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QUATERNIONIC INTEGRABILITY

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Dedicated to Antonio Degasperis on the occasion of his 70th anniversary

Standard (Arnold–Liouville) integrable systems are intimately related to complex rotations. One can define a generalization of these, sharing many of their properties, where complex rotations are replaced by quaternionic ones, and more generally by the action of a Clifford group. Such a generalization is not limited to integrable systems but — in the quaternionic case — goes over to a generalization of standard Hamilton dynamics.

Keywords: Integrable systems; quaternionic structures; hyperkahler structures; Clifford algebras; Pauli equation.

Introduction

Hamiltonian systems with compact energy manifolds which are integrable in the Arnold–Liouville sense are intimately related to a combination of uniform rotations; passing to action-angle variables, indeed, actions are constant and angle evolve with constant speed (different, in general, for different angles). Passing to complex coordinates — which is always possible locally, and also globally if the system is also Kahler — this is also seen as complex rotations acting on different complex coordinates.

It is quite natural, from this point of view, to expect that not much will change if complex rotations are replaced by quaternionic ones (when the phase space is of dimension $4n$). Our task in this note will be to develop this point of view, and discuss how the systems obtained in this way relate to — and differ from — standard Hamiltonian ones.

We will also use this point of view to go over to two further generalizations: first to use any Clifford algebra *in lieu* of the quaternionic one, albeit several features will be such to deny to these general systems the status of a true generalization of integrable Hamiltonian systems; and second to show how a quaternionic — or more precisely, *hyperkahler* — generalization of standard Hamilton mechanics is not only possible [6, 18] but also naturally obtained using these quaternionic integrable systems as a starting point.

1. Hamiltonian Integrable Systems

We start by considering Hamiltonian systems in n degrees of freedom, with compact energy manifolds, integrable in the Arnold–Liouville sense.

By definition, these can be mapped to an oscillators system, i.e.^a

$$\dot{p}_k = -\omega_k q_k, \quad \dot{q}_k = \omega_k p_k. \quad (1.1)$$

As well known, Arnold–Liouville integrable systems with n degrees of freedom are intimately related to symmetry under the group $U(1) \otimes \cdots \otimes U(1) = \mathbf{T}^n$.

If we pass to *action-angle coordinates* (I, φ) via the usual change of coordinates $p_k = \sqrt{I_k} \cos(\varphi_k)$, $q_k = \sqrt{I_k} \sin(\varphi_k)$, the evolution equations (1.1) read simply

$$\dot{I}_k = 0, \quad \dot{\varphi}_k = \omega_k(I), \quad (1.2)$$

and the \mathbf{T}^n symmetry is again immediately apparent.

We can also consider, instead of (I, φ) , *complex coordinates*

$$z_k = \sqrt{I_k} e^{i\varphi_k} = p_k + iq_k (k = 1, \dots, n); \quad (1.3)$$

each of them evolves as

$$\dot{z}_k = i\omega_k z_k, \quad (1.4)$$

which of course has solution $z_k(t) = e^{i\omega_k t} z_k(0)$.

The time evolution of an integrable system is thus given by a complex rotation (with constant speed for given initial conditions) in each \mathbf{C}^1 subspace; the frequency ω_k depends in general on all the $|z_k|^2 = I_k$ ($k = 1, \dots, n$).

It is natural to expect that not much would change if instead of a complex rotation we had a *quaternionic* one.

It is instead rather surprising that one can develop a coherent theory of dynamical systems which are in a way a quaternionic generalization of standard Hamiltonian systems. These are more precisely related to hyperkahler structures [2, 4, 10, 12, 17], and thus will be therefore called *hyperhamiltonian* [6, 18]. This hyperhamiltonian dynamics appears to provide a natural description of dynamics related to spin degrees of freedom, as described by the Pauli and the Dirac equations [8, 9].

It turns out that — albeit the quaternionic case is, in a way to be discussed below, the only possible “full” generalization of standard Hamiltonian dynamics — other generalizations are related to **Clifford algebras**. In this case we do not have complete integrability, but a “conditional” one, i.e. integrability on some (dynamically invariant) submanifold of the whole phase space; this corresponds roughly speaking to a situation already known to Levi-Civita (he referred to the constant of motion arising in this way as “invariant relations”) [16] and studied by different authors also in modern times under the name of “configurational invariants”, see e.g. [19, 20].

^aHere and in the whole paper, no sum on repeated indices will be implied.

2. Generalization: Clifford Integrable Systems

We want to write (1.1) in yet another, slightly different, form. We introduce k two-dimensional real vectors $\xi_k = (p_k, q_k)$ and denote by J the standard two-dimensional symplectic matrix; then the evolution (1.1), or equivalently (1.4), reads

$$\dot{\xi}_k = \omega_k(|\xi_1|, \dots, |\xi_n|)J\xi_k. \quad (2.1)$$

When writing the dynamics in this way, its integration goes through the observations that (i) $|\xi_k|$ is constant (due to $J^T = -J$), and (ii) $J^2 = -I$; it follows from these two facts that $\exp[\omega_k J] = \cos(\omega_k t) + J \sin(\omega_k t)$. Hence the solution $\xi_k(t) = \exp[\omega_k Jt]\xi_k(0)$ to (2.1) reads simply

$$\xi_k(t) = [\cos(\omega_k t)I + \sin(\omega_k t)J] \xi_k(0). \quad (2.2)$$

Once we have looked at oscillator dynamics in such an elementary way, it is easy to find a direct (but not entirely trivial) generalization.

Consider now m -dimensional real vectors $\xi_k \in \mathbf{R}^m$ ($k = 1, \dots, n$) and write $\rho_k = |\xi_k|^2$. Introduce the evolution equations

$$\dot{\xi}_k = \sum_{\alpha=1}^p \nu_{k\alpha} K_{\alpha} \xi_k \quad (2.3)$$

with $\nu_{k\alpha} = \nu_{k\alpha}(\rho_1, \dots, \rho_p)$ smooth functions, K_{α} ($\alpha = 1, \dots, p$) m -dimensional matrices satisfying

$$K_{\alpha}^T = -K_{\alpha}; \quad \{K_{\alpha}, K_{\beta}\} = -2\delta_{\alpha\beta}I. \quad (2.4)$$

We will denote by \mathbf{K} the Lie algebra spanned by the K_{α} ; Eq. (2.4) states that \mathbf{K} is a *Clifford algebra*, with fundamental quadratic form $-I$ [13, 15].

The ρ_k are conserved, due to $K_{\alpha}^T = -K_{\alpha}$; thus we can consider the $\nu_{k\alpha}$, and hence the whole right-hand side of (2.3), as constant on each trajectory of the system.

Remark 1. We could as well consider matrices K_{α} which depend on the ρ_i and satisfy, for all values of the ρ , the conditions (2.4); in this case we should consider the (Clifford) module generated by them. Note that, as the ρ_i are constant on the dynamics, these matrices will however be constant on each realization of the dynamics. We will only consider constant K_{α} , for ease of discussion.

The solution to (2.3) is of course $\xi_k(t) = \exp[\nu_{k\alpha} K_{\alpha} t] \xi_k(0)$. Writing

$$\omega_k = \left(\sum_{\alpha=1}^p \nu_{k\alpha}^2 \right)^{1/2}, \quad A_k := \sum_{\alpha=1}^p \frac{\nu_{k\alpha}}{\omega_k} K_{\alpha}, \quad (2.5)$$

and denoting by I the identity matrix, we have at once that

$$\xi_k(t) = [\cos(\omega_k t)I + \sin(\omega_k t)A_k] \xi_k(0). \quad (2.6)$$

Thus, to any Clifford algebra \mathbf{K} we can associate a generalization of oscillator dynamics, integrable (explicitly and elementarily solvable) by construction. We will also refer to these as **Clifford integrable systems**.

The $\xi_k(t)$ have constant norm under this dynamics; the ρ_k play the role of action variables I_k . In this scheme, angles are replaced by variables on S^{m-1} , and more precisely on the Lie group \mathcal{K} generated by \mathbf{K} , i.e. the corresponding Clifford group.

Remark 2. It should be stressed that in general the space $\mathbf{K}(\xi)$ is a proper subspace of $T_\xi S^{m-1}$ (i.e. $p < m + 1$). This means that not all the directions of motion on S^{m-1} are allowed for a given initial position. In other words, if the configuration space is $M = \mathbf{R}^{n \cdot m}$, the phase space will not be the whole tangent bundle TM , but a proper subspace of it: not all the initial velocities are allowed, but (for any initial position $\xi_0 \in M$) only those along the directions identified by $\mathbf{K}\xi_0$.

Remark 3. In the framework described in the previous remark, in order to have a dynamics defined on the whole of TM we should have some additional components of the dynamical vector fields (not belonging to the Clifford algebra); the submanifolds on which these vanish — assuming they are dynamically invariant — would lead to the appearance of configurational invariants [19, 20], or invariant relations [16], as mentioned above.

The cases in which $\mathbf{K}(\xi) = T_\xi S^{m-1}$ (i.e. $p = m - 1$) and all initial velocities are allowed, correspond to the existence of a Clifford algebra of dimension $m - 1$ acting in \mathbf{R}^m ; a necessary (but not sufficient) condition for this to happen is that the sphere S^{m-1} is parallelizable. This happens only for S^1, S^3, S^7 [13, 15].

The case $m = 2$, S^1 corresponds to standard Hamilton dynamics; in the case $m = 4$, S^3 we are dealing with $S^3 \subset \mathbf{R}^4$ and with the Clifford algebra $Cl(2)$, isomorphic to the quaternion algebra \mathbf{H} ; in the following we will concentrate on this case.

The case $m = 8$ has not been explored yet as for the corresponding extension of Hamilton dynamics; note however that the corresponding Clifford algebra ($Cl(3) \approx \mathbf{H} \oplus \mathbf{H}$) does *not* parallelize S^7 .

Thus the quaternionic case is the *only* full extension of standard Hamilton integrable dynamics (and, as we will mention in a moment, see Sec. 4, also non-integrable dynamics) along the lines considered here.

3. Quaternionic Integrable Systems

The first extension of standard Hamiltonian integrable systems along the lines sketched above would correspond to $Cl(2) \simeq \mathbf{H}$, i.e. to *quaternionic* systems; we are going to deal in some detail with this case.

We want to set our systems in the form (2.3), i.e. in \mathbf{R}^{4n} ; we should hence give a representation of the quaternionic imaginary units i, j, k over \mathbf{R} . This is e.g. provided by the matrices

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (3.1)$$

These are a real representation of the Pauli matrices and satisfy the quaternion relations

$$K_\alpha K_\beta = \varepsilon_{\alpha\beta\gamma} K_\gamma - \delta_{\alpha\beta} I. \quad (3.2)$$

In this case the evolution is given by^b

$$\dot{\xi}_k = \sum_{\alpha=1}^3 \nu_{\alpha k}(\rho) K_\alpha \xi_k. \quad (3.3)$$

The general solution, see (2.6), reads

$$\xi_k(t) = [\cos(\omega_k t) I + \sin(\omega_k t) K] \xi_k(0). \quad (3.4)$$

For $\xi_k(0) \neq 0$, this describes great circles S^1 on the sphere S^3 of radius $\sqrt{\rho_k} = |\xi_k(0)|$. (This dynamics realizes the *Hopf fibration* $S^1 \rightarrow S^3 \rightarrow S^2$.) Note that the K_α are constant, and the dependence on the ξ_k (actually, on the $\rho_k = |\xi_k|^2$) is only through the scalar functions $\nu_{\alpha k}$.

We can rewrite (3.3) in a slightly different form; we will now set $n = 1$ for ease of discussion. Introduce three functions $\mathcal{H}^\alpha : \mathbf{R}^4 \rightarrow \mathbf{R}$, with $\mathcal{H}^\alpha = h^\alpha(\rho) \in F$ (where F is some suitable space of smooth functions); now $\nabla \mathcal{H}^\alpha = f^\alpha(\rho) \xi$, and we can rewrite (3.3) as

$$\dot{\xi} = \sum_{\alpha=1}^3 K_\alpha \nabla \mathcal{H}^\alpha, \quad (3.5)$$

which makes clear the relation with the Hamiltonian case.

The flow X described by (3.5) can be seen as the superposition of three Hamiltonian flows X_α , each of them defined by the Hamiltonian \mathcal{H}^α with the symplectic structure ω_α associated to K_α , see above. The X_α do not commute, but generate a module over F .

4. Hyperhamiltonian Dynamics

The Eq. (3.5) can be taken as the starting point for the extension of this setting to the non-integrable case [6, 7], which we briefly recall for the sake of completeness. This can be defined on an arbitrary Riemannian manifold (M, g) of dimension $4n$ equipped with a *hyperkahler structure* $\{Y_1, Y_2, Y_3\}$ [2, 4, 10, 12, 17]. Here the Y_α are almost complex structures, covariantly constant under the Levi-Civita connection defined by the metric g , satisfying the quaternionic relations $Y_a Y_b = \epsilon_{abc} Y_c - \delta_{ab} I$.

The Kahler relations associates a symplectic form to each Y_α ,

$$\omega_\alpha(v, w) := g(Y_\alpha v, w). \quad (4.1)$$

Consider an ordered triple of arbitrary smooth functions $\mathcal{H}^\alpha : M \rightarrow \mathbf{R}$; we associate to these a triple of vector fields by

$$X_\alpha \lrcorner \omega_\alpha = d\mathcal{H}^\alpha \quad (4.2)$$

^bNote that the vector field defined by (3.3) does not belong to the Clifford algebra \mathbf{K} , but instead to the *Clifford module* [3] generated by it; however, as the ρ_k are constant on the flow, the vector field is in \mathbf{K} for any realization of the flow.

and define the *hyperhamiltonian vector field* X on M associated to the triple $\{\mathcal{H}^\alpha\}$ as the sum of these,

$$X := \sum_{\alpha=1}^3 X_\alpha. \quad (4.3)$$

In local coordinates, $X_\alpha = (K_\alpha \nabla \mathcal{H}^\alpha)^i \partial_i$, and the hyperhamiltonian vector field is

$$X = \left(\sum_{\alpha} \sum_j K_\alpha^{ij} \partial_j \mathcal{H}^\alpha \right) \partial_i. \quad (4.4)$$

Hyperhamiltonian dynamics shares many properties with standard Hamilton dynamics; in particular a variational formulation, based on maximal degree forms [6].

5. Hamiltonian Versus Hyperhamiltonian Dynamics

Two natural questions immediately arise after defining hyperhamiltonian dynamics: (i) is this really more general than Hamiltonian one? (ii) are quaternionic integrable systems more general than Hamiltonian ones? In this and the next section, we will try to briefly answer these.

As for the first, we note that every Hamiltonian system is trivially hyperhamiltonian: in the hyperhamiltonian framework, it suffices to set two of the three Hamiltonian functions \mathcal{H}_α equal to zero to recover the standard Hamiltonian case. On the other hand, there could be systems which are hyperhamiltonian but cannot be written in Hamiltonian form with respect to any symplectic structure. In order to show this, we recall a result by Giordano, Marmo and Rubano [11]: *Given a linear vector field $X = A_j^i x^j \partial_i$, if there is $k \in N$ such that $\text{Tr}(A^{2k+1}) \neq 0$, then X is not Hamiltonian with respect to any symplectic structure.*

The vanishing of $\text{Tr}(A)$ corresponds to the condition of zero divergence, which is also satisfied by hyper-Hamiltonian flows. Thus we have to find an example where $\mathcal{H}_\alpha = (1/2)(D_\alpha)_{ij} x_i x_j$ (with D_α symmetric matrices, and writing all indices as lower ones to avoid confusion with powers) and $A := \sum_{\alpha} K_\alpha D_\alpha$ satisfies $\text{Tr}(A^3) = 0$. This is obtained e.g. if $\mathcal{H}_1 = (1/2)[(x_1)^2 - (x_2)^2 + (x_3)^2 - (x_4)^2 + 2(x_1 x_4 - x_2 x_3)]$, $\mathcal{H}_2 = (1/2)|x|^2$ and $\mathcal{H}_3 = 0$.

Thus we have shown that: *There are Hyperhamiltonian vector fields which are not Hamiltonian with respect to any symplectic structure.*

6. Hamiltonian Versus Hyperhamiltonian Integrability

We would like to discuss the relation between hyperhamiltonian integrability and standard Hamiltonian integrability for the class of systems considered here. It will be convenient to mainly restrict to the case of dimension four, as this will suffice to make our point; see the last subsection for higher dimension.

6.1. Quaternionic oscillators

It may be useful to first discuss the case given by $\mathcal{H}^1 = |x|^2/2$, $\mathcal{H}^2 = \mathcal{H}^3 = 0$; this corresponds to two uncoupled and identical harmonic oscillators with conserved energies $E_a = (1/2)[(x^1)^2 + (x^2)^2]$ and $E_b = (1/2)[(x^3)^2 + (x^4)^2]$. The solutions of nonzero energy

$E = E_a + E_b = r_0^2/2$ describe a circle S^1 lying on the sphere S^3 of radius r_0 . When E_a and E_b are both nonzero (i.e. both oscillators are actually excited) these also lie on a torus $\mathbf{T}^2 \subset S^3$, and the circle S^1 corresponding to the solution is a combination of the two fundamental cycles of the torus. The cases $\{E_a = 0, E_b \neq 0\}$ and $\{E_a \neq 0, E_b = 0\}$ correspond to degenerate situations in which the common level set of E_a and E_b is not a torus \mathbf{T}^2 , but is reduced to a circle $\mathbf{T}^1 = S^1$, which is just the trajectory of the solution. Needless to say, these two ways (hyperhamiltonian and standard Hamiltonian) of describing the situation are immediately related, as it should be.

Remark 4. We have mentioned above the Hopf fibration of S^3 ; it should be recalled that this can indeed be described as a singular fibration of S^3 in \mathbf{T}^2 tori, with two singular fibers, corresponding to the special cases in which all the energy is on one oscillator and the other is not excited.

6.2. General quaternionic integrable systems

Let us now consider the general (nonlinear) integrable case, with $\mathcal{H}_\alpha = \mathcal{H}_\alpha(\rho)$ but with possibly different functional dependence on ρ for the three Hamiltonians; on each S^3 sphere of radius $r_0 \neq 0$, i.e. on each nonzero level manifold (we can speak of energy level manifolds as the three Hamiltonians depend on a single scalar function ρ) for the energy $E = \rho$ we can indeed reduce to a two-oscillators description. Such a system is integrable in the Arnold–Liouville sense, since the set on which the fibration in tori is singular is of zero measure in the phase space.

In this sense, *when we restrict to a given invariant sphere*, Hyperhamiltonian integrable systems are not more general than standard Hamiltonian ones.

However, two points should be stressed, one local staying on a given sphere and the other global.

6.2.1. Flow on an invariant sphere

The “local” point is that it should be noticed that in considering this system as an integrable two-oscillator system, we are completely overlooking the quaternionic structure of the system. In particular, this system is strongly degenerate if seen in terms of two oscillators: indeed the two oscillators are in 1:1 resonance for all values of H , i.e. all values of the action variables $I_1 = E_a$ and $I_2 = E_b$. Such a degeneration is of course enforced by the quaternionic structure, and thus generic in the frame of “quaternionic oscillators”.

On the other hand, if we recognize the quaternionic structure and the fact that we need therefore only the global constant of motion ρ to guarantee integrability, we have at once a much stronger information on the structure of the system and also need an easier construction to guarantee integrability. The situation is similar to the one met when we represent a quaternion by a pair of complex numbers (or a complex number by a pair of real ones): this is possible and correct, but in this way we are overlooking an additional and relevant structure, which we must then introduce by suitable relations between complex (or real) quantities.

Thus, in order to guarantee integrability in the sense of standard Hamiltonian mechanics we need two constants of motion and we have to construct a system of two action and two

angle coordinates; using the quaternionic structure we only need one constant of motion, i.e. ρ , and we have to construct a system of coordinates in which to the “action” coordinate ρ are associated three coordinates on the sphere S^3 ; as $S^3 \simeq \mathbf{H}^1 \simeq SU(2)$ (here \mathbf{H}^1 is the set of quaternions of unit norm), these are of quaternionic nature. We call them *spin coordinates*.^c

6.2.2. Flow in the full phase space

The equivalence between the families of hyperhamiltonian and of Hamiltonian systems was established only when we consider the restriction to a given invariant sphere.

We will now show, by means of an explicit (and very simple) example, that when we consider the full phase space there are integrable hyperhamiltonian flows which cannot be globally described in terms of Hamiltonian ones. In order to do so, we will consider R^4 with Euclidean metric, and the Hypercomplex structure given by the constant matrices (3.1). As Hamiltonians, we choose

$$\mathcal{H}_1 = \rho, \quad \mathcal{H}_2 = \rho^2/2, \quad \mathcal{H}_3 = 0; \quad (6.1)$$

we will also write, for later reference, $f_\alpha = (d\mathcal{H}_\alpha/d\rho)$; hence $f_1 = 1$, $f_2 = \rho$, $f_3 = 0$. The hyperhamiltonian flow is hence given by

$$X = \sum_{\alpha} K_{\alpha} \nabla \mathcal{H}_{\alpha} = \sum_{\alpha} f_{\alpha} K_{\alpha} \xi = (K_1 \xi + \rho K_2 \xi)^i \partial_i. \quad (6.2)$$

As for the symplectic structure, we know it may be written as

$$\omega = \frac{1}{2} A_{ij}(x) dx^i \wedge dx^j \quad (6.3)$$

for some antisymmetric matrix field A (subject to some further constraint as we should require $d\omega = 0$; we will look at these later on). We thus have

$$X \lrcorner \omega = \sum_{\alpha} (K_{\alpha})^{i\ell} (\partial_{\ell} \mathcal{H}_{\alpha}) A_{ij} dx^j = - \sum_{\alpha} [f_{\alpha} A_{im} (K_{\alpha})^{mj} x_j] dx^i := W_i dx^i. \quad (6.4)$$

If X is Hamiltonian with respect to the symplectic structure ω , we have $X \lrcorner \omega = dH$ for some scalar function H ; as we are in R^4 , the existence of such H is equivalent to

$$d(X \lrcorner \omega) = 0. \quad (6.5)$$

It follows from (6.4) that

$$d(X \lrcorner \omega) = \frac{1}{2} (\partial_i W_j - \partial_j W_i) dx^i \wedge dx^j, \quad (6.6)$$

hence we are reduced to looking for solutions to the system of equations

$$\partial_i W_j - \partial_j W_i = 0 \quad \forall i, j. \quad (6.7)$$

^cNotice that the evolutions along spin coordinates do not commute; thus the equivalent of the familiar integrable Hamiltonian evolution equations $\dot{I}_k = 0$, $\dot{\varphi}_k = \omega_k(I)$, related to the abelian group \mathbf{T}^2 , is now given by $\dot{I} = 0$ ($I \equiv \rho$), $\dot{\psi} = \alpha(I)$, where ψ represents coordinates on the group $SU(2) \simeq S^3$, and $\alpha(I) \in su(2)$ is an element of the algebra $su(2)$, constant on each level set of $I \equiv \rho$. This more involved (and not separable) structure is unavoidable, due to the non-abelian nature of $SU(2)$.

Note the unknowns are here coded in the (antisymmetric) matrix function A . As we are in dimension four, A can be described by means of six scalar functions $c_i(\xi)$, and we should of course require these are not all zero. However, we know that A will be constant on each sphere of radius ρ ; we can therefore assume $c_i = c_i(\rho)$.

We want now to show that with the specific choice of \mathcal{H}_α , and hence of the f_α , given above the system (6.7) does not admit nontrivial solutions.^d

In order to do so, we first write A in the form

$$A = \begin{pmatrix} 0 & c_1(\rho) & c_2(\rho) & c_3(\rho) \\ -c_1(\rho) & 0 & c_4(\rho) & c_5(\rho) \\ -c_3(\rho) & -c_4(\rho) & 0 & c_6(\rho) \\ -c_3(\rho) & -c_5(\rho) & -c_6(\rho) & 0 \end{pmatrix}; \quad (6.8)$$

with this we can explicitly compute the

$$P_{ij} := (\partial_i W_j - \partial_j W_i); \quad (6.9)$$

note that now all the unknown functions depend on ρ alone. Other types of dependencies on the x^i are explicit, and we can proceed by requiring the coefficient of any polynomial in the x^i (other than ρ itself) to vanish.

Proceeding in this way, we first compute P_{12} (which we do not write explicitly). The coefficient of $x_1 x_3$ in it is just $c'_5(\rho)$, so we must require $c_5(\rho) = \kappa_5$ (we will always denote by κ_i arbitrary constants, and omit from now on to indicate the dependence of the c_i on ρ); with this choice the coefficient of x_1^2 in P_{12} reads κ_5 , i.e. we must have $\kappa_5 = 0$ and hence $c_5 = 0$. Similarly, the coefficient of $x_2 x_4$ in P_{12} reads now c'_2 , and upon setting $c_2 = \kappa_2$ the coefficient of x_2^2 in P_{12} reads κ_2 , so we get $\kappa_2 = 0$ and hence $c_2 = 0$. The coefficient of $x_2 x_3$ reads now $c_1 + \rho c'_1 - c'_3$, which yields $c_3 = \rho c_1$. The coefficient of $x_1 x_4$ is $c_1 + \rho c'_1 - c'_4$ and hence we get also $c_4 = \rho c_1$. At this point we have $P_{12} \equiv 0$.

We pass to consider P_{14} ; the coefficient of $x_1 x_4$ in it reads $c'_6 - c'_1$, and hence we have $c_6 = c_1 + \kappa_6$; with this, the coefficient of $x_1 x_2$ is κ_6 , and hence we get $c_6 = \kappa_6 = 0$. At this point we have $P_{14} \equiv 0$, and actually all of the P_{ij} vanish identically.

The resulting antisymmetric matrix A is

$$A = \begin{pmatrix} 0 & c_1 & 0 & \rho c_1 \\ -c_1 & 0 & \rho c_1 & 0 \\ 0 & -\rho c_1 & 0 & c_1 \\ -\rho c_1 & 0 & -c_1 & 0 \end{pmatrix}. \quad (6.10)$$

This identifies a two-form ω via (6.3); more precisely, we get

$$\omega = c_1(\rho)[\omega_1 + \rho\omega_2]. \quad (6.11)$$

Note that choosing a function c_1 which is nowhere zero, e.g. $c_1(\rho) = 1$, we are guaranteed ω is nondegenerate. The form ω satisfies, by construction, $d(X \lrcorner \omega) = 0$ and hence

^dA more elegant proof of the fact (6.7) does not always admit solutions surely exists, but as we are just looking for an example showing this is the case we will be satisfied with a proof by explicit computation.

$d(X \lrcorner \omega) = dH$ for some function H . However, we have not yet required that the ω built in this way is a *closed* form — which is necessary for it to be a symplectic form — i.e. $d\omega = 0$.

It follows by tedious straightforward algebra that

$$\begin{aligned} d\omega = & [x_3 c'_1 + x_1 (c_1 + \rho c'_1)] dx^1 \wedge dx^2 \wedge dx^3 \\ & + [x_4 c'_1 - x_2 (c_1 + \rho c'_1)] dx^1 \wedge dx^2 \wedge dx^4 \\ & + [x_1 c'_1 - x_3 (c_1 + \rho c'_1)] dx^1 \wedge dx^3 \wedge dx^4 \\ & + [x_2 c'_1 + x_4 (c_1 + \rho c'_1)] dx^2 \wedge dx^3 \wedge dx^4; \end{aligned}$$

recalling that $c_1 = c_1(\rho)$, it is immediately clear that $d\omega = 0$ if and only if $c_1 = 0$, i.e. if and only if $\omega = 0$.

We have thus shown that in this case the hyperhamiltonian vector field X in R^4 cannot be described in Hamiltonian terms, for any symplectic structure. In other words, *there are hyperhamiltonian integrable systems which are not Hamiltonian*.

6.3. Higher dimension

We would finally like to briefly discuss the case of quaternionic oscillators in higher dimensional spaces R^{4n} ($n > 1$). In the standard Hamiltonian integrable case with m degrees of freedom we have invariant \mathbf{T}^m tori, and the solutions will cover densely $\mathbf{T}^k \subset \mathbf{T}^m$ tori, with $k \leq m$ depending on the rational relations between the frequencies; in the hyperhamiltonian integrable case (for n quaternionic oscillators) we have a similar situation, as we now discuss.

First of all we remark that, since $\rho = (\rho_1, \dots, \rho_n)$ are constants of motion, the common level sets of the ρ_p are invariant manifolds under the dynamics we are considering; these level sets $\rho^{-1}(b_1, \dots, b_n)$ will be, when all the b_p are nonzero, manifolds

$$\mathcal{V}^n := S^3 \times \dots \times S^3 = (S^3)^{\times n};$$

these \mathcal{V}^n represent a generalization of tori, in that in the same way as \mathbf{T}^n is the topological product of n (distinct) S^1 factors, \mathcal{V}^n is the topological product of n (distinct) S^3 factors. If k out of the n numbers b_p are zero, the level set $\rho^{-1}(b_1, \dots, b_n)$ will be a \mathcal{V}^{n-k} manifold.

Consider the trajectory with initial datum $x(0)$. The previous discussion shows that the projection of this to each \mathbf{R}^4 block, given by $\xi_{(p)}(t)$, will be periodic.

If $m \leq n$ degrees of freedom are excited and the m frequencies corresponding to $b_p \neq 0$ split in $k \leq m$ sets, each ν_p being rational with respect to frequencies in the same set and irrational with respect to frequencies in different sets, the solutions γ will densely cover tori $\mathbf{T}^k \subset \mathcal{V}^m$. We can always choose the generators S^1 of these \mathbf{T}^k so that each one lies in a different factor S^3 for \mathcal{V}^m .

7. Example. The Pauli Spin Equation

We will now briefly discuss a physically relevant equation which can be set in hyperhamiltonian form, and which under suitable conditions (spatially homogeneous magnetic field) corresponds to a quaternionic oscillator system.

The nonrelativistic evolution equation for particles with spin one-half is provided by the Pauli equation. Considering only the spin degrees of freedom, this is written as

$$\frac{d\Psi}{dt} = i\kappa(\mathbf{B} \cdot \mathbf{S})\Psi. \quad (7.1)$$

Here $\kappa = 4\pi\mu/h$ is a dimensional constant ($\kappa = 1$ in the following), Ψ is a two-components spinor,

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi_{\pm}(t) \in \mathbf{C}, \quad \|\Psi\|^2 = 1,$$

the real vector $\mathbf{B}(t)$ is the magnetic field, and \mathbf{S} is the vector spin operator with components the Pauli σ matrices. The linear operator $\mathbf{M} := \mathbf{B} \cdot \mathbf{S}$ is given by

$$\mathbf{M} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}.$$

To set (7.1) in \mathbf{R}^4 , rewrite ψ_{\pm} as $\psi_{\pm} = \chi_{\pm} + i\zeta_{\pm}$; representing a \mathbf{C}^1 number by an \mathbf{R}^2 vector, we get

$$\psi_{\pm} = \begin{pmatrix} \chi_{\pm} \\ \zeta_{\pm} \end{pmatrix}.$$

The operator of multiplication by i is represented in \mathbf{R}^2 by the standard symplectic matrix, and we can use this to write $i\mathbf{M}$ as a real four-dimensional matrix (which we do in block notation):

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i\mathbf{M} \approx \begin{pmatrix} B_z J & B_y I + B_x J \\ -B_y I + B_x J & -B_z J \end{pmatrix}.$$

Finally, the \mathbf{R}^4 representation of the Pauli equation is given by

$$\frac{d\xi}{dt} = A\xi, \quad (7.2)$$

where

$$\xi = \begin{pmatrix} \chi_+ \\ \zeta_+ \\ \chi_- \\ \zeta_- \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -B_z & B_y & -B_x \\ B_z & 0 & B_x & B_y \\ -B_y & -B_x & 0 & B_z \\ B_x & -B_y & -B_z & 0 \end{pmatrix}. \quad (7.3)$$

We can rewrite A in terms of the matrices \hat{K}_{α} as

$$A(t) = B_y(t)\hat{K}_1 + B_x(t)\hat{K}_2 + B_z(t)\hat{K}_3. \quad (7.4)$$

Therefore, from (7.3) and (7.4), the (\mathbf{R}^4 representation of the) Pauli equation can be described as a hyperhamiltonian system, with

$$\begin{aligned}\mathcal{H}^1(\xi, t) &= (1/2) B_y(t) \|\xi\|^2, \\ \mathcal{H}^2(\xi, t) &= (1/2) B_x(t) \|\xi\|^2, \\ \mathcal{H}^3(\xi, t) &= (1/2) B_z(t) \|\xi\|^2.\end{aligned}$$

When $d\mathbf{B}/dt = 0$ we have an integrable system. If \mathbf{B} varies with t , we have explicitly time-dependent Hamiltonians $\mathcal{H}^\alpha(|\xi|^2; t)$: the system is not integrable, but $|\xi|$ is still constant.

A similar but more complex construction allows to describe the Dirac equation for a particle in terms of (nonintegrable) hyperhamiltonian dynamics [8].

8. Higher Order Clifford Algebras

A generalization of Hamilton dynamics somehow similar to hyperhamiltonian dynamics can be defined on Kahler–Clifford manifolds; that is, on Riemannian manifolds (M, g) of dimension m equipped with k complex structures Y_α which satisfy the relations of a Clifford algebra, i.e. such that $Y_\alpha^2 = -I$ (which is required by being complex structures) and $\{Y_\alpha, Y_\beta\} = 2\delta_{\alpha\beta}I$. These have been studied by Joyce [14] building on previous investigations by Atiyah, Bott and Shapiro [3]; see also [5].

In this case the Kahler relation again associates to each Y_α a symplectic structure ω_α , and defining k Hamiltonian functions $\mathcal{H}_\alpha : M \rightarrow R$, we have k Hamiltonian vector fields X_α defined by $X_\alpha \lrcorner \omega_\alpha = d\mathcal{H}_\alpha$. Then the Clifford–Hamilton flow would be defined as

$$X = \sum_{\alpha=1}^k X_\alpha; \tag{8.1}$$

if the Y_α are represented in local coordinates x^i by matrices K_α , then the dynamics will be described by

$$\dot{x}^i = \sum_{\alpha} (K_\alpha)^{ij} (\nabla_j \mathcal{H}_\alpha). \tag{8.2}$$

This Clifford–Hamilton dynamics would share many of the properties of hyperhamiltonian (and hence standard Hamilton) dynamics; however the problems stressed in Remarks 2 and 3 are still present, and prevent from considering this as a meaningful generalization of Hamilton dynamics.

9. Conclusions

We have introduced and characterized quaternionic integrable systems, discussing their relations and differences with standard (Arnold–Liouville) Hamiltonian integrable systems. We have also shown how these quaternionic integrable systems would be a natural starting point for hyperhamiltonian dynamics [6].

It was also shown by a concrete example that — in the same way as not all the hyperhamiltonian systems are Hamiltonian — there are quaternionic integrable systems in R^4 which cannot be set as Hamiltonian ones with respect to any symplectic structure, albeit on any invariant sphere S^3 a standard Hamiltonian description is possible.

The possibility of such a quaternionic extension uses not only the $SU(2)$ commutation rules, but also relies essentially on the associative algebra structure of quaternions. We have observed that similar extensions — both in the integrable and in the general case — can be associated to any Clifford algebra [13, 15]; however this holds only formally, in that the dynamics so defined concerns only certain submanifolds of the full phase space. In particular, for the integrable case we would have a “conditional” integrability if the system in the full phase space is written in Clifford form on these (invariant) submanifolds.

Finally, we note that it would be quite interesting to find examples in which the structure investigated here holds for an infinite dimensional system, i.e. examples of quaternionic integrability for PDEs.

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