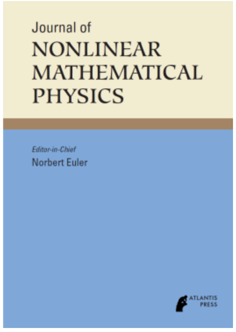

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DISPERSION RELATIONS FOR STEADY PERIODIC WATER WAVES OF FIXED MEAN-DEPTH WITH AN ISOLATED BOTTOM VORTICITY LAYER

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In this paper we obtain the dispersion relations for small-amplitude steady periodic water waves, which propagate over a flat bed with a specified mean depth, and which exhibit discontinuous vorticity. We take as a model an isolated layer of constant nonzero vorticity adjacent to the flat bed, with irrotational flow above the layer.

Keywords: Steady water waves; dispersion relation; discontinuous vorticity.

Mathematics Subject Classification 2010: 35Q31, 76D33, 34B05

1. Introduction

In this paper we derive the dispersion relations for small-amplitude two-dimensional steady periodic water waves, which propagate over a flat bed with a specified mean depth, and which have a discontinuous vorticity distribution. This discontinuity manifests itself when we assume that there is a layer of fluid which has constant nonzero vorticity adjacent to the flat bed, with irrotational flow above this layer. The existence of steady periodic water waves, which exhibit such a discontinuous vorticity, is proven in [11, 19] (using different formulations of the governing equations) by applying the Crandall–Rabinowitz theorem on local bifurcation, resulting in small-amplitude generalized solutions of the governing equations. In [19], the mean-depth of the fluid above the flat bed is a fixed quantity for the continuum of solutions of the governing equations which are given by the bifurcation curve, in [11] the mass-flux is a fixed quantity. It was observed in [24] that, for fixed mass flux, as we move along the bifurcation curve the mean-depth of the flows actually varies, and so we must make a choice between the different frameworks. Since the mean-depth of the fluid may reasonably be regarded as the more inherent physical property of a fluid flow, the framework which we adopt here is that of [19] where the mean depth is fixed. However, remarkably, we will see that the dispersion relations for the two systems coincide, which suggests that the main difference in the two frameworks is observed for larger amplitude waves [24] (see [3] for analogous dispersion relations for the fixed mass-flux system). The small-amplitude solutions which were obtained in [19], although they are in fact nonlinear, are compatible with the linear regime for water waves, and one of the most important, and most physically interesting, properties of linear water waves is their dispersion relation [25].

The presence of vorticity in a fluid domain is a physically far more relevant and interesting scenario [2, 23, 29] than that of the flow being purely irrotational. Flows with vorticity serve as models for wave-current interactions, among other intricate and physically important phenomena [2, 23, 29]. The importance of studying wave-current interaction is well recognized [23] but progress in developing this area mathematically had been slow [29] until relatively recently, due to the myriad complications which vorticity adds to the analysis of the governing equations [2]. Subsequent to the seminal paper [8], there has been an abundance of mathematically rigorous analysis for steady periodic water waves flows with general (continuous) vorticity distributions, with analytical results ranging from existence, stability, symmetry and regularity of waves, and increasingly allowing for complications such as stagnation points, critical layers, and stratification (see [1–12, 15–21, 26, 27, 31, 32] and the references therein). Recently, in [11] (and subsequently [19]) the analysis was extended to allow for discontinuous vorticity distributions. While mathematically challenging, physically these flows are of great importance, since it is known that wave-current interaction is most dramatically observed where this is a rapid change in the current strength [23], and the near-bed region is a location which commonly experiences currents (accounting for sediment transport, for instance). This is the scenario which we model in this paper.

2. Governing Equations

In this paper we deal with two-dimensional gravity water waves, where the fluid is inviscid and incompressible, and we allow for general rotational flow. We consider steady periodic traveling surface waves propagating over water of a fixed depth $d > 0$, where d is fixed, and where the dominant external restoration force is gravity, as represented by the constant of acceleration g. We take $y = 0$ to represent the location of the undisturbed water surface, then the flat bed is located at $y = -d$. The wave profile η is a free surface since it is *a priori* undetermined and thus represents an unknown in the problem. If the steady traveling waves move with a constant wavespeed $c > 0$, then the velocity field (u, v) and the free-surface profile η have an x, t functional dependence in the form $x - ct$. Since we deal with steady traveling waves, we can transform to a new reference frame moving alongside the wave, with constant speed $c > 0$, by using the change of coordinates $(x - ct, y) \mapsto (x, y)$. In this frame the flow is time independent problem. We denote the closure of the fluid domain by $\overline{D_\eta} = \{(x, y) \in \mathbb{R}^2 : -d \le y \le \eta(x)\}$. The governing equations take the form of Euler's equation, together with the boundary conditions:

$$u_x + v_y = 0 \qquad \text{in } D_\eta, \tag{2.1a}$$

$$(u - c)u_x + vu_y = -P_x \qquad \text{in } D_\eta, \tag{2.1b}$$

$$(u - c)v_x + vv_y = -P_y - g \qquad \text{in } D_\eta, \tag{2.1c}$$

$$v = (u - c)\eta_x \quad \text{on } y = \eta(x), \tag{2.1d}$$

$$P = P_{\text{atm}} \quad \text{on } y = \eta(x), \tag{2.1e}$$

$$v = 0 \quad \text{on } y = -d, \tag{2.1f}$$

where $P(x, y)$ is the pressure distribution function, and P_{atm} is the constant atmospheric pressure. For two-dimensional motion, the vorticity is given by

$$\omega = u_y - v_x. \tag{2.1g}$$

We now make the additional "non-stagnation" assumption that

$$u < c, \tag{2.1h}$$

throughout the fluid. Physically, it is known that the assumption (2.1h) is valid for flows which are not near breaking [22, 25]. The non-stagnation condition is mathematically essential to the development of our analysis of the water wave equations, as we will see below. We work with periodic waves, and without loss of generality we may choose the period to be 2π. For suppose we are dealing with water waves of wavelength L in the governing equations (2.1a)–(2.1f), then after performing the following scaling of variables

$$(x, y, t, g, \omega, \eta, u, v, P, c) \mapsto (\kappa x, \kappa y, \kappa t, \kappa^{-1} g, \kappa^{-1}\omega, \kappa\eta, u, v, P, c), \tag{2.2}$$

where $\kappa = \frac{2\pi}{L}$ is the wavenumber, we end up with a 2π-periodic system in the new variables identical to (2.1a)–(2.1f) except g, ω are replaced by $\kappa^{-1}g, \kappa^{-1}\omega$. We now use (2.1a) to define the stream function ψ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{2.3}$$

and we fix the constant by setting $\psi = 0$ on $y = \eta(x)$. Relations (2.1d) and (2.1f) tell us that ψ is constant on both boundaries of D_η, and so it follows from integrating (2.3) and using (2.1h) that $\psi = -p_0$ on $y = -d$, where

$$p_0 = \int_{-d}^{\eta(x)} (u(x, y) - c)dy < 0,$$

and p_0 is known as the relative mass flux. We can reformulate the governing equations in the moving frame in terms of the stream function [2, 19] as follows:

$$\Delta\psi = \omega \quad \text{in } -d < y < \eta(x), \tag{2.4a}$$

$$|\nabla\psi|^2 + 2g(y + d) = Q \quad \text{on } y = \eta(x), \tag{2.4b}$$

$$\psi = 0 \quad \text{on } y = \eta(x), \tag{2.4c}$$

$$\psi = -p_0 \quad \text{on } y = -d. \tag{2.4d}$$

Here Q is a constant known as the hydraulic head. The next step is our analysis to introduce the semi-Lagrangian hodograph transformation given by

$$(x, y) \mapsto (q, p) := (x, \psi(x, y)/p_0). \tag{2.5}$$

It is now apparent that the non-stagnation condition (2.1h) is vital in order to ensure that the change of variables (2.5) represents an isomorphism. The semi-hodograph transformation has the distinct advantage of transforming the fluid domain D_η, with the unknown free boundary η, into the fixed semi-infinite rectangular strip $\overline{R} = \mathbb{R} \times [-1, 0]$. It can be easily shown [18] that $\omega_q = 0$, so it follows that the vorticity is a function of p alone: $\omega = \gamma(p)$, where γ will be referred to as the vorticity function [18]. Following the transformation (2.5) we can reformulate the governing equations (2.4) in terms of the modified-height function,

$$h(q,p) = \frac{y}{d} - p, \tag{2.6}$$

as follows [18]:

$$\left(\frac{1}{d^2} + h_q^2\right) h_{pp} - 2h_q(h_p+1)h_{pq} + (h_p+1)^2 h_{qq} + \frac{\gamma(p)}{p_0}(h_p+1)^3 = 0 \quad \text{in } -1 < p < 0, \tag{2.7a}$$

$$\frac{1}{d^2} + h_q^2 + \frac{(h_p+1)^2}{p_0^2}[2gd(h+1) - Q] = 0, \quad p = 0, \tag{2.7b}$$

$$h = 0, \quad p = -1, \tag{2.7c}$$

with the non-stagnation condition (2.1h) equivalent to

$$h_p + 1 > 0. \tag{2.7d}$$

A solution $h(q,p)$ of (2.7), should be even and 2π-periodic in q. Since we wish to model an isolated region of vorticity, surrounded by irrotational flow, we must allow for the possibility of the vorticity function being discontinuous. In order to do this we consider generalized solutions [14] of the weak form of (2.7), which is expressed as [19]:

$$\left\{\frac{1+d^2h_q^2}{2d^2(1+h_p)^2} - \frac{\Gamma(p)}{2d^2}\right\}_p - \left\{\frac{h_q}{1+h_p}\right\}_q = 0, \quad -1 < p < 0, \tag{2.8a}$$

$$\frac{1+d^2h_q^2}{2d^2(1+h_p)^2} + \frac{gd(h+1)}{p_0^2} = \frac{Q}{2p_0^2}, \quad p = 0, \tag{2.8b}$$

$$h = 0, \quad p = -1. \tag{2.8c}$$

Here

$$\Gamma(p) = 2\int_0^p \frac{d^2\gamma(s)}{p_0} ds, \quad -1 \le p \le 0.$$

We understand, by a solution of (2.8), a function $h \in W^{2,r}_{\text{per}}(R) \subset C^{1,\alpha}_{\text{per}}(\overline{R})$, where $r > 2/(1-\alpha)$ for $\alpha \in (1/3, 1)$, cf. [19]. Here the per subscript indicates that our solutions are even and 2π-periodic in the q-variable, and the inclusion relation (along with the choice of r) are derived from Morrey's inequality.

Laminar flow solutions $H(p)$ of system (2.8) have no q-dependence, and the streamlines of the resulting flow are horizontal, therefore they represent parallel sheer flow with a flat surface. For $-\Gamma_{\min} < \lambda < Q$ we solve (2.8) to get [18, 19]

$$H(p) = \int_0^p \frac{ds}{\sqrt{\lambda + \Gamma(s)}} + \frac{1}{2gd}\left[Q - \frac{p_0^2}{d^2}\lambda\right] - (p+1), \quad -1 < p \leq 0.$$

The parameter λ is related to the fluid velocity at the flat surface by the relation

$$\sqrt{\lambda} = \frac{1}{H_p + 1}\bigg|_{p=0} = \frac{d(u-c)}{p_0}\bigg|_{\text{on the flat surface}}, \tag{2.9}$$

and it is implicitly related to Q by the following formula

$$Q = 2gd \int_{-1}^{0} \frac{ds}{\sqrt{\lambda + \Gamma(s)}} + \frac{p_0^2}{d^2}\lambda > 0.$$

In [19], we show using the Crandall–Rabinowitz local bifurcation theorem [13], that the necessary and sufficient condition for the existence of small amplitude waves which are perturbations of the laminar flows is that a nontrivial solution $m \in C^{1,\alpha}(-1,0)$ exists for the Sturm–Liouville problem

$$(a^3 m_p)_p = d^2 a m, \quad -1 < p < 0, \tag{2.10a}$$

$$a^3 m_p = \frac{gd^3}{p_0^2} m, \quad p = 0, \tag{2.10b}$$

$$m = 0, \quad p = -1, \tag{2.10c}$$

where $a(p;\lambda) = \frac{1}{H_p+1} = \sqrt{\lambda + \Gamma(p)} \in C^{\alpha}([-1,0])$.

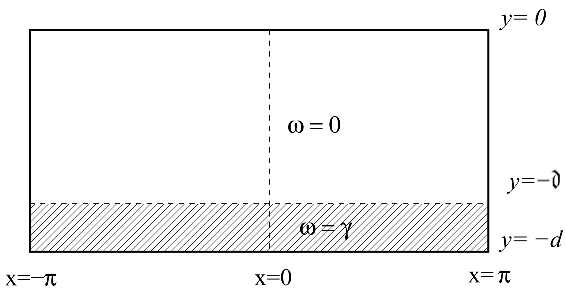
3. Dispersion Relations

We see from relation (2.9) that when small-amplitude waves exist (at the critical value λ^* of the bifurcation parameter) the speed of the laminar flows on the free surface attains the critical value

$$u^* - c = \frac{p_0\sqrt{\lambda^*}}{d}. \tag{3.1}$$

We look at the case where there is a layer of constant vorticity at the bottom of the fluid, and above this layer there is no vorticity, that is,

$$\omega = \begin{cases} 0, & -\mathfrak{d} < y < 0, \\ \gamma, & -d < y < -\mathfrak{d}. \end{cases} \tag{3.2}$$



In the laminar flow, prior to bifurcation, we have

$$u_y = \omega,$$

and so integrating we get

$$c - u(y) = \begin{cases} -\dfrac{p_0\sqrt{\lambda}}{d}, & -\mathfrak{d} < y < 0, \\ -\dfrac{p_0\sqrt{\lambda}}{d} - \gamma(y + \mathfrak{d}), & -d < y < -\mathfrak{d}. \end{cases} \tag{3.3}$$

Now, by definition we have

$$\begin{aligned} -p_0 = \int_{-d}^{0} (c-u)dy &= -\int_{-d}^{-\mathfrak{d}} \left(\frac{p_0\sqrt{\lambda}}{d} + \gamma\mathfrak{d}\right) dy - \int_{-d}^{-\mathfrak{d}} \gamma y dy - \int_{-\mathfrak{d}}^{0} \frac{p_0\sqrt{\lambda}}{d} dy \\ &= -p_0\sqrt{\lambda} + \frac{\gamma}{2}(d-\mathfrak{d})^2. \end{aligned}$$

Therefore

$$p_0(\sqrt{\lambda} - 1) = \frac{\gamma}{2}(d - \mathfrak{d})^2. \tag{3.4}$$

We note for future reference that the above relation implies that

$$\gamma < 0 \;\Rightarrow\; \lambda > 1, \quad \gamma > 0 \;\Rightarrow\; \lambda < 1. \tag{3.5}$$

Also, $\psi_y = u - c$, hence in the laminar flow we have

$$\psi(0) - \psi(y) = \int_{y}^{0} \psi_y dy = \int_{y}^{0} (u-c)\, dy,$$

and

$$\psi(-\mathfrak{d}) = \int_{-\mathfrak{d}}^{0} (c-u)dy = -\frac{p_0\mathfrak{d}\sqrt{\lambda}}{d}.$$

Let

$$\mathfrak{p} = \frac{\psi(-\mathfrak{d})}{p_0} = -\frac{\mathfrak{d}\sqrt{\lambda}}{d}, \tag{3.6}$$

and we have

$$a(p,\lambda) = \begin{cases} \sqrt{\lambda}, & \mathfrak{p} < p < 0, \\ \sqrt{\lambda + 2\dfrac{d^2\gamma(p-\mathfrak{p})}{p_0}}, & -1 < p < \mathfrak{p}. \end{cases}$$

The solution of (2.10a) in the upper irrotational region of the fluid domain, which satisfies the boundary condition (2.10b), is given by

$$m(p) = c_1 \sinh\left(\frac{dp}{\sqrt{\lambda}}\right) + c_2 \cosh\left(\frac{dp}{\sqrt{\lambda}}\right), \quad \mathfrak{p} < p < 0,$$

with the additional condition that

$$c_1 = \frac{gd^2}{\lambda p_0^2} c_2. \tag{3.7}$$

Furthermore, it can be seen by direct calculation that the solution of (2.10a) in the lower rotational region of the fluid which satisfies (2.10c) is given by

$$m(p) = \frac{c_3}{a(p)} \sinh(\Sigma(p)), \quad -1 < p < \mathfrak{p},$$

where

$$\Sigma(p) = \frac{p_0}{\gamma d}\left[\sqrt{\lambda + 2\frac{d^2\gamma(p-\mathfrak{p})}{p_0}} - \sqrt{\lambda - 2\frac{d^2\gamma(1+\mathfrak{p})}{p_0}}\right],$$

and we use the relations

$$a_p = \frac{d^2\gamma}{p_0 a}, \quad \Sigma_p = \frac{d}{a}, \quad -1 < p < \mathfrak{p}.$$

In order to have a C^1 match at $p = \mathfrak{p}$ we must have (using relation (3.6))

$$-c_1 \sinh(\mathfrak{d}) + c_2 \cosh(\mathfrak{d}) = \frac{c_3}{\sqrt{\lambda}} \sinh(\Sigma(\mathfrak{p})),$$

$$c_1 \frac{d}{\sqrt{\lambda}} \cosh(\mathfrak{d}) - c_2 \frac{d}{\sqrt{\lambda}} \sinh(\mathfrak{d}) = -\frac{c_3 d^2 \gamma}{p_0(\sqrt{\lambda})^3} \sinh(\Sigma(\mathfrak{p})) + \frac{c_3 d}{\lambda} \cosh(\Sigma(\mathfrak{p})),$$

which we solve to get

$$c_1 = \frac{c_3}{\sqrt{\lambda}} \cosh(\mathfrak{d} + \Sigma(\mathfrak{p})) - \frac{c_3 d\gamma}{p_0 \lambda} \cosh(\mathfrak{d}) \sinh(\Sigma(\mathfrak{p})),$$

$$c_2 = \frac{c_3}{\sqrt{\lambda}} \sinh(\mathfrak{d} + \Sigma(\mathfrak{p})) - \frac{c_3 d\gamma}{p_0 \lambda} \sinh(\mathfrak{d}) \sinh(\Sigma(\mathfrak{p})).$$

We now calculate

$$\Sigma(\mathfrak{p}) = \frac{p_0}{\gamma d}\sqrt{\lambda} - \frac{p_0}{\gamma d}\sqrt{\lambda - 2\frac{d^2\gamma(1+\mathfrak{p})}{p_0}}.$$

We rewrite the term under the second square root, using the relations (3.4) and (3.6), to get

$$\lambda - 2\frac{d^2\gamma(1+\mathfrak{p})}{p_0} = \lambda - 4\frac{d^2\gamma(d - \mathfrak{d}\sqrt{\lambda})(\sqrt{\lambda}-1)}{\gamma d(d-\mathfrak{d})^2} = \frac{((d+\mathfrak{d})\sqrt{\lambda} - 2d)^2}{(d-\mathfrak{d})^2}.$$

In order to choose the correct sign when taking the square root we refer to the relations (2.1h), (3.4), and evaluate the second relation in (3.3) at $y = -d$, to get

$$\frac{d}{\sqrt{\lambda}}(d - \mathfrak{d}) < \frac{p_0}{\gamma} = \frac{(d-\mathfrak{d})^2}{2(\sqrt{\lambda}-1)},$$

from which we deduce that

$$\frac{(d+\mathfrak{d})\sqrt{\lambda} - 2d}{d - \mathfrak{d}} < 0.$$

Therefore,

$$\Sigma(\mathfrak{p}) = \frac{p_0}{\gamma d}\left[\sqrt{\lambda} + \frac{(d+\mathfrak{d})\sqrt{\lambda}-2d}{d-\mathfrak{d}}\right] = \frac{(d-\mathfrak{d})^2}{2d(\sqrt{\lambda}-1)}\left[\frac{2d(\sqrt{\lambda}-1)}{d-\mathfrak{d}}\right] = d-\mathfrak{d}. \tag{3.8}$$

Hence

$$c_1 = \frac{c_3}{\sqrt{\lambda}}\cosh(d) - \frac{c_3 d\gamma}{p_0\lambda}\cosh(\mathfrak{d})\sinh(d-\mathfrak{d}), \tag{3.9}$$

$$c_2 = \frac{c_3}{\sqrt{\lambda}}\sinh(d) - \frac{c_3 d\gamma}{p_0\lambda}\sinh(\mathfrak{d})\sinh(d-\mathfrak{d}). \tag{3.10}$$

Using (3.7) we have

$$\lambda^{3/2}p_0^3 - \frac{p_0^2 d\gamma}{2}\left(\tanh(d) + \frac{\sinh(d-2\mathfrak{d})}{\cosh(d)}\right)\lambda - gd^2 p_0\tanh(d)\lambda^{1/2} + \frac{gd^3\gamma}{2}\left(1 - \frac{\cosh(d-2\mathfrak{d})}{\cosh(d)}\right) = 0. \tag{3.11}$$

To summarize up to this point: there exists non-laminar small-amplitude water wave solutions to (2.8) (that is, bifurcation occurs) if and only if there exists a unique positive root $\sqrt{\lambda^*}$ for the cubic polynomial (3.11), which is compatible with condition (2.1h). Given such a root, the dispersion relation for the flow prescribed by the discontinuous vorticity is then obtained via relation (3.1).

3.1. *Uniform vorticity throughout flow*

Before analyzing in detail the full equation (3.11), let us examine the limiting cases where the vorticity distribution is continuous. Firstly, when $\mathfrak{d} \to d$, we get irrotational flow throughout the fluid, and Eq. (3.11) reduces to

$$\lambda p_0^2 = gd^2\tanh(d).$$

Hence, from (3.1) we recover the standard dispersion relation [2, 22, 25] for irrotational flows

$$c - u^* = \sqrt{g\tanh d}.$$

We mention here that the terminology "dispersion relation" comes from the fact that, for periodic waves of wavelength L which we scale using the transformation (2.2), the above relation translates to

$$c - u^* = \sqrt{\frac{g}{\kappa}\tanh(\kappa d)},$$

where $\kappa = 2\pi/L$ is the wavenumber. The above relation implies that waves of different wavelengths, over a flat bed, travel at different speeds — this is the dispersive effect. Contrary to this is the fact that, as $d \to \infty$, we get the limit

$$c - u^* = \sqrt{gd},$$

and so in deep-water the speed of a wave does not depend on the wavelength: deep-water waves are not dispersive. Some useful references on dispersion relations for small-amplitude water waves are [2, 22, 25]. For brevity, in the following, the dispersion relations which we derive will be for waves of spatial period 2π, with the understanding that dispersion relations for waves of general wavelength may be obtained following the considerations above.

In the case of constant vorticity throughout the flow ($\mathfrak{d} \to 0$) Eq. (3.11) becomes

$$\lambda p_0^2 - p_0 d\gamma \tanh(d)\lambda^{1/2} - gd^2 \tanh(d) = 0,$$

and solving this for $\sqrt{\lambda}$ we get

$$\sqrt{\lambda} = \frac{d\gamma \tanh(d) \pm \sqrt{d^2\gamma^2 \tanh^2(d) + 4gd^2 \tanh(d)}}{2p_0},$$

and using (3.1) and the fact that $c - u > 0$ everywhere, we get the dispersion relation

$$c - u^* = -\frac{\gamma}{2}\tanh(d) + \frac{1}{2}\sqrt{\gamma^2 \tanh^2(d) + 4g\tanh(d)}.$$

These relations match exactly those of [8, 18] for small amplitude waves in flows with constant vorticity (note that in [8] the vorticity is defined using the opposite sign).

3.2. *Isolated vorticity region*

From now on we exclude the limiting cases given above by assuming that $\mathfrak{d} \in (0, d)$. In order to obtain dispersion relations for the flow prescribed by the discontinuous vorticity (3.2) we need to find a unique positive root for the cubic polynomial (3.11) which fulfills the condition (2.1h). Let us examine the cubic polynomial

$$\begin{aligned} p(x) = x^3 - \frac{d\gamma}{2p_0}\left(\tanh(d) + \frac{\sinh(d-2\mathfrak{d})}{\cosh(d)}\right)x^2 - \frac{gd^2}{p_0^2}\tanh(d)x \\ + \frac{gd^3\gamma}{2p_0^3}\left(1 - \frac{\cosh(d-2\mathfrak{d})}{\cosh(d)}\right) = 0. \end{aligned} \tag{3.12}$$

We note that

$$p(0)\begin{cases} < 0, & \gamma > 0, \\ > 0, & \gamma < 0. \end{cases}$$

Furthermore, upon substituting the value

$$x_0 = \frac{d\gamma}{2p_0}\left(\frac{1}{\tanh(d)} - \frac{\cosh(d-2\mathfrak{d})}{\sinh(d)}\right) \quad \begin{pmatrix} > 0, & \gamma > 0, \\ < 0, & \gamma < 0. \end{pmatrix}$$

into (3.12), we eliminate the two lower order terms and get

$$p(x_0) = x_0^2 \cdot \frac{d\gamma}{2p_0}\left(\frac{1}{\tanh(d)} - \frac{\cosh(d-2\mathfrak{d})}{\sinh(d)} - \tanh(d) - \frac{\sinh(d-2\mathfrak{d})}{\cosh(d)}\right)$$
$$= x_0^2 \cdot \frac{d\gamma}{2p_0}\frac{1-\cosh(2d-2\mathfrak{d})}{\sinh(d)\cosh(d)}\begin{cases} > 0, & \gamma > 0, \\ < 0, & \gamma < 0. \end{cases}$$

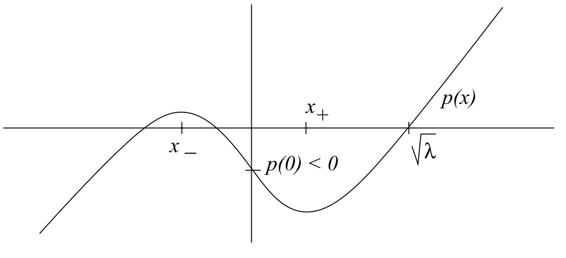
We have

$$p'(x) = 3x^2 - \frac{d\gamma}{p_0}\left(\tanh(d) + \frac{\sinh(d-2\mathfrak{d})}{\cosh(d)}\right)x - \frac{gd^2}{p_0^2}\tanh(d),$$

and since the discriminant of this quadratic is strictly positive, and the last term is strictly negative, so it follows that $p'(x)$ has one negative real root, x_-, which corresponds to a local maximum of $p(x)$, and a positive real root x_+, which is a local minimum of $p(x)$.

Region of positive constant vorticity: $\gamma > 0$

If $\gamma > 0$, then $p(x_0) > 0$ for $x_0 < 0$, and $p(0) < 0$. This implies that there is a unique positive root $\sqrt{\lambda}$ to $p(x) = 0$, since the graph of $p(x)$ must be of the form:



Cardano's formula [28, 30] gives the roots of the cubic polynomial $p(x)$ to be of the form

$$-\frac{A}{3} - \frac{\theta}{3}\left(\frac{m+\sqrt{n}}{2}\right)^{1/3} - \frac{\theta^2}{3}\left(\frac{m-\sqrt{n}}{2}\right)^{1/3}, \tag{3.13}$$

where

$$m = 2A^3 - 9AB + 27C, \quad n = m^2 - 4(A^2-3B)^3,$$

for

$$A = -\frac{d\gamma}{2p_0}\left(\tanh(d) + \frac{\sinh(d-2\mathfrak{d})}{\cosh(d)}\right),$$
$$B = -\frac{gd^2}{p_0^2}\tanh(d),$$
$$C = \frac{gd^3\gamma}{2p_0^3}\left(1 - \frac{\cosh(d-2\mathfrak{d})}{\cosh(d)}\right).$$

Here θ is a root of unity, and to find $\sqrt{\lambda}$ we choose the unique value $\theta \in \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ which gives us a positive root in (3.13). Using (3.1) we get

$$u^* - c = -\frac{A^*}{3} - \frac{\theta}{3}\left(\frac{m^* + \sqrt{n^*}}{2}\right)^{1/3} - \frac{\theta^2}{3}\left(\frac{m^* - \sqrt{n^*}}{2}\right)^{1/3}, \tag{3.14}$$

where

$$\begin{aligned} A^* &= -\frac{\gamma}{2}\left(\tanh(d) + \frac{\sinh(d - 2\mathfrak{d})}{\cosh(d)}\right), \\ B^* &= -g\tanh(d), \\ C^* &= \frac{g\gamma}{2}\left(1 - \frac{\cosh(d - 2\mathfrak{d})}{\cosh(d)}\right), \end{aligned}$$

and m^*, n^* take the same form as m, n above, except with A, B, C replaced by A^*, B^*, C^*. We observe from (3.11) that

$$\sqrt{\lambda}\left(\lambda - \frac{gd^2}{p_0^2}\tanh(d)\right) = \frac{d\gamma\sinh(d - \mathfrak{d})\cosh(\mathfrak{d})}{p_0\cosh(d)}\left(\lambda - \frac{gd^2}{p_0^2}\tanh(\mathfrak{d})\right), \tag{3.15}$$

and since $p_0 < 0$ it follows that

$$\sqrt{gd^2\tanh(\mathfrak{d})/p_0^2} < \sqrt{\lambda} < \sqrt{gd^2\tanh(d)/p_0^2},$$

or from (3.1) we get bounds on the dispersion relation as follows:

$$\sqrt{g\tanh(\mathfrak{d})} < c - u^* < \sqrt{g\tanh(d)}.$$

In particular, we see that the solution $\sqrt{\lambda}$ satisfies (2.1h). Also, we recover the dispersion relation for irrotational flow in the limiting case $\mathfrak{d} \to d$.

Region of negative constant vorticity: $\gamma < 0$

As we saw above, $p(0) > 0$ when $\gamma < 0$, and also $p(x_0) < 0$ for $x_0 > 0$. This implies that $p(x)$ has got two positive real roots and one negative one, since $p(x)$ must be of the form given by the figure below. What we now aim to show is that, of the two positive roots of $p(x)$, only the largest value is compatible with the non-stagnation condition (2.1h), and so this unique positive root $\sqrt{\lambda}$ leads, via (3.1), to the dispersion relation for our flow. It follows from (3.3) that the non-stagnation condition (2.1h) is equivalent to

$$\sqrt{\lambda} > \frac{\gamma d(d - \mathfrak{d})}{p_0}. \tag{3.16}$$

Since the first positive root of (3.12) must be smaller than $x_0 > 0$, if we show that $x_0 \leq \gamma d(d - \mathfrak{d})/p_0$ then this proves that there is a unique positive root $\sqrt{\lambda}$ of $p(x)$ which satisfies (2.1h), and the dispersion relation follows from (3.1). To show that this inequality, we must show that

$$2(d - \mathfrak{d})(e^d - e^{-d}) + e^{d-2\mathfrak{d}} + e^{2\mathfrak{d}-d} - e^d - e^{-d} \geq 0, \quad \text{for } d \geq \mathfrak{d} \geq 0.$$

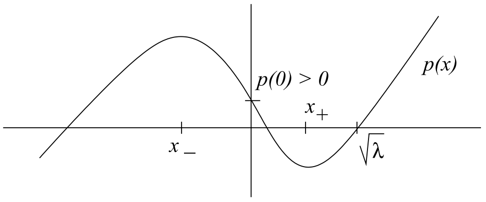


Fig. 1.

When $d = \mathfrak{d}$ we get zero on both sides of the above relation. Differentiating the left-hand side with respect to d we get

$$\begin{aligned} 2(e^d - e^{-d}) + 2(d-\mathfrak{d})(e^d + e^{-d}) + e^{d-2\mathfrak{d}} - e^{2\mathfrak{d}-d} - e^d + e^{-d} \\ = e^d - e^{-d} + 2(d-\mathfrak{d})(e^d + e^{-d}) + e^{d-2\mathfrak{d}} - e^{2\mathfrak{d}-d} > 0, \end{aligned}$$

since $\sinh(d) > \sinh(2\mathfrak{d} - d)$. Therefore $x_0 \leq \gamma d(d-\mathfrak{d})/p_0$ and we have shown that the only admissible solution to $p(x) = 0$ is the largest positive root, which we denote $\sqrt{\lambda}$ and which is found using Cardano's formula (3.13), giving us the dispersion relation (3.14).

Finally, we note that for $\gamma > 0$ local bifurcation always occurs, since there are no stagnation points (as we see from (3.3)). For $\gamma < 0$ things are a little more complicated. In this case, a necessary and sufficient condition that there be no stagnation points is given by (3.16), and since $x_0 \leq \frac{\gamma d(d-\mathfrak{d})}{p_0}$, we can see from looking at the graph of $p(x)$ in the second diagram in Fig. 1 that this necessary and sufficient condition is equivalent to $p(\frac{\gamma d(d-\mathfrak{d})}{p_0}) < 0$. We can rephrase this condition, using (3.11), as follows:

$$\begin{aligned} d^3\gamma^3(d-\mathfrak{d})^3\cosh(d) - d^3\gamma^3\cosh(\mathfrak{d})\sinh(d-\mathfrak{d})(d-\mathfrak{d})^2 \\ - gd^3\gamma\sinh(d)(d-\mathfrak{d}) + gd^3\gamma\sinh(\mathfrak{d})\sinh(d-\mathfrak{d}) > 0, \end{aligned}$$

or equivalently

$$|\gamma| < \frac{\sqrt{g(\sinh(d)(d-\mathfrak{d}) - \sinh(\mathfrak{d})\sinh(d-\mathfrak{d}))}}{(d-\mathfrak{d})\sqrt{(d-\mathfrak{d})\cosh(d) - \gamma^2\cosh(\mathfrak{d})\sinh(d-\mathfrak{d})}}.$$

We note that, upon taking the limit $\mathfrak{d} \to 0$, this necessary and sufficient condition for bifurcation particularizes to the equivalent condition for flows of constant negative vorticity which was first obtained in [12] (see also [18]), namely

$$\gamma^2 d^2 < (g + \gamma^2 d)\tanh(d).$$

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