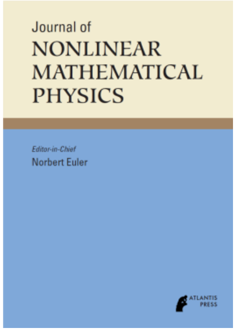

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TODA–HEISENBERG CHAIN: INTERACTING σ-FIELDS IN TWO DIMENSIONS

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We study a $(2+1)$-dimensional system that can be viewed as an infinite number of $O(3)$ σ-fields coupled by a nearest-neighbour Heisenberg-like interaction. We reduce the field equations of this model to an integrable system that is closely related to the two-dimensional relativistic Toda chain and the Ablowitz–Ladik equations. Using this reduction we obtain the dark-soliton solutions of our model.

Keywords: Toda–Heisenberg chain; interacting sigma-fields; relativistic Toda chain; Ruijsenaars–Toda lattice; Ablowitz–Ladik hierarchy; dark-soliton solutions.

Mathematics Subject Classification 2010: 37J35, 35Q51, 37K10, 37K35, 11C20

1. Introduction

The model considered in this paper can be viewed as a generalization of the classical $O(3)$ σ-model in two dimensions, described by the Hamiltonian function

$$\mathcal{E} = \mathcal{E}[\boldsymbol{\sigma}] = \int_{\mathbb{R}^2} dx\, dy (\nabla\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) \tag{1.1}$$

where $\boldsymbol{\sigma}$ is a three-component vector of unit length,

$$(\boldsymbol{\sigma}, \boldsymbol{\sigma}) = 1 \tag{1.2}$$

and braces denote the standard scalar product. The energy of our system is given by

$$\mathcal{H} = \sum_n \mathcal{E}[\boldsymbol{\sigma}_n] + \mathcal{H}_{\text{int}} \tag{1.3}$$

with nearest-neighbour interaction

$$\mathcal{H}_{\text{int}} = \frac{1}{2}\sum_n \sum_{p=n\pm1} \mathcal{U}_{np} \tag{1.4}$$

of the Heisenberg type:

$$\mathcal{U}_{np} = \mathcal{U}[\boldsymbol{\sigma}_n, \boldsymbol{\sigma}_p] = \int_{\mathbb{R}^2} dx\, dy\, F((\boldsymbol{\sigma}_n, \boldsymbol{\sigma}_p)). \tag{1.5}$$

The models of this type can appear, for example, in the studies of the lamellar (graphite-like) magnetics when the spin interaction inside one layer can be described in the framework of the Landau–Lifshitz theory with effective Heisenberg interaction between adjacent layers.

The stationary structures of our system are governed by the (2+1)-dimensional equation

$$\frac{\delta \mathcal{H}}{\delta \boldsymbol{\sigma}_n} = 0, \quad (\delta \boldsymbol{\sigma}_n, \boldsymbol{\sigma}_n) = 0. \tag{1.6}$$

In what follows we use a function F which is peculiar to integrable nonlinear mathematics (see e.g. [1, 2]),

$$F(x) = g^2 \ln(1 + x). \tag{1.7}$$

The resulting equations are given by

$$[\Delta \boldsymbol{\sigma}_n, \boldsymbol{\sigma}_n] = \frac{g^2}{4} \sum_{p=n\pm 1} f_{np} [\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_n] \tag{1.8}$$

where

$$f_{np} = \frac{2}{1 + (\boldsymbol{\sigma}_n, \boldsymbol{\sigma}_p)}. \tag{1.9}$$

The factor g^2 can be eliminated by rescaling the coordinates, so we take

$$g = 4 \tag{1.10}$$

and write the central equation of our study as

$$\frac{1}{4}[\Delta \boldsymbol{\sigma}_n, \boldsymbol{\sigma}_n] = \sum_{p=n\pm 1} f_{np} [\boldsymbol{\sigma}_p, \boldsymbol{\sigma}_n]. \tag{1.11}$$

In the following sections, after re-parametrization of (1.11), we split it in Sec. 2 into a first-order system, bilinearize it (Sec. 3) and derive the dark-soliton solutions (Sec. 4).

2. Parametrization and Splitting

Using the vector-matrix correspondence

$$\boldsymbol{\sigma} = (s_1, s_2, s_3)^T \quad \rightarrow \quad \mathbb{S} = \begin{pmatrix} s_3 & s_1 - is_2 \\ s_1 + is_2 & -s_3 \end{pmatrix} = \sum_{j=1}^{3} s_j \sigma^j \tag{2.1}$$

where σ^j $(j = 1, 2, 3)$ are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

and introducing complex variables

$$z = x + iy, \quad \bar{z} = x - iy \tag{2.3}$$

one can rewrite Eq. (1.11) as

$$[\partial\bar{\partial}\,\mathbb{S}_n, \mathbb{S}_n] = \sum_{p=n\pm 1} f_{np}[\mathbb{S}_p, \mathbb{S}_n] \tag{2.4}$$

with $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial \bar{z}$ and

$$f_{np} = \frac{2}{1 + \frac{1}{2}\,\mathrm{tr}\,\mathbb{S}_n\mathbb{S}_p}. \tag{2.5}$$

In what follows we use the parametrization of the vectors $\boldsymbol{\sigma}_n$ based on the presentation of the matrices $\mathbb{S}_n$ in the form

$$\mathbb{S}_n = \Psi_n^{-1}\sigma^3\Psi_n. \tag{2.6}$$

Using the invariance of this representation with respect to transformations $\Psi_n \to \mathbb{D}_n\Psi_n$ with arbitrary diagonal matrices $\mathbb{D}_n$ one can choose

$$\Psi_n = \begin{pmatrix} 1 & B_n \\ C_n & 1 \end{pmatrix} \tag{2.7}$$

which leads to

$$\mathbb{S}_n = \frac{1}{1 - B_nC_n}\begin{pmatrix} 1 + B_nC_n & 2B_n \\ -2C_n & -1 - B_nC_n \end{pmatrix}. \tag{2.8}$$

Calculating $\partial\bar{\partial}\,\mathbb{S}_n$ and f_{np},

$$f_{np} = \frac{(1 - B_nC_n)(1 - B_pC_p)}{(1 - B_nC_p)(1 - C_nB_p)}, \tag{2.9}$$

one comes to the following system of equations:

$$\begin{cases} A_n\mathcal{L}_n^B = \bar{Y}_n(B_{n+1} - B_n) - Y_{n-1}(B_n - B_{n-1}) \\ A_n\mathcal{L}_n^C = Y_n(C_{n+1} - C_n) - \bar{Y}_{n-1}(C_n - C_{n-1}) \end{cases} \tag{2.10}$$

where

$$A_n = \frac{1}{1 - B_nC_n}, \tag{2.11}$$

$$\mathcal{L}_n^B = \partial\bar{\partial}B_n + 2A_n(\partial B_n)(\bar{\partial}B_n)C_n, \tag{2.12}$$

$$\mathcal{L}_n^C = \partial\bar{\partial}C_n + 2A_nB_n(\partial C_n)(\bar{\partial}C_n) \tag{2.13}$$

and

$$Y_n = \frac{1}{1 - B_nC_{n+1}}, \quad \bar{Y}_n = \frac{1}{1 - B_{n+1}C_n}. \tag{2.14}$$

The crucial step of our proceeding is the following *ansatz*: we split the above system into two first-order ones,

$$\begin{cases} iA_n\partial\,B_n = Z_{n-1}(B_n - B_{n-1}) \\ iA_n\partial\,C_n = Z_n(C_{n+1} - C_n) \end{cases} \tag{2.15}$$

and

$$\begin{cases} -iA_n\bar{\partial}\, B_n = \bar{Z}_n(B_{n+1} - B_n) \\ -iA_n\bar{\partial}\, C_n = \bar{Z}_{n-1}(C_n - C_{n-1}). \end{cases} \tag{2.16}$$

By direct calculations one can show that this can be done provided we can find the functions Z_n and $\bar{Z}_n$ that (i) make (2.15) and (2.16) compatible and (ii) lead to (2.10). It is demonstrated in the appendix that the functions Z_n and $\bar{Z}_n$ that meet these conditions can be chosen as

$$Z_n = \zeta Y_n, \quad \bar{Z}_n = \bar{\zeta}\bar{Y}_n \tag{2.17}$$

where ζ and $\bar{\zeta}$ are arbitrary constants related by

$$\zeta\bar{\zeta} = 1. \tag{2.18}$$

To summarize, one can obtain a large number of solutions of (2.4) by solving the system

$$\begin{cases} i\partial B_n = \zeta \dfrac{1 - B_nC_n}{1 - B_{n-1}C_n}(B_n - B_{n-1}) \\ i\partial C_n = \zeta \dfrac{1 - B_nC_n}{1 - B_nC_{n+1}}(C_{n+1} - C_n) \end{cases} \tag{2.19}$$

and

$$\begin{cases} -i\bar{\partial} B_n = \bar{\zeta} \dfrac{1 - B_nC_n}{1 - B_{n+1}C_n}(B_{n+1} - B_n) \\ -i\bar{\partial} C_n = \bar{\zeta} \dfrac{1 - B_nC_n}{1 - B_nC_{n-1}}(C_n - C_{n-1}). \end{cases} \tag{2.20}$$

Before proceed further, we would like to give some comments on this system. After introducing new variables,

$$\widetilde{B}_n = 1/C_n. \tag{2.21}$$

Equations (2.19), (2.20) can be cast into the Hamiltonian form

$$\begin{cases} i\partial B_n = (B_n - \widetilde{B}_n)^2 \dfrac{\partial H}{\partial \widetilde{B}_n} \\ -i\partial \widetilde{B}_n = (B_n - \widetilde{B}_n)^2 \dfrac{\partial H}{\partial B_n} \end{cases} \tag{2.22}$$

with

$$H = \zeta \sum_{n=-\infty}^{\infty} \ln \frac{B_n - \widetilde{B}_n}{B_n - \widetilde{B}_{n+1}} \tag{2.23}$$

and

$$\begin{cases} i\bar{\partial} B_n = (B_n - \widetilde{B}_n)^2 \dfrac{\partial \bar{H}}{\partial \widetilde{B}_n} \\ -i\bar{\partial} \widetilde{B}_n = (B_n - \widetilde{B}_n)^2 \dfrac{\partial \bar{H}}{\partial B_n} \end{cases} \tag{2.24}$$

with

$$\bar{H} = \bar{\zeta} \sum_{n=-\infty}^{\infty} \ln \frac{B_n - \widetilde{B}_n}{B_n - \widetilde{B}_{n-1}} \tag{2.25}$$

and can be identified with the (X_1, Y_1) equations (with $a(u,v) = (u-v)^2$) from the list of the paper by Adler and Shabat [3].

At the same time both B_n and C_n solve the $(2+1)$-dimensional version of the Ruijsenaars–Toda lattice [4, 5]

$$\partial\bar{\partial} U_n + (\partial U_n)(\bar{\partial} U_n)\left[\frac{1}{U_{n+1} - U_n} - \frac{1}{U_n - U_{n-1}}\right] = 0. \tag{2.26}$$

Note that Eq. (2.26) are different from (and complementary to) the Ruijsenaars–Toda lattice (R_1) that appears in a natural way in the framework of [3].

Finally, calculating from Eqs. (2.15), (2.16) derivatives of the functions f_n defined by

$$f_n = f_{n,n+1} = \frac{(1 - B_n C_n)(1 - B_{n+1} C_{n+1})}{(1 - B_n C_{n+1})(1 - B_{n+1} C_n)} \tag{2.27}$$

one can demonstrate that these functions satisfy

$$\partial\bar{\partial} \ln f_n = f_{n+1} - 2f_n + f_{n-1}. \tag{2.28}$$

Thus one can see the relationship of the model discussed in this paper with the famous two-dimensional Toda lattice.

3. Bilinearization

To bilinearize Eqs. (2.19), (2.20) we introduce $\check{\rho}_n$, $\check{\tau}_n$, $\hat{\tau}_n$ and $\hat{\sigma}_n$ by

$$B_n = \frac{\check{\rho}_{n-1}}{\hat{\tau}_n}, \quad C_n = -\frac{\hat{\sigma}_n}{\check{\tau}_{n-1}} \tag{3.1}$$

and another set of tau-functions by

$$\begin{cases} iD\, \check{\rho}_{n-1} \cdot \hat{\tau}_n = \alpha\, \rho_{n-1} \tau_n \\ iD\, \check{\tau}_{n-1} \cdot \hat{\sigma}_n = \alpha\, \tau_{n-1} \sigma_n \end{cases} \tag{3.2}$$

and

$$\begin{cases} -i\bar{D}\, \check{\rho}_{n-1} \cdot \hat{\tau}_n = \bar{\alpha}\, \tau_{n-1} \rho_n \\ -i\bar{D}\, \check{\tau}_{n-1} \cdot \hat{\sigma}_n = \bar{\alpha}\, \sigma_{n-1} \tau_n \end{cases} \tag{3.3}$$

where α and $\bar{\alpha}$ are constants, D and $\bar{D}$ are the Hirota's bilinear differential operators, $D\, u \cdot v = (\partial u)v - u(\partial v)$ and $\bar{D}\, u \cdot v = (\bar{\partial} u)v - u(\bar{\partial} v)$. Now, to finish the bilinearization of our equations, we impose the restrictions

$$\begin{cases} \check{\rho}_n \hat{\tau}_n - \check{\rho}_{n-1} \hat{\tau}_{n+1} = \beta\, \rho_n \tau_n \\ \check{\tau}_n \hat{\sigma}_n - \check{\tau}_{n-1} \hat{\sigma}_{n+1} = \beta\, \tau_n \sigma_n \end{cases} \tag{3.4}$$

and

$$\begin{cases} \check{\tau}_{n-1}\hat{\tau}_n + \check{\rho}_{n-1}\hat{\sigma}_n = \gamma^A\ \tau_{n-1}\tau_n \\ \check{\tau}_n\hat{\tau}_n + \check{\rho}_{n-1}\hat{\sigma}_{n+1} = \gamma^B\ \tau_n^2 \\ \check{\tau}_{n-1}\hat{\tau}_{n+1} + \check{\rho}_n\hat{\sigma}_n = \gamma^C\ \tau_n^2 \end{cases} \tag{3.5}$$

where β, γ^A, γ^B and γ^C are again some constants. It can be shown by direct calculations that Eqs. (3.2)–(3.5) imply that B_n, C_n satisfy Eqs. (2.19), (2.20). Indeed, noting that Eqs. (3.4) and (3.5) are nothing but

$$B_{n+1} - B_n = \beta\ \frac{\rho_n\tau_n}{\hat{\tau}_n\hat{\tau}_{n+1}} \tag{3.6}$$

$$C_{n+1} - C_n = \beta\ \frac{\tau_n\sigma_n}{\check{\tau}_{n-1}\check{\tau}_n} \tag{3.7}$$

and

$$1 - B_nC_n = \gamma^A\ \frac{\tau_{n-1}\tau_n}{\check{\tau}_{n-1}\hat{\tau}_n} \tag{3.8}$$

$$1 - B_nC_{n+1} = \gamma^B\ \frac{\tau_n^2}{\check{\tau}_n\hat{\tau}_n} \tag{3.9}$$

$$1 - B_{n+1}C_n = \gamma^C\ \frac{\tau_n^2}{\check{\tau}_{n-1}\hat{\tau}_{n+1}}, \tag{3.10}$$

calculating f_n,

$$f_n = \frac{(\gamma^A)^2}{\gamma^B\gamma^C}\ \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \tag{3.11}$$

and substituting the above formulae into (3.2) and (3.3) one can obtain

$$i\partial B_n = \Gamma^B \frac{1 - B_nC_n}{1 - B_{n-1}C_n}(B_n - B_{n-1}) \tag{3.12}$$

$$i\partial C_n = \Gamma^C\ \frac{1 - B_nC_n}{1 - B_nC_{n+1}}(C_{n+1} - C_n) \tag{3.13}$$

and

$$-i\bar{\partial} B_n = \bar{\Gamma}^B\ \frac{1 - B_nC_n}{1 - B_{n+1}C_n}(B_{n+1} - B_n) \tag{3.14}$$

$$-i\bar{\partial} C_n = \bar{\Gamma}^C\ \frac{1 - B_nC_n}{1 - B_nC_{n-1}}(C_n - C_{n-1}) \tag{3.15}$$

where

$$\Gamma^B = \frac{\gamma^B}{\gamma^A}\frac{\alpha}{\beta}, \quad \Gamma^C = \frac{\gamma^B}{\gamma^A}\frac{\alpha}{\beta}, \quad \bar{\Gamma}^B = \frac{\gamma^C}{\gamma^A}\frac{\bar{\alpha}}{\beta}, \quad \bar{\Gamma}^C = \frac{\gamma^C}{\gamma^A}\frac{\bar{\alpha}}{\beta}. \tag{3.16}$$

Thus, to finish solution of our problem one has impose the condition

$$\Gamma^B = \Gamma^C = \zeta, \quad \bar{\Gamma}^B = \bar{\Gamma}^C = \bar{\zeta} = 1/\zeta. \tag{3.17}$$

In this way we have reduced Eqs. (2.19), (2.20), and hence Eq. (2.4), to the set of the bilinear equations (3.2)–(3.5). An important question that arises now is the question about compatibility of this system. We do not present here an explicit proof of the fact that Eqs. (3.2)–(3.5) are compatible because (i) we present (in the next section) their explicit solutions and (ii) show their relation to a well-known nonlinear compatible system — the Ablowitz–Ladik hierarchy (ALH) [6]. To do the latter let us consider the matrix

$$\Phi_n = \frac{1}{\tau_{n-1}} \begin{pmatrix} \hat{\tau}_n & \check{\rho}_{n-1} \\ -\hat{\sigma}_n & \check{\tau}_{n-1} \end{pmatrix}. \tag{3.18}$$

Calculating its determinant,

$$\det \Phi_n = \gamma^A \frac{\tau_n}{\tau_{n-1}}, \tag{3.19}$$

and inverse one can obtain

$$\Phi_{n+1} = U_n \Phi_n \tag{3.20}$$

with

$$U_n = \frac{1}{\gamma^A} \begin{pmatrix} \gamma^C & \beta \dfrac{\rho_n}{\tau_n} \\ \beta \dfrac{\sigma_n}{\tau_n} & \gamma^B \end{pmatrix} \tag{3.21}$$

and

$$i\partial \Phi_n = V_n \Phi_n, \quad i\bar{\partial} \Phi_n = \bar{V}_n \Phi_n \tag{3.22}$$

with

$$V_n = \begin{pmatrix} i\partial \ln \dfrac{\hat{\tau}_n}{\tau_{n-1}} + \dfrac{\alpha}{\gamma^A} \dfrac{\rho_{n-1}\hat{\sigma}_n}{\tau_{n-1}\hat{\tau}_n} & \dfrac{\alpha}{\gamma^A} \dfrac{\rho_{n-1}}{\tau_{n-1}} \\ \dfrac{\alpha}{\gamma^A} \dfrac{\sigma_n}{\tau_n} & i\partial \ln \dfrac{\check{\tau}_{n-1}}{\tau_{n-1}} - \dfrac{\alpha}{\gamma^A} \dfrac{\check{\rho}_{n-1}\sigma_n}{\check{\tau}_{n-1}\tau_n} \end{pmatrix}. \tag{3.23}$$

and

$$\bar{V}_n = \begin{pmatrix} i\bar{\partial} \ln \dfrac{\hat{\tau}_n}{\tau_{n-1}} - \dfrac{\bar{\alpha}}{\gamma^A} \dfrac{\rho_n\hat{\sigma}_n}{\tau_n\hat{\tau}_n} & -\dfrac{\bar{\alpha}}{\gamma^A} \dfrac{\rho_n}{\tau_n} \\ -\dfrac{\bar{\alpha}}{\gamma^A} \dfrac{\sigma_{n-1}}{\tau_{n-1}} & i\bar{\partial} \ln \dfrac{\check{\tau}_{n-1}}{\tau_{n-1}} + \dfrac{\bar{\alpha}}{\gamma^A} \dfrac{\check{\rho}_{n-1}\sigma_{n-1}}{\check{\tau}_{n-1}\tau_{n-1}} \end{pmatrix}. \tag{3.24}$$

Inspecting (3.20)–(3.24) one can conclude, after eliminating the unnecessary constants, introducing

$$q_n = \frac{\sigma_n}{\tau_n}, \quad r_n = \frac{\rho_n}{\tau_n} \tag{3.25}$$

and making some simple gauge transformations, that (3.20) with (3.21) is nothing but the spectral problem of the ALH whereas Eqs. (3.22) with (3.23) and (3.24) describe its first positive and negative flows. So, bilinear equations (3.2)–(3.5) belong to the ALH. This leads to two important results: (i) they are compatible and (ii) we can use already known solutions for the ALH to get solutions of our equations.

To expose the inner structure of Eqs. (3.2)–(3.5) and to make the following formulae more readable it seems useful to introduce instead of the triplet ρ_n, τ_n and σ_n an infinite set of tau-functions τ_n^m,

$$\rho_n = \tau_n^{-1}, \quad \tau_n = \tau_n^0, \quad \sigma_n = \tau_n^1. \tag{3.26}$$

In new terms equations (3.2)–(3.5) become

$$iD\, \check{\tau}_{n-1}^{m-1} \cdot \hat{\tau}_n^m = \alpha\, \tau_{n-1}^{m-1} \tau_n^m \tag{3.27}$$

$$-i\bar{D}\, \check{\tau}_{n-1}^{m-1} \cdot \hat{\tau}_n^m = \bar{\alpha}\, \tau_{n-1}^m \tau_n^{m-1} \tag{3.28}$$

$$\check{\tau}_n^{m-1} \hat{\tau}_n^m - \check{\tau}_{n-1}^{m-1} \hat{\tau}_{n+1}^m = \beta\, \tau_n^{m-1} \tau_n^m \tag{3.29}$$

for $m = 0, 1$ and

$$\check{\tau}_{n-1}^m \hat{\tau}_n^m + \check{\tau}_{n-1}^{m-1} \hat{\tau}_n^{m+1} = \gamma^A\, \tau_{n-1}^m \tau_n^m \tag{3.30}$$

$$\check{\tau}_n^m \hat{\tau}_n^m + \check{\tau}_{n-1}^{m-1} \hat{\tau}_{n+1}^{m+1} = \gamma^B\, (\tau_n^m)^2 \tag{3.31}$$

$$\check{\tau}_n^{m-1} \hat{\tau}_n^{m+1} + \check{\tau}_{n-1}^m \hat{\tau}_{n+1}^m = \gamma^C\, (\tau_n^m)^2 \tag{3.32}$$

for $m = 0$. These equations are a part of the generalized ALH [7] and can be solved without imposing restrictions on m, for $-\infty < m < \infty$.

4. Dark Solitons

4.1. *Dark solitons of the ALH*

Here we would like to present some basic formulae describing the dark-soliton solutions of the ALH that we then use to obtain solutions of our problem.

The dark solitons for the AL equations were obtained in [8] using the inverse scattering method. In [9] these solutions were derived, using purely algebraic method based on the Fay-like identities for the determinants of some special matrices. Here we use notation slightly different from one of [9], which makes the following formulae more simple and clear.

The key objects behind the dark-soliton solutions of the ALH are the determinants

$$\omega(A) = \det |\mathbb{I} + A| \tag{4.1}$$

with matrices A satisfying

$$LA - AR = |\ell\rangle\langle a|. \tag{4.2}$$

Here $\mathbb{I}$ is the $N \times N$ unit matrix, L and R are constant diagonal matrices,

$$\begin{aligned} L &= \operatorname{diag}(L_1, \ldots, L_N), \\ R &= \operatorname{diag}(R_1, \ldots, R_N), \end{aligned} \tag{4.3}$$

$|\ell\rangle$ is a constant N-column, $|\ell\rangle = (\ell_1, \ldots, \ell_N)^T$, and $\langle a|$ is a N-row depending on the coordinates describing the ALH flows: in our case $\langle a| = \langle a(z, \bar{z})| = (a_1(z, \bar{z}), \ldots, a_N(z, \bar{z}))$.

In what follows we use "shifted" determinants

$$\omega_\zeta = \mathbb{T}_\zeta \, \omega, \quad \omega_{\xi\eta} = \mathbb{T}_\xi \mathbb{T}_\eta \, \omega \tag{4.4}$$

where

$$\mathbb{T}_\zeta^l \, \omega = \omega(A \, H_\zeta^l), \quad l = \pm 1 \tag{4.5}$$

with

$$H_\zeta = (L - \zeta\mathbb{I})(R - \zeta\mathbb{I})^{-1}. \tag{4.6}$$

An important property of these determinants, that we repeatedly use below, is the Fay's identity

$$(\xi - \eta)\omega_\zeta\omega_{\xi\eta} + (\eta - \zeta)\omega_\xi\omega_{\eta\zeta} + (\zeta - \xi)\omega_\eta\omega_{\zeta\xi} = 0 \tag{4.7}$$

which can be proved directly.

Using the limit procedure one can introduce differential operators ∂_ζ as

$$\mathbb{T}_\zeta^{-1} \, \mathbb{T}_{\zeta+\delta} \, \omega = \omega + i\delta \, \partial_\zeta\omega + O(\delta^2) \tag{4.8}$$

or

$$i\partial_\zeta A = AX_\zeta \tag{4.9}$$

where

$$X_\zeta = (L - R)(L - \zeta\mathbb{I})^{-1}(R - \zeta\mathbb{I})^{-1}. \tag{4.10}$$

One can obtain from (4.7) many differential Fay's identities of the following type:

$$i(\zeta - \alpha)(\zeta - \beta) \; D_\zeta \, \omega_\alpha \cdot \omega_\beta = (\alpha - \beta)[(\mathbb{T}_\zeta^{-1}\omega_{\alpha\beta})(\mathbb{T}_\zeta\omega) - \omega_\alpha\omega_\beta] \tag{4.11}$$

where

$$D_\zeta \, \omega_\alpha \cdot \omega_\beta = (\partial_\zeta\omega_\alpha)\omega_\beta - \omega_\alpha(\partial_\zeta\omega_\beta). \tag{4.12}$$

The matrices L and R used in the ALH context are not independent: they are related by

$$(L - \kappa\mathbb{I})(R - \kappa\mathbb{I}) = -\rho^2 \, \mathbb{I} \tag{4.13}$$

with constant parameters κ and ρ. Relations of this kind play crucial role in the construction of dark solitons for the ALH, so it seems useful introduce the notion of "duality": two complex numbers ξ and ξ^* are said to be dual if

$$(\zeta - \kappa)(\zeta^* - \kappa) = -\rho^2 \tag{4.14}$$

which leads to an alternative definition

$$H_\zeta H_{\zeta^*} = H_\kappa, \quad \kappa = \text{const}. \tag{4.15}$$

Many Fay-like formulae can be simplified when rewritten in terms of dual numbers. In particular, Eq. (4.11) leads to

$$iD_\zeta\, \omega_\alpha \cdot \omega_{\alpha^*} = \frac{\alpha - \alpha^*}{(\zeta - \alpha)(\zeta - \alpha^*)}[(\mathbb{T}_\zeta^{-1}\omega_\kappa)(\mathbb{T}_\zeta \omega) - \omega_\alpha \omega_{\alpha^*}]. \tag{4.16}$$

Given some fixed number μ and its dual, which will be denoted by ν, $\nu = \mu^*$, one can construct an doubly infinite set of matrices/determinants

$$\omega_n^m = \omega(A_n^m) \tag{4.17}$$

where

$$A_n^m = AH_\mu^m H_\nu^n \tag{4.18}$$

and derive from (4.7) a lot of lattice Fay's identities the most important of which is

$$(\omega_n^m)^2 = \rho_\mu^2 \omega_n^{m-1}\, \omega_n^{m+1} + \rho_\nu^2 \omega_{n-1}^m\, \omega_{n+1}^m \tag{4.19}$$

with

$$\rho_\mu = \sqrt{\frac{\mu - \kappa}{\mu - \nu}}, \quad \rho_\nu = \sqrt{\frac{\kappa - \nu}{\mu - \nu}}. \tag{4.20}$$

In order to ensure the involution $\overline{\tau_n^m} = \tau_n^{-m}$ (where overline stands for the complex conjugation) which appears in physical applications of the Ablowitz–Ladik model one has to restrict himself to the case of real κ, μ and ν,

$$\nu < \kappa < \mu \tag{4.21}$$

(which leads to $\operatorname{Im} \rho_\mu = \operatorname{Im} \rho_\nu = 0$) and to choose the matrices L to be of the form

$$L = \mu + \sqrt{\mu - \nu}\sqrt{\mu - \kappa}\, E, \quad E = \operatorname{diag}(e^{i\psi_j}) \tag{4.22}$$

with real angles ψ_j (compare with the parametrization (2.27) of the eigenvalues of the scattering problem for the ALH used in [8]).

Calculating R from (4.22) and (4.13) one can verify that in this case

$$\overline{H_\mu} = H_\mu^{-1}, \quad \overline{H_\nu} = H_\nu \tag{4.23}$$

and

$$\overline{A_n^m} = A_n^{-m} \tag{4.24}$$

provided

$$\overline{A} = A. \tag{4.25}$$

The last condition can be met by choosing properly the constants ℓ_j and a_j in (4.2). By straightforward algebra one can get

$$L_j - R_k = (\mu - \nu)\, \frac{e^{i(\psi_j + \psi_k)} + \rho_\mu(e^{i\psi_j} + e^{i\psi_k}) + 1}{1 + \rho_\mu^{-1} e^{i\psi_k}} \tag{4.26}$$

and rewrite the matrix A, after eliminating excessive constants, as

$$A(z, \bar{z})_{jk} = D_{jk} c_k(z, \bar{z}) \tag{4.27}$$

where

$$D_{jk} = \left[\cos\left(\frac{\psi_j + \psi_k}{2}\right) + \rho_\mu \cos\left(\frac{\psi_j - \psi_k}{2}\right) \right]^{-1} \tag{4.28}$$

and $c_k(z, \bar{z})$ are some real functions.

4.2. *Dark soliton solutions of* (3.27)–(3.32)

After we have established the relation of our model with the ALH and knowing the structure of the ALH dark solitons, we can reformulate the *ansatz* we use as follows: all tau-functions are related by the $\mathbb{T}$-shifts ($\mathbb{T}_\nu$, $\mathbb{T}_\mu$ and $\mathbb{T}_{\xi^*}$ for some given ξ). The sequence $\rho \to \tau \to \sigma$ is generated by $\mathbb{T}_\mu$, the sequence $\check{\tau} \to \tau \to \hat{\tau}$ is generated by $\mathbb{T}_{\xi^*}$, as is depicted in Fig. 1, while the nodes n and $n+1$ are related by $\mathbb{T}_\nu$. Thus we can say that our tau-functions occupy sites of a three-dimensional lattice. However in what follows we do not use the three-indices and adhere to the τ_n^m notation.

To find solutions of (3.27)–(3.32) we look for our tau-functions in the form

$$\tau_n^m = \rho_\mu^{m^2} \rho_\nu^{n^2} u^m v^n \omega_n^m \tag{4.29}$$

with similar formulae for $\check{\tau}_n^m$ and $\hat{\tau}_n^m$

$$\check{\tau}_n^m = \rho_\mu^{m^2} \rho_\nu^{n^2} \check{u}^m \check{v}^n \check{\omega}_n^m \tag{4.30}$$

$$\hat{\tau}_n^m = \rho_\mu^{m^2} \rho_\nu^{n^2} \hat{u}^m \hat{v}^n \hat{\omega}_n^m \tag{4.31}$$

where $\check{\omega}_n^m$ and $\hat{\omega}_n^m$ are related to ω_n^m by means of $\mathbb{T}_{\xi^*}^{\pm 1}$ shifts:

$$\check{\omega}_n^m = \mathbb{T}_{\xi^*} \omega_n^m, \quad \hat{\omega}_n^m = \mathbb{T}_{\xi^*}^{-1} \omega_n^m \tag{4.32}$$

or

$$\check{\omega}_n^m = \mathbb{T}_\xi^{-1} \omega_{n+1}^{m+1}, \quad \hat{\omega}_n^m = \mathbb{T}_\xi \omega_{n-1}^{m-1}. \tag{4.33}$$

$$\begin{tikzcd}
\hat{\rho} \arrow[r, "\mathbb{T}_\mu"] & \hat{\tau} \arrow[r, "\mathbb{T}_\mu"] & \hat{\sigma} \\
\rho \arrow[u, "\mathbb{T}_{\xi^*}"'] \arrow[r, "\mathbb{T}_\mu"] & \tau \arrow[u, "\mathbb{T}_{\xi^*}"'] \arrow[r, "\mathbb{T}_\mu"] & \sigma \arrow[u, "\mathbb{T}_{\xi^*}"'] \\
\check{\rho} \arrow[u, "\mathbb{T}_{\xi^*}"'] \arrow[r, "\mathbb{T}_\mu"] & \check{\tau} \arrow[u, "\mathbb{T}_{\xi^*}"'] \arrow[r, "\mathbb{T}_\mu"] & \check{\sigma} \arrow[u, "\mathbb{T}_{\xi^*}"']
\end{tikzcd}$$

Fig. 1. $\mathbb{T}_\mu$-$\mathbb{T}_{\xi^*}$ lattice of tau-functions (for $n = \text{constant}$).

Taking ξ, η, ζ in (4.7) being equal to different triples from $\{\mu, \nu, \kappa, \xi, \xi^*\}$ one can conclude that restrictions

$$\left(\rho_\mu \frac{\hat{u}}{u}\right)^2 = -\frac{\xi - \mu}{\xi^* - \mu}, \quad \left(\rho_\nu \frac{\hat{v}}{v}\right)^2 = \frac{\xi - \nu}{\xi^* - \nu}, \quad \left(\rho_\mu \rho_\nu \frac{\hat{u}}{u}\frac{\hat{v}}{v}\right)^2 = -\frac{\xi - \kappa}{\xi^* - \kappa} \tag{4.34}$$

lead to (3.29)–(3.32) with

$$\beta = \frac{\xi^* - \xi}{\xi^* - \nu}\frac{\hat{u}}{u} \tag{4.35}$$

$$\gamma^A = \frac{\xi^* - \xi}{\xi^* - \mu}\frac{\hat{v}}{v} \tag{4.36}$$

$$\gamma^B = \frac{\xi^* - \xi}{\xi^* - \kappa} \tag{4.37}$$

$$\gamma^C = \frac{(\xi^* - \xi)(\mu - \nu)}{(\xi^* - \mu)(\xi^* - \nu)}. \tag{4.38}$$

Considering the dependence on z and $\bar{z}$ it can be shown that to meet (3.27) and (3.28) one has to take

$$i\partial A = AX, \quad -i\bar{\partial} A = A\widetilde{X} \tag{4.39}$$

where

$$X = \mathrm{const} \cdot X_\kappa, \quad \widetilde{X} = \mathrm{const} \cdot X_\nu \tag{4.40}$$

We will write the explicit value of the corresponding constants later, after discussing the involution (complex conjugation) and reality requirement.

4.3. *Complex conjugation and parametrization*

Recalling the definitions of our tau-functions we can present B_n and C_n as

$$B_n = B_n^0, \quad C_n = C_n^0 \tag{4.41}$$

where

$$B_n^m = \frac{1}{uv}\left(\frac{\mu - \xi^*}{\xi - \mu}\right)^{m-\frac{1}{2}}\left(\frac{\xi^* - \nu}{\xi - \nu}\right)^{n-\frac{1}{2}}\frac{\breve{\omega}_{n-1}^{m-1}}{\hat{\omega}_n^m} \tag{4.42}$$

$$C_n^m = -uv\left(\frac{\xi - \mu}{\mu - \xi^*}\right)^{m+\frac{1}{2}}\left(\frac{\xi - \nu}{\xi^* - \nu}\right)^{n-\frac{1}{2}}\frac{\hat{\omega}_n^{m+1}}{\breve{\omega}_{n-1}^m}. \tag{4.43}$$

The involution that ensures reality of $\boldsymbol{\sigma}_n$ and that is consistent with the ALH is

$$B_n^m = -\overline{C_n^{-m}} \tag{4.44}$$

where overline stands for complex conjugation. A simple analysis of (4.42), (4.43) and (4.18) leads to the restrictions

$$\mathrm{Im}\,\frac{\xi^* - \mu}{\xi - \mu} = 0, \quad \left|\frac{\xi^* - \nu}{\xi - \nu}\right| = 1 \tag{4.45}$$

and

$$H_\xi \overline{H_\xi} = H_\nu \tag{4.46}$$

that should be added to the restriction $\overline{A_n^m} = A_n^{-m}$ discussed above.

The family of ξ and ξ^* that satisfy (4.14), (4.46) and (4.45) can be parametrized as

$$\begin{aligned} \xi &= \nu + ae^{i(\phi+\theta)} \\ \xi^* &= \nu + ae^{i(\phi-\theta)} \end{aligned} \tag{4.47}$$

where

$$a = \sqrt{\mu - \nu}\sqrt{\kappa - \nu} = \frac{\rho}{\rho_\mu} \tag{4.48}$$

and the angles ϕ and θ are related by

$$\cos\phi = \rho_\nu \cos\theta. \tag{4.49}$$

Using another parametrization of the matrices L and R, stemming from (4.22) and (4.13),

$$\begin{aligned} L_j &= \nu + a\exp(\chi_j + i\phi_j) \\ R_j &= \nu + a\exp(-\chi_j + i\phi_j) \end{aligned} \tag{4.50}$$

where the quantities χ_j and ϕ_j are defined by

$$\exp(\chi_j + i\theta_j) = \frac{1}{\rho_\nu}(\rho_\mu + e^{i\psi_j}) \tag{4.51}$$

one can obtain

$$H_\xi = -\mathrm{diag}(e^{\chi_j - i\gamma_j^+}), \quad H_{\xi^*} = -\mathrm{diag}(e^{\chi_j - i\gamma_j^-}) \tag{4.52}$$

with

$$\gamma_j^\pm = \phi - \phi_j \pm \theta - 2\arg[e^{\chi_j} - e^{i(\phi-\phi_j\pm\theta)}]. \tag{4.53}$$

4.4. *Dark solitons of the Toda–Heisenberg chain*

Now we have all necessary to write dark soliton solutions of the Toda–Heisenberg chain. Their structure is given by (4.42) and (4.43). The dependence on z and $\bar{z}$ enters through the matrices A and the factor uv,

$$i\partial A = AX, \quad -i\bar{\partial} A = A\widetilde{X} \tag{4.54}$$

where

$$X = -\frac{\lambda\rho^2}{\mu - \nu}X_\kappa, \quad \widetilde{X} = \lambda^{-1}(\kappa - \nu)X_\nu \tag{4.55}$$

or, explicitly,

$$X = \frac{\lambda}{\mu - \nu}(L - R), \quad \widetilde{X} = \frac{\lambda^{-1}}{\mu - \nu}\overline{(L - R)}. \tag{4.56}$$

This dependence of A, as follows from the differential Fay's identity (4.11), leads to

$$iD\,\omega_{\alpha^*}\cdot\omega_\alpha = \frac{\lambda(\alpha^*-\alpha)}{\mu-\nu}[\omega\,\omega_\kappa - \omega_{\alpha^*}\omega_\alpha] \tag{4.57}$$

$$i\bar{D}\,\omega_{\alpha^*}\cdot\omega_\alpha = \frac{\lambda^{-1}\overline{(\alpha^*-\alpha)}}{\mu-\nu}[\omega_\mu\omega_\nu - \omega_{\alpha^*}\omega_\alpha] \tag{4.58}$$

and hence to

$$iD\,\breve{\omega}_{n-1}^{m-1}\cdot\hat{\omega}_n^m = \frac{\lambda(\xi^*-\xi)}{\mu-\nu}[\omega_{n-1}^{m-1}\,\omega_n^m - \breve{\omega}_{n-1}^{m-1}\,\hat{\omega}_n^m] \tag{4.59}$$

$$iD\,\hat{\omega}_n^{m+1}\cdot\breve{\omega}_{n-1}^m = \frac{\lambda(\xi^*-\xi)}{\mu-\nu}[\breve{\omega}_{n-1}^m\,\hat{\omega}_n^{m+1} - \omega_{n-1}^m\,\omega_n^{m+1}] \tag{4.60}$$

and

$$i\bar{D}\,\breve{\omega}_{n-1}^{m-1}\cdot\hat{\omega}_n^m = \frac{\lambda^{-1}\overline{(\xi^*-\xi)}}{\mu-\nu}[\omega_n^{m-1}\,\omega_{n-1}^m - \breve{\omega}_{n-1}^{m-1}\,\hat{\omega}_n^m] \tag{4.61}$$

$$i\bar{D}\,\hat{\omega}_n^{m+1}\cdot\breve{\omega}_{n-1}^m = \frac{\lambda^{-1}\overline{(\xi^*-\xi)}}{\mu-\nu}[\breve{\omega}_{n-1}^m\,\hat{\omega}_n^{m+1} - \omega_n^m\,\omega_{n-1}^{m+1}]. \tag{4.62}$$

The extra terms in the right-hand sides of the above formulae can be eliminated by taking

$$uv = \mathrm{const}\times e^{i\varphi} \tag{4.63}$$

with

$$\varphi = \lambda\frac{\xi^*-\xi}{\mu-\nu}z + \lambda^{-1}\frac{\overline{(\xi^*-\xi)}}{\mu-\nu}\bar{z}. \tag{4.64}$$

It is easy to verify that all reality conditions are met provided

$$|\lambda| = 1. \tag{4.65}$$

Now one can check that tau-functions τ_n^m satisfy Eqs. (3.27) and (3.28) with

$$\alpha = \lambda\frac{\xi^*-\xi}{\mu-\nu}\frac{\hat{u}\hat{v}}{uv}, \quad \bar{\alpha} = -\lambda^{-1}\frac{\overline{(\xi^*-\xi)}}{\mu-\nu}\frac{\hat{u}\hat{v}}{uv}. \tag{4.66}$$

Gathering all constants, one comes to the conclusion that for $m=0$ the quantities B_n, C_n solve Eqs. (3.12)–(3.15) with

$$\Gamma^B = \Gamma^C = -\lambda\, e^{2i\arg(\xi^*-\xi)} \tag{4.67}$$

and

$$\bar{\Gamma}^B = \bar{\Gamma}^C = -\lambda^{-1}\, e^{-2i\arg(\xi^*-\xi)}. \tag{4.68}$$

Finally, calculating from (4.47) and (4.50) the coefficients that describe the z-, $\bar{z}$-dependence,

$$\frac{L_j - R_j}{\mu-\nu} = 2\rho_\mu\,\cos\theta_j\, e^{i\phi_j} \tag{4.69}$$

and

$$\frac{\xi^* - \xi}{\mu - \nu} = -2i\rho_\nu \sin\theta\, e^{i\phi}, \tag{4.70}$$

one can present B_n and C_n as follows:

$$B_n(x,y) = B_* \exp\{-2in\theta - i\varphi(x,y)\} \frac{\Delta_n^+(x,y)}{\Delta_n^-(x,y)} \tag{4.71}$$

and

$$C_n(x,y) = -\overline{B_n(x,y)}. \tag{4.72}$$

Here

$$B_* = \frac{1}{\rho_\mu} |1 - \rho_\nu \exp(i\psi)|, \tag{4.73}$$

$$\Delta_n^\pm(x,y) = \det|\delta_{jk} - D_{jk} \exp\{2n\chi_k + a_k(x,y) + i\gamma_k^\pm\}|, \tag{4.74}$$

with the coefficients D_{jk} and $\gamma_k^\pm$ being defined in (4.28), (4.53). The phase φ and the functions a_k are given by

$$\varphi(x,y) = 4\rho_\nu \sin\theta(x\sin\alpha_0 + y\cos\alpha_0) + \varphi^{(0)} \tag{4.75}$$

with

$$\alpha_0 = \phi + \arg\lambda \tag{4.76}$$

and

$$a_k(x,y) = 4\rho_\mu \cos\theta_k(x\sin\alpha_k + y\cos\alpha_k) + a_k^{(0)} \tag{4.77}$$

with

$$\alpha_k = \phi_k + \arg\lambda \tag{4.78}$$

where $\varphi^{(0)}$ and $a_k^{(0)}$ are arbitrary real constants.

These formulae, together with the vector-matrix correspondence (2.1), lead to the dark soliton solutions of the Toda–Heisenberg chain:

$$\boldsymbol{\sigma}_n = \frac{1}{1+|B_n|^2} \begin{pmatrix} 2\,\mathrm{Re}\,B_n \\ -2\,\mathrm{Im}\,B_n \\ 1-|B_n|^2 \end{pmatrix}. \tag{4.79}$$

5. Conclusion

We have studied the (2+1)-dimensional system that was reduced to the integrable Ablowitz–Ladik equations. Using this reduction we have derived its soliton solutions. It is clear that using this approach one can also derive a wide range of other solutions starting from the ones already known for the ALH. Thus Eqs. (1.11), (1.9) possess a set of solutions that are typical for the integrable systems (solitons, algebro-geometric solutions etc.). At the same time it is not clear whether this model is integrable or we deal with another example of soliton equation that is not integrable (see, e.g., [10]), which can occur in multidimensions, contrary (as is presumed) to the $(1+1)$-dimensional case. However, this very interesting question, as well as other related questions (such as, e.g., the Painlevé test, the symmetry analysis), is out of the scope of this paper and may constitute the subject of the subsequent studies.

Appendix A

Rewriting the *ansatz* (2.15), (2.16) in the form

$$\begin{cases} i\partial\, B_n = \zeta_{n-1} Y_{n-1}(1 - B_n C_n)(B_n - B_{n-1}) \\ i\partial\, C_n = \zeta_n Y_n (1 - B_n C_n)(C_{n+1} - C_n) \end{cases} \tag{A.1}$$

and

$$\begin{cases} -i\bar\partial\, B_n = \bar\zeta_n \bar Y_n (1 - B_n C_n)(B_{n+1} - B_n) \\ -i\bar\partial\, C_n = \bar\zeta_{n-1} \bar Y_{n-1}(1 - B_n C_n)(C_n - C_{n-1}) \end{cases} \tag{A.2}$$

one can calculate $\partial\bar\partial\, B_n$ as

$$\partial\bar\partial\, B_n = i\partial(-i\bar\partial\, B_n) = i\partial\, \bar\zeta_n \bar Y_n (1 - B_n C_n)(B_{n+1} - B_n) \tag{A.3}$$

which leads, after repeated usage of (A.1) to

$$A_n \mathcal{L}_n^B = (i\partial\bar\zeta_n + \zeta_n\bar\zeta_n)\bar Y_n (B_{n+1} - B_n) - \zeta_{n-1}\bar\zeta_n Y_{n-1}(B_n - B_{n-1}). \tag{A.4}$$

Interchanging the ∂ and $\bar\partial$ derivatives, $\partial\bar\partial\, B_n = -i\bar\partial(i\partial\, B_n)$, one can obtain

$$A_n \mathcal{L}_n^B = \zeta_{n-1}\bar\zeta_n \bar Y_n (B_{n+1} - B_n) - (i\bar\partial\zeta_{n-1} + \zeta_{n-1}\bar\zeta_{n-1}) Y_{n-1}(B_n - B_{n-1}). \tag{A.5}$$

Similar calculations for $\partial\bar\partial\, C_n$ lead to

$$A_n \mathcal{L}_n^C = (-i\bar\partial\zeta_n + \zeta_n\bar\zeta_n) Y_n (C_{n+1} - C_n) - \zeta_n\bar\zeta_{n-1}\bar Y_{n-1}(C_n - C_{n-1}) \tag{A.6}$$

$$= \zeta_n\bar\zeta_{n-1} Y_n (C_{n+1} - C_n) + (i\partial\bar\zeta_{n-1} - \zeta_{n-1}\bar\zeta_{n-1})\bar Y_{n-1}(C_n - C_{n-1}). \tag{A.7}$$

Comparing the right-hand sides of the above equations with each other and with the right-hand sides of Eq. (2.10) one can conclude that conditions

$$\zeta_{n\pm1} = \zeta_n, \tag{A.8}$$

$$\bar\zeta_{n\pm1} = \bar\zeta_n, \tag{A.9}$$

$$\partial\zeta_n = \bar\partial\zeta_n = \partial\bar\zeta_n = \bar\partial\bar\zeta_n = 0 \tag{A.10}$$

and

$$\zeta_n\bar\zeta_n = 1 \tag{A.11}$$

validate the *ansatz* (2.19), (2.20).

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