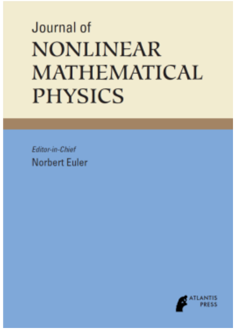

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 ISSN (Print): 1402-9251
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

Linearization of a Second-Order Stochastic Ordinary Differential Equation

Sergey V. Meleshko, Eckart Schulz

To cite this article: Sergey V. Meleshko, Eckart Schulz (2011) Linearization of a Second-Order Stochastic Ordinary Differential Equation, Journal of Nonlinear Mathematical Physics 18:3, 427–441, DOI: https://doi.org/10.1142/S1402925111001696

To link to this article: https://doi.org/10.1142/S1402925111001696

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 3 (2011) 427–441

DOI: 10.1142/S1402925111001696

LINEARIZATION OF A SECOND-ORDER STOCHASTIC ORDINARY DIFFERENTIAL EQUATION

SERGEY V. MELESHKO* and ECKART SCHULZ†

School of Mathematics, Suranaree University of Technology
Nakhon Ratchasima 30000, Thailand
and
Center of Excellence in Mathematics, CHE
Si Ayutthaya Rd., Bangkok 10400, Thailand
*sergey@math.sut.ac.th
†eckart@math.sut.ac.th

Received 31 January 2011
Accepted 14 March 2011

Necessary and sufficient conditions which allow a second-order stochastic ordinary differential equation to be transformed to linear form are presented. The transformation can be chosen in a way so that all but one of the coefficients in the stochastic integral part vanish. The linearization criteria thus obtained are used to determine the general form of a linearizable Langevin equation.

Keywords: Brownian motion; linearization; stochastic ordinary differential equation.

Mathematics Subject Classification 2000: 60H10

1. Introduction

Physical phenomena of interest in science are very often simulated by means of models which correspond to differential equations. These equations are in general nonlinear, and their solutions difficult to obtain. In addition, many of these mathematical models are given in terms of stochastic nonlinear differential problems. In chemistry and physics for example, one frequently encounters models based on the second order equation

$$\ddot{X} = f(t, X, \dot{X}) + g(t, X, \dot{X})\,\dot{W} \tag{1.1}$$

where $\dot{W}$ is white noise. The second order Langevin equation

$$\ddot{X} = f(t, X, \dot{X}) + \sigma\dot{W} \tag{1.2}$$

is the simplest of these, and it describes the motion of a particle in a noise-perturbed force field. In particular, for the case of the harmonic oscillator,

$$f(t, x, \dot{x}) = -\nu^2 x - \beta\dot{x},$$

Eq. (1.2) becomes a linear second-order stochastic differential equation. The Langevin equation is also encountered frequently in the theory of lasers, chemical kinetics and population dynamics.

While solving problems involving deterministic differential equations it is often expedient to simplify an equation by a suitable change of variables. The simplest form of a second-order ordinary differential equation

$$\ddot{x} = f(t, x, \dot{x})$$

is the linear form. Sophus Lie [7][a] showed that this equation is linearizable by a change of both the independent and dependent variables if, and only if, f is a polynomial of third degree with respect to the first-order derivative,

$$\ddot{x} + a\dot{x}^3 + b\dot{x}^2 + c\dot{x} + d = 0,$$

where the coefficients $a(t, x)$, $b(t, x)$, $c(t, x)$ and $d(t, x)$ satisfy the conditions

$$\begin{aligned} L_1 &= 3a_{tt} - 2b_{tx} + c_{xx} - 3a_t c + 3a_x d + 2b_t b - 3c_t a - c_x b + 6d_x a = 0, \\ L_2 &= b_{tt} - 2c_{tx} + 3d_{xx} - 6a_t d + b_t c + 3b_x d - 2c_x c - 3d_t a + 3d_x b = 0. \end{aligned} \tag{1.3}$$

In addition, it is linearizable by a change of the dependent variable x only if, and only if, $a = 0$ in which case conditions (1.3) become

$$\begin{aligned} L_1 &= (2f_{2t} - f_{1x})_x + f_2(2f_{2t} - f_{1x}) = 0, \\ L_2 &= (f_{2t} - 2f_{1x})_t - f_1(f_{2t} - 2f_{1x}) + 3(f_{0x} - f_2 f_0)_x = 0, \end{aligned} \tag{1.4}$$

where $f_0 = -d$, $f_1 = -c$, $f_2 = -b$.

In the realm of stochastic ordinary differential equations, linear equations play a role similar to that of linear equations in the classical theory of ordinary differential equations. For example, the reduction of a stochastic ordinary differential equation to linear form allows one to construct an exact solution of the original equation [5, 6, 12]. Hence, one can state the linearization problem for stochastic ordinary differential equations: find a change of variables which transforms a given equation to linear form.

In general, the change of variables in stochastic differential equations differs from that in ordinary differential equations owing to the necessity of using the Itô formula instead of the chain rule. The d-dimensional version of the Itô formula with one-dimensional Brownian motion $\{W_t : t \geq 0\}$ can be stated as follows. Let $X = (X_1, X_2, \ldots, X_d)$ be a d-dimensional Itô process defined on a filtered probability space $(\Omega, \mathcal{F}, P)$, which has the stochastic differentials

$$dX_i = f_i(t, X)\, dt + g_i(t, X)\, dW, \quad (i = 1, 2, \ldots, d).$$

Here $f_i(t, x)$ and $g_i(t, x)$, $(i = 1, 2, \ldots, d)$ are deterministic functions. Suppose that the function $\varphi(t, x)$ has continuous derivatives φ_t, φ_{x_i}, $\varphi_{x_i x_j}$, $(i, j = 1, 2, \ldots, d)$. Then the

[a]See [8] for details and references.

process $\varphi(t, X)$ has the stochastic differential[b]

$$d\varphi(t, X) = \left(\varphi_t + f_i\varphi_{x_i} + \frac{1}{2}g_ig_j\varphi_{x_ix_j}\right)(t, X)\,dt + (g_i\varphi_{x_i})(t, X)\,dW.$$

To the authors' knowledge the linearization problem for scalar second-order stochastic ordinary differential equations has not yet been studied. Sections 3 and 4 of the present manuscript discuss the solution of this task.

2. Linearization of First-Order Stochastic ODEs

Let us begin by reviewing the known linearization criteria for first-order equations.

2.1. *Strong linearization*

In [2, 5, 6] the Itô formula was applied to solving the linearization problem of a variety of scalar first-order stochastic ordinary differential equations

$$dX_t = f(t, X)\,dt + g(t, X)\,dW, \tag{2.1}$$

and some particular criteria for the existence of a change of the dependent variable

$$y = \varphi(t, x) \tag{2.2}$$

turning Eq. (2.1) into a linear equation

$$dY_t = (a_1(t)Y + a_0(t))\,dt + (b_1(t)Y + b_0(t))\,dW \tag{2.3}$$

were presented. For example, in [5] linearization criteria for autonomous ($f = f(x)$ and $g = g(x)$) equations were found and furthermore, conditions for reducibility to an explicitly integrable equation ($a_1 = 0$ and $b_1 = 0$) were obtained. Various examples of stochastic ordinary differential equations satisfying these criteria are given in [5, 6]. Based on this analysis, the author of [2] developed a MAPLE package containing routines which return explicit solutions of stochastic differential equations. General necessary and sufficient linearization conditions were finally presented in [9] and [13], and can be summarized as follows.

Let us set

$$N = g^{-1}\left(g_t + fg_x + \frac{g^2}{2}g_{xx} - gf_x\right).$$

Theorem 2.1. *Suppose, the coefficients of stochastic differential equation (2.1) satisfy the condition $N_x = 0$. Then the backward Kolmogorov equation corresponding to (2.1) is equivalent to the heat equation, and Eq. (2.1) is reducible to the linear stochastic differential equation*

$$dY_t = e^J\,dW$$

[b]The usual convention on summation with respect to a repeated index is used here.

where

$$J(t) = \int N\,dt.$$

The transition function $y = \varphi(t, x)$ is found by integrating the compatible system of partial differential equations

$$\varphi_t = e^J\left(\frac{g_x}{2} - \frac{f}{g}\right), \quad \varphi_x = \frac{e^J}{g}.$$

Theorem 2.2. *Assume that $N_x \neq 0$, and the function N satisfies equations*[c]

$$\frac{\partial}{\partial x}\left(\frac{(N_x g)_x}{N_x}\right) = 0, \quad \frac{\partial}{\partial t}\left(\frac{(N_x g)_x}{N_x}\right) - N\frac{(N_x g)_x}{N_x} + N_x g = 0. \tag{2.4}$$

Then the change

$$y = -\frac{\beta_1 e^J}{\beta_{1t} - \beta_1 N}$$

transforms any solution of Eq. (2.1) *to a solution of the linear stochastic differential equation*

$$dY_t = e^J\,dt + \beta_1 Y_t\,dW.$$

Here $\beta_1 = -N_x^{-1}(N_x g)_x,\ J(t) = \int q\,dt,$ *and*

$$q = \beta_1(g_x - \beta_1)/2 + (g_t - f\beta_1)/g + N_{tx}/N_x - 2N_x g/\beta_1 - N.$$

Remark. For stochastic differential equations with fractional Brownian motion (fBm) W^h of Hurst parameter $h \in (0, 1)$,

$$dX = f(t, X)\,dt + g(t, X)\,dW^h,$$

similar linearization conditions were also obtained in [14], now with the function

$$N = g^{-1}(g_t + fg_x + ht^{2h-1}g_{xx}g^2 - gf_x).$$

Remark. Considering stochastic differential equation (2.1) as the system of stochastic ordinary differential equations

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW,$$
$$dZ = dW,$$

one may choose to include Brownian motion in the linearizing transformation,

$$Y = \varphi(t, X, Z).$$

[c]The second of the conditions (2.4) is missing in [14]. This was corrected in [9].

In this case, an investigation of linearizability leads to the study of the deterministic overdetermined system of equations

$$\varphi_t + f\varphi_x + \frac{1}{2}g^2\varphi_{xx} + g\varphi_{xz} + \frac{1}{2}\varphi_{zz} = \alpha_1\varphi + \alpha_0,$$
$$g\varphi_x + \varphi_z = \beta_1\varphi + \beta_0,$$

where $\varphi = \varphi(t, x, z)$. The analysis of these equations is similar to the proofs of Theorems 2.1 and 2.2 (using the Itô formula for a pair of stochastic processes) and shows that this extension to a wider class of transformations does not lead to a larger collection of linearizable stochastic differential equations (2.1).

2.2. *Weak linearization*

Since transformation (2.2) does not change the Brownian motion, the linearization to Eq. (2.3) may be called strong linearization. In contrast, the transformation of a stochastic differential equation to a linear equation with changed Brownian motion may be called weak linearization, similar to the definitions of strong and weak solutions. For example, the random time change τ with time change rate $h^2(t, \omega)$,

$$y = x, \quad \tau(t, \omega) = \int_0^t h^2(s, \omega)\, ds, \tag{2.5}$$

leads to the change of Brownian motion by the formula [11]

$$\widetilde{W}_t = \int_0^{\alpha(t,\omega)} h(s, \omega)\, dW_s,$$

where $\tau(\alpha(t, \omega), \omega) = t$. This type of transformation can be used to further simplify the coefficients of a linear stochastic differential equation (2.3). For example, one of the functions $b_1(t)$ or $b_0(t)$ of a diffusion coefficient can be reduced to one. Transformations of this type were used in [1, 3, 4, 10] for defining fiber preserving admitted Lie groups of stochastic differential equations.

In [13] it was proven that the generalization of (2.5)[d],

$$\tau(t) = \int_0^t \eta^2(s, X(s))\, ds, \quad \widetilde{W}_t = \int_0^{\alpha(t)} \eta(s, X(s))\, dW_s, \tag{2.6}$$

also gives a transformation of the Brownian motion W_s to Brownian motion $\widetilde{W}_t$. Recall now that any first-order deterministic ordinary differential equation can be mapped into the simplest equation $y' = 0$ by a suitable change of the dependent and independent variables. It is not difficult to show that by virtue of its generality, transformation (2.6) renders the weak linearization problem equally trivial: given any first-order stochastic differential equation (2.1), there exists a transformation of type (2.6) mapping it to the simplest equation

$$dY_t = d\widetilde{W}_t.$$

[d]Considering $h = h(t, x, b)$, this generalization can be extended to include the Brownian motion W in the transformation.

Among other weak transformations one can also mention applications of the Girsanov theorems.

3. The Linearization Problem for Second Order Equations

A scalar second-order stochastic ordinary differential equation (1.1) is written in differential form as

$$d\dot{X} = f(t, X, \dot{X})\, dt + g(t, X, \dot{X})\, dW. \tag{3.1}$$

Here, the first term on the right-hand side represents a Riemann integral, and the second term an Itô integral. The aim is to find a change of the dependent variable $y = \varphi(t, x)$ which transforms this equation to a linear stochastic differential equation

$$d\dot{Y} = (a_1(t)Y + b_1(t)\dot{Y} + c_1(t))\, dt + (a_2(t)Y + b_2(t)\dot{Y} + c_2(t))\, dW. \tag{3.2}$$

Equation (3.1) can be rewritten as a system of first-order stochastic differential equations

$$\begin{aligned} dX &= V\, dt, \\ dV &= f(t, X, V)\, dt + g(t, X, V)\, dW. \end{aligned} \tag{3.3}$$

Similarly, the linear system corresponding to (3.2) is

$$\begin{aligned} dY &= Z\, dt, \\ dZ &= (a_1(t)Y + b_1(t)Z + c_1(t))\, dt + (a_2(t)Y + b_2(t)Z + c_2(t))\, dW. \end{aligned} \tag{3.4}$$

Just as with deterministic differential equations, a change of the dependent variable $y = \varphi(t, x)$ in (3.1) determines a corresponding change in the system of stochastic differential equations (3.3),

$$y = \varphi(t, x), \quad z = \varphi_2(t, x, v),$$

where the function $\varphi_2(t, x, v)$ is defined by the formula

$$\varphi_2 = \varphi_t + v\varphi_x.$$

We assume here that φ is three times continuously differentiable, and that $\varphi_x \neq 0$. Applying the Itô formula to the system of stochastic differential equations (3.3), one obtains

$$\begin{aligned} dY &= \alpha(t, X, V)\, dt + \beta(t, X, V)\, dW, \\ dZ &= \widetilde{f}(t, X, V)\, dt + \widetilde{g}(t, X, V)\, dW, \end{aligned} \tag{3.5}$$

where

$$\alpha(t, x, v) = \varphi_2(t, x, v), \quad \beta(t, x, v) = 0,$$

$$\widetilde{f}(t, x, v) = \left(\varphi_{2t} + \varphi_{2x}v + \varphi_{2v}f + \frac{1}{2}\varphi_{2vv}g^2\right)(t, x, v), \quad \widetilde{g}(t, x, v) = (\varphi_{2v}g)(t, x, v).$$

Equating the integrands of the Riemann and Itô integrals in (3.4) and (3.5) one obtains the two equations

$$\varphi_{2t} + \varphi_{2x}v + \varphi_{2v}f + \frac{1}{2}\varphi_{2vv}g^2 = a_1\varphi + b_1\varphi_2 + c_1,$$

$$\varphi_{2v}g = a_2\varphi + b_2\varphi_2 + c_2.$$

After substituting the function φ_2 into these, one has

$$\varphi_{xx}v^2 + (2\varphi_{tx} - \varphi_x b_1)v + \varphi_{tt} - \varphi_t b_1 + \varphi_x f - a_1\varphi - c_1 = 0, \tag{3.6}$$

$$\varphi_x g = b_2(\varphi_t + \varphi_x v) + a_2\varphi + c_2. \tag{3.7}$$

Thus, the pair of conditions (3.6) and (3.7) is necessary and sufficient for stochastic differential equation (3.1) to be linearizable.

4. Linearization Criteria

We now investigate in detail what type of coefficient functions f and g allow Eq. (3.1) to be linearized. Differentiating (3.7) with respect to v, one obtains that $g_v = b_2$, hence the function $g(t,x)$ has to be of the form

$$g = b_2 v + \psi, \tag{4.1}$$

where $\psi = \psi(t,x)$. In particular, $g_{xv} = 0$, $g_{vv} = 0$. Differentiating (3.6) three times with respect to v, by virtue of $\varphi_x \neq 0$, one finds that $f_{vvv} = 0$ or

$$f = f_2 v^2 + f_1 v + f_0, \tag{4.2}$$

where $f_i = f_i(t,x)$ $(i = 0,1,2)$. Notice that knowledge of the functions $g(t,x)$ and $f(t,x)$ uniquely determines the functions $b_2(t)$, $\psi(t,x)$ and $f_i(t,x)$.

Splitting Eq. (3.6) with respect to v, one obtains

$$\varphi_{xx} = -\varphi_x f_2, \quad \varphi_{tx} = \varphi_x(b_1 - f_1)/2, \quad \varphi_{tt} = \varphi_t b_1 - \varphi_x f_0 + a_1\varphi + c_1. \tag{4.3}$$

Applying the identities $(\varphi_{tx})_t - (\varphi_{tt})_x = 0$ and $(\varphi_{tx})_x - (\varphi_{xx})_t = 0$ one has, respectively,

$$a_1 = (2b_{1t} + 4f_{0x} - 2f_{1t} - b_1^2 - 4f_0f_2 + f_1^2)/4 \tag{4.4}$$

and

$$f_{1x} = 2f_{2t}. \tag{4.5}$$

Since a_1 is independent of x, differentiation of (4.4) with respect to x leads to

$$f_{0xx} = f_{0x}f_2 + f_{2tt} - f_{2t}f_1 + f_{2x}f_0. \tag{4.6}$$

Comparing the two conditions (4.5) and (4.6) with (1.4), one immediately obtains:

Theorem 4.1. *If a second-order stochastic differential equation*

$$d\dot{X} = f(t,X,\dot{X})\,dt + g(t,X,\dot{X})\,dW,$$

is linearizable, then the associated second-order ordinary differential equation

$$\ddot{x} = f(t, x, \dot{x})$$

is also linearizable, and by a change of the dependent variable only.

Suppose now, that Eq. (3.1) can be linearized. As the form of the coefficient g is known, Eq. (3.7) becomes

$$\varphi_x\psi - \varphi_t b_2 - a_2\varphi - c_2 = 0. \tag{4.7}$$

Differentiating this equation with respect to x, one finds

$$a_2 = (2\psi_x - b_1 b_2 + b_2 f_1 - 2f_2\psi)/2. \tag{4.8}$$

By virtue of $(a_2)_x = 0$ and because of (4.5), one has

$$(\psi_x - f_2\psi)_x + f_{2t} b_2 = 0. \tag{4.9}$$

Thus far, we have shown that identities (4.1), (4.2), (4.5), (4.6) and (4.9) are necessary conditions for linearizability of Eq. (3.1). Conditions (4.1), (4.5) and (4.6) apply to deterministic equations as well, while conditions (4.2) and (4.9) are specific to stochastic equations. The value of the coefficient b_2 is already determined by (4.1), while the coefficients a_1, a_2 and b_1 of the linearized equation are required to satisfy the relations (4.4) and (4.8).

In order to obtain sufficiency conditions, assume that all the identities listed above are satisfied. The two relations (4.4) and (4.8) leave three degrees of freedom for the choice of coefficients of the linearized equation. Observe that Eqs. (3.6) and (3.7) are equivalent to (4.3) and (4.7), respectively, and further analysis of compatibility of this overdetermined system of equations depends on the value of b_2 as determined by (4.1). In fact, we will show that the nonlinear equation can be transformed to a linear equation with simple stochastic part, provided the above conditions are satisfied.

4.1. *Case* $b_2 \neq 0$

Since b_2 is nonzero, one finds φ_t from Eq. (4.7),

$$\varphi_t = (\varphi_x\psi - a_2\varphi - c_2)/b_2. \tag{4.10}$$

Substituting φ_t into the second and third equations of (4.3), one obtains

$$\varphi_x\lambda + p\varphi + u = 0, \tag{4.11}$$

where

$$\lambda = (\psi_x\psi - \psi_t b_2 - b_2^2 f_0 + \psi(b_{2t} + b_2 f_1 - f_2\psi))/b_2^2, \quad p = -\lambda_x + f_2\lambda,$$
$$u = (2c_{2t}b_2 + c_2(2f_2\psi - 2b_{2t} - 2\psi_x - b_1 b_2 - b_2 f_1) + 2b_2^2 c_1)/(2b_2^2).$$

In addition, the relations $u_x = 0$ and $p_x = 0$ hold. There are now several possibilities to consider.

Suppose first that $\lambda = 0$. Then $p = 0$ and Eq. (4.11) reduces to $u = 0$, which constitutes a linear first-order ordinary differential equation in the coefficient $c_2(t)$. Thus, the stochastic differential equation (3.1) is linearizable. In fact, the coefficients of the linear equation (3.2) are defined by Eqs. (4.4), (4.8) and $u = 0$, and the function $\varphi(t, x)$ is obtained by integrating the involutive system of partial differential equations consisting of Eq. (4.10) and the first equation of (4.3).

Without loss of generality one may choose $a_2 = 0$, $c_1 = 0$, $c_2 = 0$. Then the function $\varphi(t, x)$ is obtained by integrating the involutive system

$$\varphi_t = \varphi_x \psi / b_2, \quad \varphi_{xx} = -\varphi_x f_2,$$

and the remaining coefficients of the linear equation (3.2) are uniquely defined by the functions f and g:

$$a_1 = 0, \quad b_1 = f_1 + 2(\psi_x - f_2\psi)/b_2, \quad c_1 = 0, \quad a_2 = 0, \quad b_2 = (g - \psi)/v, \quad c_2 = 0.$$

Observe that the linear equation obtained is of first order.

Next assume that $\lambda \neq 0$. Solving Eq. (4.11) with respect to φ_x, the first-order derivatives of the function $\varphi(t, x)$ are all determined. The relation $(\varphi_x)_t - (\varphi_t)_x = 0$ gives

$$2\varphi\lambda_2 - w\lambda = 0, \tag{4.12}$$

where

$$\lambda_2 = \lambda_t p - p_t \lambda + \frac{p}{b_2}(\psi_x \lambda + \psi(p - f_2\lambda)),$$

$$w = 2u_t + u(f_1 - b_1) - \frac{2}{b_2\lambda}(c_2 p\lambda + u(\lambda_t b_2 + p\psi)).$$

The functions $\lambda_2(t, x)$ and $w(t, x)$ also satisfy the relations

$$\lambda_{2x} = f_2\lambda_2, \quad w_x = -2u\lambda_2/\lambda^2.$$

If $\lambda_2 = 0$, then Eq. (4.12) leads to $w = 0$. Thus stochastic differential equation (3.1) is linearizable, and the coefficients of the linear equation (3.2) must satisfy (4.4), (4.8) and $w = 0$. The function $\varphi(t, x)$ is obtained by integrating the involutive system of partial differential equations consisting of the equations

$$\varphi_t = \varphi\left(-\frac{\psi_x\lambda + \psi(p - f_2\lambda)}{b_2\lambda} + \frac{1}{2}(b_1 - f_1)\right) - \frac{1}{b_2\lambda}(c_2\lambda + u\psi), \quad \varphi_x = -\frac{1}{\lambda}(u + p\varphi).$$

If $p \neq 0$ then one may choose $a_2 = 0$, $c_1 = 0$, $c_2 = 0$. Consequently, $u = 0$ and

$$a_1 = p, \quad b_1 = f_1 + 2(\psi_x - f_2\psi)/b_2, \quad c_1 = 0, \quad a_2 = 0, \quad b_2 = (g - \psi)/v, \quad c_2 = 0,$$

while $\varphi(t, x)$ is found by solving

$$\varphi_t = -\varphi p\psi/(b_2\lambda), \quad \varphi_x = -\varphi p/\lambda.$$

If on the other hand $p = 0$, then one can only choose $a_2 = 0$ and $c_2 = 0$ as $u \neq 0$ is required. The linear equation (3.2) reduces to a first order equation, as its coefficients are

$$a_1 = 0, \quad b_1 = f_1 + 2(\psi_x - f_2\psi)/b_2, \quad c_1 = -\varphi_x\lambda, \quad a_2 = 0, \quad b_2 = (g - \psi)/v, \quad c_2 = 0,$$

where $\varphi(t, x)$ is an arbitrary solution of the involutive system of equations

$$\varphi_t = \varphi_x \psi / b_2, \quad \varphi_{xx} = -\varphi_x f_2$$

satisfying $\varphi_x \neq 0$.

Finally, if $\lambda_2 \neq 0$, then (4.12) defines the function $\varphi(t, x)$. Substituting this function into Eq. (4.10), one obtains that w must satisfy

$$\begin{aligned} w_t = (&-4c_2\lambda\lambda_2^2 - 4u\psi\lambda_2^2 + w\lambda(-2\psi_x\lambda\lambda_2 - 2\lambda_t b_2\lambda_2 + 2\lambda_{2t}b_2\lambda - 2v\psi\lambda_2 \\ &+ b_1 b_2\lambda\lambda_2 - b_2 f_1\lambda\lambda_2 + 2f_2\psi\lambda\lambda_2))/(2b_2\lambda^2\lambda_2). \end{aligned} \tag{4.13}$$

Notice that the equations (4.11) and $(w_x)_t - (w_t)_x = 0$ are satisfied.

Thus, the stochastic differential equation (3.1) is linearizable, and the coefficients of the linear equation (3.2) are determined by Eqs. (4.4), (4.8) and (4.13). Since it is assumed that $\varphi_x \neq 0$, then the coefficients also have to satisfy the condition $2u\lambda_2 + wp\lambda \neq 0$. Without loss of generality, one can choose $a_2 = 0$, $c_2 = 0$. Then the coefficients of the linear equation (3.2) are

$$a_1 = p, \quad b_1 = f_1 + 2(\psi_x - f_2\psi)/b_2, \quad c_1 = -(\varphi_x\lambda + \varphi p), \quad a_2 = 0,$$
$$b_2 = (g - \psi)/v, \quad c_2 = 0,$$

where the function $\varphi(t, x)$ is any solution of the involutive system of equations

$$\varphi_t = \varphi_x \psi / b_2, \quad \varphi_{xx} = -\varphi_x f_2,$$

satisfying the condition $\varphi_x \neq 0$.

We have thus shown that in case $b_2 \neq 0$ one can obtain a linear equation whose coefficients a_2 and c_2 vanish.

4.2. *Case $b_2 = 0$*

The assumption $b_2 = 0$ implies that $g_v = 0$ and $\psi = g$. Hence the coefficient a_2 is determined by the functions f and g only,

$$a_2 = g_x - f_2 g,$$

and Eq. (4.9) guarantees that $a_2 = a_2(t)$.

From (3.7) one finds

$$\varphi_x = \frac{\varphi a_2 + c_2}{g}.$$

Substituting φ_x into the first and second equations of (4.3), one obtains

$$a_2\left(\varphi_t - \varphi\left(\frac{g_t}{g} + \frac{b_1 - f_1}{2}\right)\right) + a_{2t}\varphi + c_{2t} - c_2\left(\frac{g_t}{g} + \frac{b_1 - f_1}{2}\right) = 0. \tag{4.14}$$

Suppose first that $a_2 = 0$. Because of $\varphi_x \neq 0$ one has $c_2 \neq 0$, and Eq. (4.14) becomes

$$\frac{c_{2t}}{c_2} = \left(\frac{g_t}{g} + \frac{b_1 - f_1}{2}\right). \tag{4.15}$$

Thus, stochastic differential equation (3.1) is linearizable. The remaining coefficients of the linear equation (3.2) are determined by Eqs. (4.4), (4.8) and (4.15), and the function $\varphi(t,x)$ is a solution of the involutive system of equations

$$\varphi_x = c_2/g, \quad \varphi_{tt} = \varphi_t b_1 + a_1\varphi + c_1 - c_2 f_0/g.$$

Without loss of generality one may choose $b_1 = 0$, $c_1 = 0$. Then

$$a_1 = f_{0x} - f_0 f_2 - (2f_{1t} - f_1^2)/4, \quad b_1 = 0, \quad c_1 = 0, \quad a_2 = 0, \quad b_2 = 0,$$

and c_2 is a nontrivial solution of Eq. (4.15).

If $a_2 \neq 0$, then Eq. (4.14) can be solved with respect to the derivative φ_t:

$$\varphi_t = \frac{\varphi\tilde{v}a_2 + \tilde{u}}{2ga_2^2}, \tag{4.16}$$

where

$$\tilde{u} = -2c_{2t}ga_2 + 2a_{2t}c_2 g + \tilde{v}c_2, \quad \tilde{v} = 2g_t a_2 - 2a_{2t}g + b_1 g a_2 - f_1 g a_2.$$

The functions $\tilde{u}(t,x)$ and $\tilde{v}(t,x)$ satisfy the conditions

$$\tilde{u}_x = \tilde{u}\left(f_2 + \frac{a_2}{g}\right) + 2c_2 a_2\left(a_{2t} - \frac{g_t}{g}a_2\right), \quad \tilde{v}_x = \tilde{v}\left(f_2 + \frac{a_2}{g}\right) + 2a_2\left(a_{2t} - \frac{g_t}{g}a_2\right).$$

Substitution of φ_t into the third equation of (4.3) gives

$$\varphi\lambda_3 + \tilde{w} = 0, \tag{4.17}$$

where

$$\lambda_3 = \frac{a_{2tt}}{a_2} - \frac{g_{tt}}{g} - \left(2\frac{a_{2t}}{a_2^2} + \frac{f_1}{a_2}\right)\left(a_{2t} - \frac{g_t}{g}a_2\right) + f_{0x} - f_0\left(f_2 + \frac{a_2}{g}\right),$$

$$\tilde{w} = \frac{1}{4g^2a_2^3}(-2\tilde{u}_t g a_2 + \tilde{u}(8a_{2t}g - 2g_t a_2 + 2f_1 g a_2 + \tilde{v})) - c_2\frac{f_0}{g} + c_1.$$

Notice that $(\varphi_t)_x - (\varphi_x)_t = 0$ and $\lambda_{3x} = -a_2\lambda_3/g$. In addition,

$$\tilde{w}_x = -c_2\lambda_3/g. \tag{4.18}$$

Assuming first that $\lambda_3 = 0$, Eq. (4.17) implies that $\tilde{w} = 0$. Thus, the stochastic differential equation (3.1) is linearizable, and the coefficients of the linear equation (3.2) will satisfy the conditions (4.4), (4.8) and $\tilde{w} = 0$. The function $\varphi(t,x)$ is found from the involutive system of equations

$$\varphi_t = \frac{\varphi\tilde{v}a_2 + \tilde{u}}{2ga_2^2}, \quad \varphi_x = \frac{\varphi a_2 + c_2}{g}.$$

Without loss of generality one may choose $b_1 = 0$, $c_1 = 0$, $c_2 = 0$. These assumptions imply that $\tilde{u} = 0$, and

$$\tilde{v} = 2g_t a_2 - 2a_{2t}g - f_1 g a_2.$$

Thus

$$a_1 = f_{0x} - f_0 f_2 - (2f_{1t} - f_1^2)/4, \quad b_1 = 0, \quad c_1 = 0, \quad a_2 = g_x - f_2 g, \quad b_2 = 0, \quad c_2 = 0,$$

and in order to obtain the function $\varphi(t,x)$ one has to integrate the involutive system of equations

$$\varphi_t = \varphi \frac{\tilde{v}}{2ga_2}, \quad \varphi_x = \varphi \frac{a_2}{g}.$$

When $\lambda_3 \neq 0$ then one finds from (4.17) that

$$\varphi = -\tilde{w}/\lambda_3,$$

and substitution into Eq. (4.16) yields

$$\tilde{w}_t = \tilde{w}\left(\frac{\lambda_{3t}}{\lambda_3} + \frac{\tilde{v}}{2ga_2}\right) - \tilde{u}\frac{\lambda_3}{2ga_2^2}. \tag{4.19}$$

Thus, stochastic differential equation (3.1) is linearizable, and the coefficients of the linearized equation (3.2) need to satisfy (4.4), (4.8), (4.18) and (4.19). Because of the assumption $\varphi_x \neq 0$ the coefficients have to satisfy the condition $c_2\lambda_3 - a_2\tilde{w} \neq 0$ as well. One can therefore choose only two of the coefficients arbitrarily, say $b_1 = 0$ and $c_2 = 0$, which gives $\tilde{u} = 0$, $\tilde{w} = c_1$ and

$$\tilde{v} = 2g_t a_2 - 2a_{2t} g - f_1 g a_2.$$

Then by (4.19), the coefficient $c_1(t)$ can be found by solving

$$\frac{c_{1t}}{c_1} = \frac{\lambda_{3t}}{\lambda_3} + \frac{\tilde{v}}{2ga_2}.$$

The remaining coefficients are

$$a_1 = (4f_{0x} - 2f_{1t} - 4f_0 f_2 + f_1^2)/4, \quad b_1 = 0, \quad a_2 = g_x - f_2 g, \quad b_2 = 0, \quad c_2 = 0.$$

Thus, in case $b_2 = 0$ we are also able to obtain a linear equation where all but one of the coefficients of the stochastic part vanish.

We summarize the above:

Theorem 4.2. *A second-order stochastic differential equation*

$$\ddot{X} = f(t, X, \dot{X}) + g(t, X, \dot{X})\,\dot{W}$$

is linearizable by a change of the dependent variables if and only if

$$f(t,x,v) = v^2 f_2(t,x) + v f_1(t,x) + f_0(t,x), \quad g(t,x,v) = v b_2(t) + \psi(t,x),$$

where the functions f_i, g, ψ and b_2 satisfy the conditions

$$f_{1x} = 2f_{2t}, \quad f_{0xx} = f_{0x}f_2 + f_{2tt} - f_{2t}f_1 + f_{2x}f_0, \quad (\psi_x - f_2\psi)_x + f_{2t}b_2 = 0. \tag{4.20}$$

5. Linearizable Langevin Equations

We illustrate our results by two examples.

The first example furnishes the class of second order Langevin equations (1.2) which are linearizable. By Theorem 4.2, these equations are necessarily of the form

$$\ddot{X} = f_2(t, X)\dot{X}^2 + f_1(t, X)\dot{X} + f_0(t, X) + \sigma\dot{W}, \tag{5.1}$$

with $\sigma \neq 0$.

We may assume that $f_2 \neq 0$, for otherwise conditions (4.20) ensure that this equation is already linear. Since $g = \sigma \neq 0$, Eqs. (4.20) reduce to

$$f_2 = p(t), \quad f_1 = q(t) + 2p'(t)x, \quad f_0 = h(t)e^{p(t)x} + \alpha(t)x^2 + \beta(t)x + r(t),$$

where

$$\alpha = (p')^2/p, \quad \beta = (qp' - p'')/p + 2(p'/p)^2. \tag{5.2}$$

This is the case $b_2 = 0$, $a_2 \neq 0$ discussed in the previous section, with $\lambda_3 = hpe^{px}$.

If $h = 0$, then a linearizing transformation is given by

$$\varphi(t, x) = \frac{\varphi_0(t)}{p(t)} e^{-p(t)x}$$

where $\varphi_0(t)$ is a nontrivial solution of the equation

$$2\varphi_0' + \varphi_0 q = 0,$$

yielding an equivalent linear stochastic differential equation

$$\ddot{Y} = a_1(t)Y - \sigma p(t)Y\dot{W}$$

where

$$a_1 = \beta - pr - q'/2 + q^2/4. \tag{5.3}$$

On the other hand, if $h \neq 0$, then a linearizing transformation is given by

$$\varphi(t, x) = -\frac{c_1(t)}{p(t)h(t)} e^{-p(t)x}$$

where $c_1(t)$ is a nontrivial solution of the equation

$$\frac{c_1'}{c_1} = \frac{h'}{h} - \frac{q}{2}.$$

The equivalent linear stochastic differential equation is

$$\ddot{Y} = a_1(t)Y + c_1(t) - \sigma p(t)Y\dot{W},$$

where a_1 is again defined by formula (5.3).

Both cases may be combined to:

Theorem 5.1. *A nonlinear Langevin equation is linearizable if, and only if, it is of the form*

$$\ddot{X} = p(t)\dot{X}^2 + [q(t) + 2p'(t)X]\dot{X} + h(t)e^{p(t)X} + \alpha(t)X^2 + \beta(t)X + r(t) + \sigma\dot{W}$$

where α and β are as defined in (5.2)*. Any linearizable Langevin equation can be reduced to the linear equation*

$$\ddot{Y} = a_1(t)Y + c_1(t) - \sigma p(t)Y\dot{W},$$

where a_1 is given by (5.3) *and*

$$c_1(t) = h(t)\exp\left(-\frac{1}{2}\int q(t)\,dt\right),$$

by means of the transformation

$$y = \varphi(t,x) = \frac{-1}{p(t)}\exp\left(-\frac{1}{2}\int q(t)\,dt - p(t)x\right).$$

A similar analysis can be applied to the following stochastic equation with multiplicative noise,

$$\ddot{X} = f(t,X,\dot{X}) + b_2(t)\dot{X}\dot{W}, \tag{5.4}$$

by using the discussion of the $b_2 \neq 0$ case of the previous section. Details are omitted as they are easy to verify.

Theorem 5.2. *Equation* (5.4) *is linearizable if, and only if, it is of the form*

$$\ddot{X} = q(X)\dot{X}^2 + r(t)\dot{X} + f_0(t,X) + b_2(t)\dot{X}\dot{W}$$

where

$$f_0(t,x) = \frac{1}{\beta(x)}\left[s(t) + h(t)\int \beta(x)\,dx\right]$$

with

$$\beta(x) = e^{-\int q(x)\,dx}.$$

The transformation

$$y = \varphi(x) = \int \beta(x)\,dx$$

maps any linearizable equation to the equation

$$\ddot{Y} = h(t)Y + r(t)\dot{Y} + s(t) + b_2(t)\dot{Y}\dot{W}.$$

Remark. All computations were initially performed using the REDUCE symbolic program, but later carefully verified by hand.

Acknowledgments

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References

[1] O. V. Alexandrova, Group analysis of the Itô stochastic system, *Differential Equations Dyn. Systems* **14**(3/4) (2006) 255–279.

[2] S. Cyganowski, Solving stochastic differential equations with Maple, *Maple-Tech Newsletter* **3**(2) (1996) 38–40.

[3] G. Gaeta, Symmetry of stochastic equations, ed. A. M. Samoilenko, in *Proceedings of the Institute of Mathematics of NAS of Ukraine, SYMMETRY in Nonlinear Mathematical Physics*, Institute of Mathematics of NAS of Ukraine, Kyiv, 2004, pp. 98–109.

[4] G. Gaeta and N. R. Quinter, Lie-point symmetries and differential equations, *J. Phys. A* **32** (1999) 8485–8505.

[5] T. C. Gard, *Introduction to Stochastic Differential Equations* (Marcel Dekker, New York and Basel, 1988).

[6] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, Berlin, 1992).

[7] S. Lie, Klassifikation und Integration von gewöhnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten III, *Archiv for Matematik og Naturvidenskab* **8**(4) (1883) 371–42. [Reprinted in Lie's Gesammelte Abhandlungen, 1924, 5, paper XIY, pp. 362–427].

[8] S. V. Meleshko, *Methods for Constructing Exact Solutions of Partial Differential Equations, Mathematical and Analytical Techniques with Applications to Engineering* (Springer, New York, 2005).

[9] S. V. Meleshko and E. Schulz, Linearization of first order stochastic differential equations, *Contributions in Mathematics and Applications III, East–West J. Math.*, special volume (2010), 281–289.

[10] S. A. Melnick, The group analysis of stochastic partial differential equations, *Theory Stoch. Process.* **9**(25) (2003) 99–107.

[11] B. Øksendal, *Stochastic Ordinary Differential Equations, an Introduction with Applications*, 5th edn. (Springer, Berlin, 1998).

[12] M. M. Rao, Higher order stochastic differential equations, ed. M. M. Rao, in *Real and Stochastic Analysis, Recent Advances*, CRC Press, Boca Raton and New York, 1977, pp. 225–302.

[13] B. Srihirun, S. V. Meleshko and E. Schulz, On the definition of an admitted Lie group for stochastic differential equations, *Commun. Nonlinear Sci. Numer. Simul.* **12** (2007) 1379–1389.

[14] G. Unal and A. Dinler, Exact linearization of one dimensional Itô equations driven by fBm: Analytical and numerical solutions, *Nonlinear Dynam.* **53** (2008) 251–259.