/dx$ the spatial derivative. The study of this example lead to develop a more general theory [18] when A is a *Fourier multiplier*, including the important case of H^s metrics with $s \geq 1/2$.

The goal of this short paper is to present an overview of this theory and to discuss, after having recalled the geometrical framework, important examples arising as approximation models in the mathematical description of water-waves.

The paper is organized as follows: In Sec. 2, we review the geometric framework in which Euler equations on a Lie group can be studied. Section 3 offers a large selection of examples with hydrodynamical background arising as Euler equations on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ or related groups. In Sec. 4, we reduce the local existence problem for geodesics to the smoothness of the conjugates of the inertia operator on some extended Banach approximation manifold of $\mathrm{Diff}^\infty(\mathbb{S}^1)$. Section 5 is devoted to the study of inertia operators for which this smoothness condition is fulfilled. In Sec. 6 finally, further geometric considerations about the minimization problem for geodesics are discussed.

2. Geometric Framework

It is known since Euler [22] in 1765, that the free motions of a rigid body correspond to the geodesics of a left-invariant metric on the rotation group (which represents the kinetic energy of the rigid body). As noticed by Poincaré [43] in 1901, this approach can be generalized to any mechanical system, provided there is a *Lie group* which acts *transitively* on the *configuration space*. This theory has been extended by Arnold [1] in 1966 to continuum mechanics. He has recast the evolution equations of an ideal fluid (*with fixed boundary*) as the geodesic equations for a *right-invariant metric* on the diffeomorphism group.

2.1. *Semi-invariant metrics on a Lie group*

A right-invariant (or left-invariant) Riemannian metric on a Lie group G is defined by an inner product on its Lie algebra $\mathfrak{g}$, or equivalently, by a symmetric invertible operator $A : \mathfrak{g} \to \mathfrak{g}^*$. In this context the operator A is called an *inertia operator*.

The corresponding *Riemannian connection* is also right-invariant and is given by

$$\nabla_{\xi_u}\xi_v = \frac{1}{2}[\xi_u, \xi_v] + B(\xi_u, \xi_v),$$

where ξ_u is the right-invariant vector field generated by $u \in \mathfrak{g}$ and B is the right-invariant tensor field generated by the bilinear operator

$$B(u, v) = \frac{1}{2}\left[(\mathrm{ad}_u)^\top(v) + (\mathrm{ad}_v)^\top(u)\right],$$

where $u, v \in \mathfrak{g}$. In this formula, $\mathrm{ad}_u^\top$ is the *adjoint* (relatively to the inner product given by A) of the natural action of the Lie algebra on itself given by

$$\mathrm{ad}_u : w \mapsto [u, w].$$

Remark 2.1. Notice that B vanishes if the metric is bi-invariant, because then

$$\mathrm{ad}_u^\top = -\,\mathrm{ad}_u$$

for all $u \in \mathfrak{g}$.

2.2. *The Euler equation*

Given a smooth path $g(t)$ in G, we define its *Eulerian velocity*, which lies in the Lie algebra $\mathfrak{g}$, by

$$u(t) = R_{g^{-1}(t)}\dot{g}(t),$$

where R_g stands for the right translation[a] in G.

Proposition 2.1. *A path $g(t)$ is a* geodesic *if and only if its Eulerian velocity u satisfies the first-order equation*

$$u_t = -B(u, u). \tag{2.1}$$

This equation for the velocities is known as the Euler equation.

Remark 2.2. This formalism can be extended to any *semi-invariant symmetric linear connection* on a Lie group G. The fact that this connection is derived from a semi-invariant Riemannian metric is not essential, as explored in [17].

[a]We use the same notation R_g for the diffeomorphism of G, as well as for its tangent map.

Example 2.1 (Motion of a free rigid body around a fixed point). To give a paradigmatic example, let G be the rotation group SO(3) which acts transitively of the configuration space of a rigid body. The metric corresponds to the kinetic energy of the body, which is left-invariant. It is represented in the Lie algebra $\mathfrak{so}(3) \simeq (\mathbb{R}^3, \wedge)$ by the inertia tensor A of the rigid body

$$A_{xx} = \int_\Sigma (y^2 + z^2) d\mu, \quad A_{xy} = -\int_\Sigma xy d\mu, \dots$$

which is a positive, symmetric 3×3 matrix. The Euler equation is given by

$$\omega_t = A^{-1}(A\omega \wedge \omega),$$

where ω is the angular velocity (relative to the body).

Example 2.2 (Motion of an ideal fluid). In [1], Arnold observed that the equations which described the motion of an ideal fluid (with fixed boundary) could also be interpreted as the geodesics of a right-invariant metric but on an *infinite-dimensional Lie group*. In this case, G is the *infinite-dimensional Lie group* $\mathrm{SDiff}(D)$ of volume-preserving diffeomorphisms of the fluid domain D. The Lie algebra $\mathrm{SVect}(D)$ of $\mathrm{SDiff}(D)$ is the vector space of divergence free vector fields on D which are tangent to the boundary. The metric corresponds to the kinetic energy of the fluid which is right-invariant. It is represented by the L^2 inner product on $\mathrm{SVect}(D)$

$$\langle u, v \rangle := \int_D u(x) \cdot v(x) dx.$$

The corresponding Euler equation

$$u_t = u \wedge \operatorname{rot} u - \operatorname{grad} h, \quad \operatorname{div} u = 0$$

described the motion of perfect fluid ($\rho = 1$) where $u = \varphi_t \circ \varphi^{-1}$ is the Eulerian velocity of the fluid and the *enthalpy* h is related to the pressure by $h = \frac{1}{2}\|u\|^2 + p$.

2.3. *The Euler–Poincaré equation*

A Riemannian metric on a manifold M permits to identify its tangent bundle TM with its cotangent bundle T^*M and the *canonical symplectic structure* on T^*M can be pulled back on TM. The metric defines a function $\bar{H}$ on TM and the geodesics corresponds to the integral curves of the Hamiltonian vector field $X_{\bar{H}}$ on TM.

When the manifold M is a Lie group G, the canonical symplectic structure on T^*G is invariant by right and left translations R_g and L_g. It induces on $\mathfrak{g}^* \simeq T^*G/G$ a *Poisson* structure, called the *Lie–Poisson bracket*

$$\{H, K\}_{\mathrm{LP}}(m) = (m, [d_m H, d_m K]), \quad H, K \in C^\infty(\mathfrak{g}^*).$$

Notice that $d_m H, d_m K$ are elements of $\mathfrak{g}$.

A right- (or left-) invariant Hamiltonian function $\bar{H}$ on T^*G induces a reduced function H on $\mathfrak{g}^*$ and a Hamiltonian system

$$m_t = ad^*_{d_m \bar{H}} m, \quad m \in \mathfrak{g}^* \tag{2.2}$$

called the *Euler–Poincaré* equation. In (2.2),

$$\mathrm{ad}_u^* m(v) := -m([u,v]), \quad u,v \in \mathfrak{g},\ m \in \mathfrak{g}^*$$

is the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.

When the Hamiltonian corresponds to a right-invariant metric on G, defined by the inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$, we have $H(m) = (m, A^{-1}m)$ and we get

$$m_t = ad^*_{A^{-1}m} m, \quad m \in \mathfrak{g}^*.$$

Writing $m = Au$, the Euler–Poincaré equation (2.2) corresponds to the *covariant counterpart* of the Euler equation (2.1).

2.4. *A conservation law*

The fact that the group G is a symmetry group for the Hamiltonian system leads to a conservation law (Noether's theorem). For a right-invariant Hamiltonian, we get

$$\frac{d}{dt}(\mathrm{Ad}^*_{g^{-1}} m) = 0, \tag{2.3}$$

where $m = Au$ and

$$\mathrm{Ad}_g^* m(v) := m(L_{g-1} R_g.v), \quad g \in G,\ v \in \mathfrak{g},\ m \in \mathfrak{g}^*$$

is the coadjoint action of G on its dual Lie algebra $\mathfrak{g}^*$.

Example 2.3. In the case of the rigid body, this conservation law corresponds to *conservation of the angular momentum*. In the case of the ideal fluid, it corresponds to the *Helmholtz conservation law* (the vorticity is transported along the flow).

Remark 2.3. The conservation law (2.3) implies that every solution $m(t)$ of the Euler–Poincaré equation stays on the coadjoint-orbit of the initial data $m(0)$ (*iso-vorticity*).

3. Euler Equations with Hydrodynamical Background

After Arnolds's founding paper, a large number of evolution equations arising in mathematical physics have been shown to be derived the same way [33, 34, 41, 42]. Among them, several approximations models in hydrodynamics. In this section, we will present some examples which correspond to Euler equations on the diffeomorphism group of the circle $\mathrm{Diff}^\infty(\mathbb{S}^1)$ or related groups.

3.1. *Right-invariant metrics on* $\mathrm{Diff}^\infty(\mathbb{S}^1)$

Let $\mathrm{Diff}^\infty(\mathbb{S}^1)$ denotes the group of all smooth and orientation preserving diffeomorphisms of the circle. This group is naturally equipped with a *Fréchet manifold* structure. More precisely, we can cover $\mathrm{Diff}^\infty(\mathbb{S}^1)$ with charts taking values in the *Fréchet vector space* $\mathrm{C}^\infty(\mathbb{S}^1)$ and in such a way that the change of charts are smooth maps (see [12] or [16] for more details). Since the composition and the inverse are smooth maps for this structure we say that $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is a *Fréchet–Lie group*, cf. [26]. Its Lie algebra $\mathrm{Vect}(\mathbb{S}^1)$ is isomorphic to $\mathrm{C}^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[u, v] = u_x v - u v_x.$$

A *right-invariant* metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is defined by an inner product on the Lie algebra $\mathrm{Vect}(\mathbb{S}^1) = \mathrm{C}^\infty(\mathbb{S}^1)$. In this paper, we shall consider an inner product which is given by

$$\langle u, v\rangle = \int_{\mathbb{S}^1} (Au)v dx,$$

where $A : \mathrm{C}^\infty(\mathbb{S}^1) \to \mathrm{C}^\infty(\mathbb{S}^1)$ is a linear operator on $\mathrm{C}^\infty(\mathbb{S}^1)$, which commutes with $D := d/dx$, i.e. A is a *Fourier multiplier*, given by

$$(Au)(x) = \sum_{n\in\mathbb{Z}} a(n)\hat{u}_n \exp(2i\pi nx),$$

where $\hat{u}_n$ is the nth Fourier coefficients of u, cf. [18]. The sequence $a : \mathbb{Z} \to \mathbb{C}$ is the *symbol* of A and we shall use the notation $A = \mathbf{op}(a)$. In the sequel we shall specify conditions on a, which guarantee that A is continuous, invertible, and symmetric.

By translating the above inner product, we obtain an inner product on each tangent space $T_\varphi \mathrm{Diff}^\infty(\mathbb{S}^1)$

$$\langle \eta, \xi\rangle_\varphi = \langle \eta\circ\varphi^{-1}, \xi\circ\varphi^{-1}\rangle_{\mathrm{id}} = \int_{\mathbb{S}^1} \eta(A_\varphi \xi)\varphi_x dx,$$

where $\eta, \xi \in T_\varphi \mathrm{Diff}^\infty(\mathbb{S}^1)$ and $A_\varphi = R_\varphi \circ A \circ R_{\varphi^{-1}}$. This smooth family of pre-Hilbertian structures, indexed by $\varphi \in \mathrm{Diff}^\infty(\mathbb{S}^1)$, defines a *weak*[b] *Riemannian* metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$.

An important special case corresponds to the H^k inner product ($k \in \mathbb{N}$) on $\mathrm{C}^\infty(\mathbb{S}^1)$

$$\langle u, v\rangle_{H^k} := \int_{\mathbb{S}^1} (uv + u_x v_x + \cdots + u_x^{(k)} v_x^{(k)}) dx$$

for which the inertia operator is given by

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}}.$$

More generally, the H^s inner product on $\mathrm{C}^\infty(\mathbb{S}^1)$ ($s \in \mathbb{R}^+$) is given by

$$\langle u, v\rangle_{H^s} := \frac{1}{2}\sum_{n\in\mathbb{Z}} (1+n^2)^s (\hat{u}_n \overline{\hat{v}_n} + \overline{\hat{u}_n}\hat{v}_n).$$

The corresponding inertia operator is the Fourier multiplier

$$A_s := \mathbf{op}((1+n^2)^s).$$

[b] The metric is called *weak*, because the corresponding topology induced on each tangent space of the Fréchet manifold $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is weaker than the usual Fréchet topology.

3.2. *Euler equations on* $\mathbf{Diff}^{\infty}(\mathbb{S}^1)$

In general, the existence of a symmetric, linear connection, compatible with a *weak Riemannian metric* is far from being granted. However, in the situation we consider, the adjoint operator $\mathrm{ad}_u^{\top}$ is well-defined and given by

$$\mathrm{ad}_u^{\top} v = A^{-1}(2(Av)u_x + (Av)_x u)$$

for $u, v \in \mathrm{C}^{\infty}(\mathbb{S}^1)$. Hence, one can define

$$B(u,v) = \frac{1}{2}A^{-1}[2(Av)u_x + (Av)_x u + 2(Au)v_x + (Au)_x v]$$

and check that the expression

$$\frac{D\xi(t)}{Dt} = \left(\varphi, w_t + \frac{1}{2}[u,w] + B(u,w)\right),$$

where $\xi(t) = (\varphi(t), w(t))$ is a vector field defined along the curve $\varphi(t)$ in $\mathrm{Diff}^{\infty}(\mathbb{S}^1)$ and $u(t) := \varphi_t \circ \varphi^{-1}$, defines a right-invariant, symmetric linear connection on $\mathrm{Diff}^{\infty}(\mathbb{S}^1)$ which is compatible with the (weak) metric induced by A. The corresponding Euler equation on $\mathrm{Diff}^{\infty}(\mathbb{S}^1)$ is given by

$$u_t = -A^{-1}\{(Au)_x u + 2(Au)u_x\}.$$

Example 3.1. For $A = I$ (L^2 metric), we get the *inviscid Burgers equation*:

$$u_t + 3uu_x = 0. \tag{3.1}$$

Example 3.2. For $A = I - D^2$ (H^1 metric), we get the *dispersionless CH equation*:

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{3.2}$$

3.3. *Euler equations on* $\mathbf{Diff}^{\infty}(\mathbb{S}^1)/\mathbf{Rot}(\mathbb{S}^1)$

When the "inertia operator" A is not invertible, the theory may still be meaningful, provided we reduce to a homogeneous space G/K in place of a Lie group G. In fact, the original paper of Poincaré [43] deals with homogeneous spaces rather than Lie groups, and the *Euler-Poincaré equation* is well-defined in that case [33]. Unfortunately, there is no simple *contravariant* formulation of this equation such as the *Euler equation*, in general. Indeed, in this case, the Eulerian velocity is only defined *up to a path* in K (see [45]). Nevertheless, in some cases, the theory can be simplified by restricting to a subgroup.

This is the case, for instance, for a Fourier multiplier A on $\mathrm{C}^{\infty}(\mathbb{S}^1)$ such that $\ker A = \mathbb{R}$. Then, A defines an isomorphism

$$A : \{u \in \mathrm{C}^{\infty}(\mathbb{S}^1); u(x_0) = 0\} \to \{m \in \mathrm{C}^{\infty}(\mathbb{S}^1); \hat{m}_0 = 0\},$$

where x_0 is some arbitrary point on $\mathbb{S}^1$. Therefore, we are lead to consider A as the (non-degenerate) inertia operator for a right-invariant metric on the subgroup $\mathrm{Diff}^{\infty}_{x_0}(\mathbb{S}^1)$ of diffeomorphisms which fix x_0, whose Lie algebra is

$$\{u \in \mathrm{C}^{\infty}(\mathbb{S}^1); u(x_0) = 0\},$$

and its (regular) dual Lie algebra may be identified with

$$\{m \in \mathrm{C}^\infty(\mathbb{S}^1); \hat{m}_0 = 0\}.$$

Example 3.3. For $A = D^2$ (which corresponds to the homogeneous $\dot{H}^1$ metric), we get the *HS equation*:

$$u_{xxt} + uu_{xxx} + 2u_x u_{xx} = 0. \tag{3.3}$$

Example 3.4. For $A = \mathcal{H}D$ (which corresponds the homogeneous $\dot{H}^{1/2}$ metric), where $\mathcal{H}$ is the *Hilbert transform*, we get the modified *CLM equation*:

$$\mathcal{H}u_{xt} + u\mathcal{H}u_{xx} + 2u_x\mathcal{H}u_x = 0. \tag{3.4}$$

3.4. *Euler equation on the Bott–Virasoro group*

The Virasoro group **Vir** is a central extension of $\mathrm{Diff}^\infty(\mathbb{S}^1)$ by $\mathbb{R}$. The group multiplication is given by the formula

$$(\varphi, \alpha) \circ (\psi, \beta) = \left(\varphi \circ \psi, \alpha + \beta - \frac{1}{2}\int_{\mathbb{S}^1} \log(\varphi(\psi(x)))_x d\log \psi_x\right),$$

where $(\varphi, \alpha), (\psi, \beta) \in \mathbf{Vir}$, see [25]. It is a Fréchet Lie group whose Lie algebra $\mathfrak{vir}$ can be identified with the space $\mathrm{C}^\infty(\mathbb{S}^1) \times \mathbb{R}$ with Lie bracket

$$[(u, a), (v, b)] = \left(u_x v - uv_x, \int_{\mathbb{S}^1} uv_{xxx} dx\right),$$

where $(u, a), (u, b) \in \mathfrak{vir} \simeq \mathrm{C}^\infty(\mathbb{S}^1) \times \mathbb{R}$.

We consider an inner product on $\mathfrak{vir}$ given by

$$\langle (u, a), (v, b)\rangle = \int_{\mathbb{S}^1} A(u)v dx + ab,$$

where A is a Fourier multiplier on $\mathrm{C}^\infty(\mathbb{S}^1)$. The corresponding Euler equation on $\mathfrak{vir}$ is given by

$$u_t = -A^{-1}[u(Au)_x + 2(Au)u_x - au_{xxx}], \quad a_t = 0.$$

Example 3.5. For $A = I$ (L^2 metric on $\mathfrak{vir}$), we get the *KdV equation*:

$$u_t + 3uu_x - au_{xxx} = 0, \quad a \in \mathbb{R}. \tag{3.5}$$

Example 3.6. For $A = I - D^2$ (H^1 metric on $\mathfrak{vir}$), we get the *general CH equation*:

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} - au_{xxx} = 0, \quad a \in \mathbb{R}. \tag{3.6}$$

3.5. *Semi-direct products*

It is also interesting to consider Euler equations on semi-direct products. For instance, let $\mathrm{Diff}^\infty(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$ be the semi-direct product of the diffeomorphism group $\mathrm{Diff}^\infty(\mathbb{S}^1)$ with the Abelian group $\mathrm{C}^\infty(\mathbb{S}^1)$. The group operation is given by

$$(\varphi, f) \cdot (\psi, g) := (\varphi \circ \psi, f + g \circ \varphi^{-1}),$$

where $\varphi, \psi \in \mathrm{Diff}^\infty(\mathbb{S}^1)$ and $f, g \in \mathrm{C}^\infty(\mathbb{S}^1)$. For the $H^1 \times L^2$ metric on the Lie algebra $\mathrm{C}^\infty(\mathbb{S}^1) \ltimes \mathrm{C}^\infty(\mathbb{S}^1)$, the corresponding Euler equation corresponds to the two-component CH equation:

$$\rho_t = -(\rho u)_x, \quad u_t - u_{xxt} = -3uu_x + 2u_x u_{xx} + uu_{xxx} - \rho\rho_x, \tag{3.7}$$

a generalization of the CH equation (see [4, 8, 23, 30]).

3.6. *Non-metric Euler equations*

As noticed before, the Euler equation is still meaningful for a semi-invariant, symmetric linear connection, even if it is not derived from a semi-invariant metric. This generalization of the theory permits to interpret geometrically [17] the so-called *b-equations* [15, 28]

$$m_t = -(m_x u + bmu_x), \quad m := Au = u - u_{xx}, \tag{3.8}$$

where $b \in \mathbb{R}$. For $b = 2$ we recover the CH equation and for $b = 3$ we obtain the *DP equation*

$$u_t - u_{txx} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0, \tag{3.9}$$

another integrable model which has been extensively studied in the last decade. It is admitted that the b-equations are integrable only for $b = 2$ and $b = 3$.

Each of them corresponds to the Euler equation (with real parameter b) for the right-invariant, symmetric linear connection on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ induced by

$$B(u, v) = \frac{1}{2} A^{-1}[(Av)_x u + (Au)_x v + b(Av)u_x + b(Au)v_x].$$

Remark 3.1. This connection derives from a semi-invariant *metric* only for $b = 2$ (see [35, 20]). In all cases, nevertheless, the solutions of (3.8) satisfy the conservation law

$$(m(t) \circ \varphi(t))\varphi_x^b(t) = m(0).$$

In the non-metric case, however, this does not derive from (2.3).

4. Local Existence of Geodesics

Although the geometric theory presented above is appealing and can be used to state some stability results, using for instance sectional curvatures [1], the theory remains somewhat formal, due to the fact that analysis on Fréchet manifold is not an easy task. To circumvent the serious analytical difficulties encountered in working rigorously with the actual configuration space (smooth diffeomorphisms), Ebin and Marsden [16] enlarged this configuration space to spaces with a more convenient structure where a rigorous study can be pursued.

4.1. *The group of diffeomorphisms of class H^q*

For $q > 3/2$, let $\mathcal{D}^q(\mathbb{S}^1)$ be the set of C^1-diffeomorphisms of the circle which are of class H^q. This set has the structure of a *Banach manifold* (modeled on $H^q(\mathbb{S}^1)$). It is a *topological group* but *not a Lie group*, composition and inversion in $\mathcal{D}^q(\mathbb{S}^1)$ are continuous but *not differentiable*.

From an analytic point of view, $\mathrm{Diff}^\infty(\mathbb{S}^1)$ may be viewed as an *inverse limit* of these *Banach manifolds*

$$\mathrm{Diff}^\infty(\mathbb{S}^1) = \bigcap_{q>\frac{3}{2}} \mathcal{D}^q(\mathbb{S}^1).$$

The scales of space $\{\mathcal{D}^q(\mathbb{S}^1)\}_{q>3/2}$ is called a *Banach manifold approximation* of $\mathrm{Diff}^\infty(\mathbb{S}^1)$.

4.2. *The geodesic equation as an ODE*

Notice that the right-hand side of the Euler equation

$$u_t = -A^{-1}[u(Au)_x + 2(Au)u_x]$$

is of *order* 1 because if $u \in H^q(\mathbb{S}^1)$ then $A^{-1}[u(Au_x)] \in H^{q-1}(\mathbb{S}^1)$. Hence the Euler equation cannot be realized as a dynamical system on any of the Banach spaces $H^q(\mathbb{S}^1)$.

It is however quite surprising that in *Lagrangian coordinates*, this problem can be overcame, provided the *order* of A is not less than 1. In fact, let φ be the flow of the time dependent vector field u and let $v = \varphi_t$. Then $v_t = (u_t + uu_x) \circ \varphi$ and u solves the Euler equation if and only if (φ, v) is a solution of

$$\begin{cases} \varphi_t = v, \\ v_t = S_\varphi(v), \end{cases} \tag{4.1}$$

where

$$S_\varphi(v) := (R_\varphi \circ S \circ R_{\varphi^{-1}})(v),$$

and

$$S(u) := A^{-1}\{[A, u]u_x - 2(Au)u_x\}.$$

The main observation is that if A is a *differential operator* of order $r \geq 1$ then the quadratic operator

$$S(u) := A^{-1}\{[A, u]u_x - 2(Au)u_x\}$$

is of order 0 because the commutator $[A, u]$ is of order less than $\leq r - 1$. One might expect, that for a larger class of operators A, the quadratic operator S to be of order 0 and the second order system (4.1) to be the local expression of an ODE on the Banach manifold $T\mathcal{D}^q(\mathbb{S}^1)$.

Remark 4.1. For the Euler equation on the Bott–Virasoro group

$$u_t = -A^{-1}[u(Au)_x + 2(Au)u_x - au_{xxx}], \quad a_t = 0,$$

the same process leads to a second-order system of the form (4.1) but for which

$$S(u) = -A^{-1}(-[A,u](u_x) + 2(Au)u_x - au_{xxx}).$$

In that case, the quadratic operator S might be expected to be of order 0 provided that $r \geq 3$ rather than $r \geq 1$ (see [10]).

4.3. *The geodesic spray*

The *second-order vector field* on $\mathrm{Diff}^\infty(\mathbb{S}^1)$, defined in a local chart by

$$F : (\varphi, v) \mapsto (v, S_\varphi(v))$$

is called the *geodesic spray*, cf. [36].

Notice that, even if S can be extended to a quadratic, bounded operator $H^q(\mathbb{S}^1) \to H^q(\mathbb{S}^1)$, we can only conclude, *a priori*, that F extends continuously on $T\mathcal{D}^q(\mathbb{S}^1)$. Indeed, $(\varphi, u) \mapsto u \circ \varphi$ and $\varphi \mapsto \varphi^{-1}$ are only continuous, but not differentiable. In Sec. 5, we will give conditions on A which ensure that F extends to a smooth vector field F_q on $T\mathcal{D}^q(\mathbb{S}^1)$, for q large enough.

If the extension of F_q to $T\mathcal{D}^q(\mathbb{S}^1)$ is smooth, the application of the *Picard–Lindelöf theorem* on the Banach manifold $T\mathcal{D}^q(\mathbb{S}^1)$ ensures that, given any $(\varphi_0, v_0) \in T\mathcal{D}^q(\mathbb{S}^1)$, there is a *maximal solution* (φ, v) of (4.1), defined on an interval $J_q(\varphi_0, v_0)$, such that

$$(\varphi(0), v(0)) = (\varphi_0, v_0).$$

4.4. *A no gain, no loss result*

A remarkable observation due to Ebin and Marsden (see [16, Theorem 12.1]) states that, if the initial data (φ_0, v_0) are smooth, the maximal time interval of existence $J_q(\varphi_0, v_0)$ is independent of the parameter q. This is an essential ingredient which makes it possible to avoid Nash–Moser type schemes to establish local existence of geodesics on $\mathrm{Diff}^\infty(\mathbb{S}^1)$.

Notice first that the spray F is invariant under right translations R_η. In a local chart, we have

$$F(R_\eta\varphi, R_\eta v) = (R_\eta v, R_\eta S_\varphi(v)).$$

Indeed, if $\eta \in \mathrm{Diff}^\infty(\mathbb{S}^1)$, R_η is a diffeomorphism of $\mathcal{D}^q(\mathbb{S}^1)$ for all $q > 3/2$, and the invariance property under R_η is true for the extended spray F_q. This property, is inherited by the flow Ψ_q of F_q on $T\mathcal{D}^q(\mathbb{S}^1)$. In a local chart we have

$$\Psi_q(R_\eta\varphi, R_\eta v, t) = R_\eta\Psi_q(\varphi, v, t).$$

In particular (see [18] for the details), specializing to $\eta = \tau_s$, the spatial rotation by s on $\mathbb{S}^1$, and taking the derivative in s at $s = 0$, we get

$$D_{(\varphi,v)}\Psi_q(\varphi_0, v_0, t) \cdot (\varphi_{0x}, v_{0x}) = (\varphi_x(t), v_x(t)).$$

Now, since $D_{(\varphi,v)}\Psi_q(\varphi_0, v_0, t)$ is a bounded, linear operator on the Banach space $H^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$, we obtain that if the initial data (φ_0, v_0) is of class H^{q+1} then $(\varphi(t), v(t))$ is of class

H^{q+1} (*no loss*). Going backward, we get that if $(\varphi(t), v(t))$ is of class H^{q+1} for some $t > 0$ then (φ_0, v_0) is of class H^{q+1} (*no gain*).

Proposition 4.1 (Ebin–Marsden [16]). *Let (φ, v) be a solution of (4.1) on $T\mathcal{D}^q(\mathbb{S}^1)$ with initial data (φ_0, v_0).*

(1) *If $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ then $(\varphi(t), v(t)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ for all $t \in J_q(\varphi_0, v_0)$.*
(2) *If there exists $t \in J_q(\varphi_0, v_0)\backslash\{0\}$ such that $(\varphi(t), v(t)) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$ then $(\varphi_0, v_0) \in T\mathcal{D}^{q+1}(\mathbb{S}^1)$.*

A consequence of Proposition 4.1 is the following local existence theorem for geodesics on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ (see [18]).

Theorem 4.1. *Suppose that the spray F extends to a smooth vector field F_q on the Banach manifold $T\mathcal{D}^q(\mathbb{S}^1)$, for all q large enough. Then, given any $(\varphi_0, v_0) \in T\mathrm{Diff}^\infty(\mathbb{S}^1)$, there exists a unique non-extendable solution*

$$(\varphi, v) \in C^\infty(J, T\mathrm{Diff}^\infty(\mathbb{S}^1))$$

of (4.1), *with initial data (φ_0, v_0), defined on the maximal interval of existence $J = (t^-, t^+)$. Moreover, the solution depends smoothly of the initial data.*

5. Smoothness of the Spray

In this section, we shall investigate under which conditions on the inertia operator A, the spray F extends to a smooth vector field F_q on $T\mathcal{D}^q(\mathbb{S}^1)$, for q large enough. In the following, we suppose that A extends, for all q large enough, to a bounded, linear isomorphism

$$A : H^q(\mathbb{S}^1) \to H^{q-r}(\mathbb{S}^1),$$

where $r \geq 1$ is fixed. In terms of the symbol a of the Fourier multiplier A, this is equivalent to assume that a does not vanish, and that

$$a(n) = O(|n|^r), \quad \frac{1}{a(n)} = O(|n|^{-r}).$$

As already stated, even if A is a bounded operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$, one cannot conclude directly that the mapping

$$(\varphi, v) \mapsto A_\varphi(v) := R_\varphi \circ A \circ R_{\varphi^{-1}}(v)$$

is smooth from $\mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$, because $(\varphi, v) \mapsto R_\varphi(v)$ is *not differentiable*. A first reduction of the problem consists to show, however, that if this maps is smooth then F extends to a smooth vector field on $T\mathcal{D}^q(\mathbb{S}^1)$ (see [18] for the proof).

Lemma 5.1. *Let $q > r + 1/2$. Suppose that the mapping*

$$(\varphi, v) \mapsto A_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \to H^{q-r}(\mathbb{S}^1)$$

is of class C^{m+1}. Then the mapping

$$(\varphi, v) \mapsto S_\varphi(v), \quad \mathcal{D}^q(\mathbb{S}^1) \times H^q(\mathbb{S}^1) \to H^q(\mathbb{S}^1)$$

is of class C^m, where $S(u) = A^{-1}\{[A, u]u_x - 2(Au)u_x\}$.

Remark 5.1. This proposition has to be compared to the well-known fact, in classical Riemannian geometry, that the spray is of class C^m provided the metric is of class C^{m+1}.

We are therefore reduced to investigate for which Fourier multiplier A of order r is the mapping

$$\varphi \mapsto A_\varphi, \qquad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

smooth, for sufficiently large q. If A is a differentiable operator, then $A_\varphi(v)$ is a rational expression of φ_x, v and their derivatives (see [17], for instance). Hence for $q > r + 1/2$, the map is smooth (because $H^s(\mathbb{S}^1)$ is a Banach algebra for $s > 1/2$).

Example 5.1. For $D = d/dx$ we get

$$D_\varphi(v) = \frac{v_x}{\varphi_x}, \quad D_\varphi^2(v) = \frac{v_{xx}}{(\varphi_x)^2} - \frac{v_x \varphi_{xx}}{(\varphi_x)^3}, \ldots.$$

However, this argument *does not apply* to a general *Fourier multiplier*. In [18], we have established smoothness of the conjugates for a larger class of (nonlocal) Fourier multiplier.

Theorem 5.1 (Escher–Kolev [17]). *Let $A = \mathbf{op}(a(\xi))$ be a Fourier multiplier of order $r \geq 1$. Suppose that, for each $n \geq 1$, the symbol a satisfies the following conditions*:

(1) *$f_n(\xi) := \xi^{n-1} a(\xi)$ is of class C^{n-1},*
(2) *$f_n^{(n-1)}$ is absolutely continuous,*
(3) *there exists a constant $C_n > 0$ such that*

$$|f_n^{(n)}(\xi)| \leq C_n (1 + \xi^2)^{(r-1)/2}, \ \ a.e.$$

Then the map

$$\varphi \mapsto A_\varphi, \qquad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is smooth for each $q \in (r + \frac{3}{2}, \infty)$.

Remark 5.2. The hypothesis on the symbol of the operator A can easily be checked on explicit examples. It is satisfied for the H^s metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ for which

$$a(\xi) = (1 + |\xi|^2)^s,$$

and for the homogeneous metric $\dot{H}^s$ on $\mathrm{Diff}^\infty(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$ for which

$$a(\xi) = |\xi|^{2s}$$

provided $s \geq 1/2$. For $s = 1/2$, we get the $\dot{H}^{1/2}$ metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$, which corresponds to the modified CLM equation (3.4).

Sketch of proof. The proof consists mainly in three steps.

Step 1: *Computing the derivatives.* The map $(\varphi, v) \mapsto A_\varphi(v)$ is smooth on the Fréchet manifold $\mathrm{Diff}^\infty(\mathbb{S}^1) \times \mathrm{C}^\infty(\mathbb{S}^1)$. The nth Gâteaux partial derivative in φ can be written as

$$\partial_\varphi^n A_\varphi(v, \delta\varphi_1, \dots, \delta\varphi_n) = R_\varphi A_n R_{\varphi^{-1}}(v, \delta\varphi_1, \dots, \delta\varphi_n),$$

where A_n is a $(n+1)$-linear operator, which is given by a recursive formula involving commutators. In particular, for $n = 1$, we get:

$$A_1(u_0, u_1) = [u_1, A]u_{0x}.$$

Step 2: *Extension of A_n to the Sobolev spaces.* Let $e_m(x) = \exp(2i\pi mx)$ for $m \in \mathbb{Z}$. Then

$$A_n(e_{m_0}, \dots, e_{m_n}) = a_n(m_0, \dots, m_n)e_{m_0+\cdots+m_n},$$

where a_n is given by a recursive formula involving a. For instance

$$a_1(m_0, m_1) = (2i\pi)m_0[a(m_0) - a(m_0 + m_1)].$$

The hypothesis on the symbol a of A leads to the inequality

$$|a_n(m_0, \dots, m_n)| \le C_n(1 + m_0^2)^{r/2} \cdots (1 + m_n^2)^{r/2},$$

from which we can deduce that A_n extends to a bounded $(n+1)$-linear operator from $H^q(\mathbb{S}^1)$ to $H^{q-r}(\mathbb{S}^1)$.

Step 3: *Smoothness of A_φ.* The mapping

$$\varphi \mapsto R_\varphi, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}(H^\rho(\mathbb{S}^1))$$

is *locally bounded*[c] for $3/2 < \rho \le q$. Since $\varphi \mapsto \varphi^{-1}$ is continuous on $\mathcal{D}^q(\mathbb{S}^1)$ for $q > 3/2$, the same is true for $\varphi \mapsto R_{\varphi^{-1}}$. Therefore

$$\varphi \mapsto A_{n,\varphi} := R_\varphi A_n R_{\varphi^{-1}}, \quad \mathcal{D}^q(\mathbb{S}^1) \to \mathcal{L}^{n+1}(H^q(\mathbb{S}^1), H^{q-r}(\mathbb{S}^1))$$

is *locally bounded*, for each $n \ge 0$. Using the *mean value theorem*, we deduce then inductively that

$$\{\varphi \mapsto A_{n+1,\varphi} \text{ is locally bounded}\} \;\Rightarrow\; \{\varphi \mapsto A_{n,\varphi} \text{ is locally Lipschitz}\},$$

and

$$\{\varphi \mapsto A_{n+1,\varphi} \text{ is locally Lipschitz}\} \;\Rightarrow\; \{\varphi \mapsto A_{n,\varphi} \text{ is differentiable}\}.$$

Since A_φ is a bounded, linear operator, we can then conclude that

$$\varphi \mapsto A_\varphi$$

is smooth. □

[c]But not continuous, although the mapping $(\varphi, v) \mapsto v \circ \varphi$ is continuous.

6. The Minimization Problem

6.1. *The exponential map*

The geodesic flow of a smooth spray on a Banach manifold M satisfies the following remarkable property

$$\varphi(t, x_0, sv_0) = \varphi(st, x_0, v_0),$$

which is a consequence of the quadratic nature of the spray [36]. Hence, the *exponential mapping* $\mathfrak{exp}_{x_0}$ (the time one of the flow) is well-defined on a neighborhood of 0 in $T_{x_0}M$. For a *strong metric*, this mapping is a local diffeomorphism from a neighborhood of 0 in $T_{x_0}M$ onto a neighborhood of x_0 in M. It defines a privileged chart around x_0, called a *normal neighborhood*.

On a Fréchet manifold, and in particular on the group $\mathrm{Diff}^\infty(\mathbb{S}^1)$, the existence of this chart is not granted. In that case, even the *group exponential* is not locally surjective [40]. Moreover, it was shown in [11] that the Riemannian exponential map for the L^2 metric (Burgers equation) on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ is not a local C^1-diffeomorphism near the origin. Nevertheless, for the H^k metrics ($k \geq 1$) (see [12]), the Riemannian exponential map is a smooth, local diffeomorphism. This result is still true for H^s metrics provided $s \in [1/2, +\infty)$ (see [17, 18]).

6.2. *Minimizing the arc-length*

On a *strong Riemannian manifold*, given two nearby points x and y, there exists a unique geodesic, joining these two points, which minimizes (globally) the *arc-length*. This is a consequence of the existence of normal neighborhoods. This is no longer true for a *weak Riemannian metric* (pre-Hilbertian structure) in general. Indeed, it may happen that the lower bound of arc-lengths between any pair of points always vanishes. This is the case for the L^2 metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ as was shown in [39]. Nevertheless, for a right-invariant metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$, which extends smoothly to a weak Riemannian metric on each Banach approximation $\mathcal{D}^q(\mathbb{S}^1)$, *with a smooth spray*, there exists a unique geodesic which minimizes *locally* the arc-length (weak version) [18]. This is the case, for instance, for the $H^{1/2}$-metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$. Notice that this is not in contradiction with the fact that the *global* infimum of arc-length between each pair of diffeomorphisms, in this case, vanishes identically [2].

6.3. *The geodesic semi-distance*

On a *finite-dimensional Riemannian manifold* M, the lower bound of the arc-lengths of piecewise C^1 paths γ between two points

$$d_g(x, y) = \inf_\gamma \int_0^1 \|\dot\gamma(t)\|_g dt,$$

defines a distance on M, that is

$$x \neq y \quad \Rightarrow \quad d_g(x, y) > 0.$$

This is an essential property of a *strong Riemannian metric* (see [36, Chap. VII]). This is no longer true, in general, for a weak Riemannian metric. In that case, this function is only a *semi-distance*. It may even happen that this function vanishes identically as was proved in [39] for the right-invariant metric induced by the L^2 metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$. This result has been completed recently in [2], where it was shown that the geodesic semi-distance d_s induced by the H^s metric vanishes identically on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ if $s \in [0, 1/2]$, whereas d_s is a distance for $s > 1/2$.

7. Conclusion

In this paper, we have presented a survey of approximation models in water waves that can be recast as Euler equations for *weak*, right-invariant metrics on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ or related groups. In all relevant cases, the inertia operator is a Fourier multiplier (i.e. a continuous linear operator on $\mathrm{C}^\infty(\mathbb{S}^1)$ which commutes with d/dx) or build from a Fourier multiplier (in the case of the Bott–Virasoro group or for the two-component CH, for instance).

We have provided a condition on the symbol of the inertia operator A which ensures that its conjugates

$$\varphi \mapsto A_\varphi,$$

extends smoothly to $\mathcal{D}^q(\mathbb{S}^1)$ for sufficiently large q. We have used this result to prove that the spray extends smoothly to the Banach approximation manifolds $T\mathcal{D}^q(\mathbb{S}^1)$ and established local existence of geodesics using a *no loss*, *no gain* result. This method was successfully applied to the H^s right-invariant metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ and the homogeneous $\dot{H}^s$ right-invariant metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)/\mathrm{Rot}(\mathbb{S}^1)$ for $s \geq 1/2$.

Our methods should apply to establish the existence of smooth geodesics for the *Weil–Petersson equation* (corresponding to the $\dot{H}^{3/2}$ metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)/\mathrm{PSL}(2,\mathbb{R})$) provided we can prove a reduction Lemma 5.1. This equation was studied in [24], where it has been shown that this metric is strong on a "replacement" for $\mathcal{D}^{3/2}(\mathbb{S}^1)$ (which does not exists as a topological group) and that geodesics are complete on this extended manifold. Nevertheless, it is not clear that geodesics on $\mathrm{Diff}^\infty(\mathbb{S}^1)/\mathrm{PSL}(2,\mathbb{R})$ can be obtained from [24].

In a second part, we have discuss the minimization problem for geodesics of a *weak* Riemannian metric. It was shown that when the spray can be extended smoothly to the Banach approximation manifolds, the geodesics minimize *locally* the arc-length between two nearby points. This has to be compared to what happens for a *strong* Riemannian metric, where the geodesics minimize *globally* the arc-length between two nearby points. The problem whether the geodesics minimize *globally* the arc-length if the geodesic semi-distance is moreover a *distance* (like for CH, for instance) remains to be studied.

In all the above study, we have limited ourselves to the periodic case. It should be emphasized that this is not a serious limitation and that the theory can be extended to the non-periodic case, provided we impose some kinds of *decreasing conditions at infinity*.[d] The natural candidate for this is the group

$$\mathrm{Diff}_{H^\infty}(\mathbb{R}) = \{\mathrm{id} + f; f \in H^\infty(\mathbb{R}) \text{ and } f' > -1\},$$

[d]Because $\mathrm{Diff}^\infty(\mathbb{R})$ is not a *regular* Fréchet Lie group in the sense of Milnor (see [40]). In a regular Lie group, elements of the Lie algebra correspond to one-parameter subgroups.

where

$$H^{\infty}(\mathbb{R}) = \bigcap_{n=1}^{+\infty} H^n(\mathbb{R}).$$

It was shown in [27] that $\mathrm{Diff}_{H^\infty}(\mathbb{R})$ is a *regular* Fréchet Lie group.

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