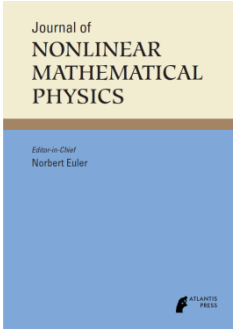

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

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To cite this article: Oleksiy O. Vakhnenko (2011) Semidiscrete Integrable Nonlinear Systems Generated by the New Fourth-Order Spectral Operator: Local Conservation Laws, Journal of Nonlinear Mathematical Physics 18:3, 401–414, DOI: https://doi.org/10.1142/S1402925111001672

To link to this article: https://doi.org/10.1142/S1402925111001672

Published online: 04 January 2021

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 3 (2011) 401–414

DOI: 10.1142/S1402925111001672

SEMIDISCRETE INTEGRABLE NONLINEAR SYSTEMS GENERATED BY THE NEW FOURTH-ORDER SPECTRAL OPERATOR. LOCAL CONSERVATION LAWS

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Received 24 September 2010
Accepted 17 March 2011

Starting with the semidiscrete integrable nonlinear Schrödinger system on a zigzag-runged ladder lattice we have presented the generalization and an essentially off-diagonal enlargement of its spectral operator which in the framework of zero-curvature equation allows to generate at least two new types of semidiscrete integrable nonlinear systems. The two types of evolutionary operators consistent with the extended spectral operator are proposed. In order to fix arbitrary sampling functions in each type of evolution operators we have to rely upon a restricted collection of lowest local conservation laws whose local densities are independent on the type of admissible evolution operators. For this purpose the modified procedure of seeking the infinite hierarchy of local conservation laws based upon several distinct generating functions has been developed and some lowest local conservation laws have been explicitly obtained.

Keywords: Zero-curvature equation; fourth-order spectral operator; generating functions; local conservation laws.

PACS Number(s): 02.30.Ik, 11.10.Lm, 45.05.+x

1. Introduction

The nonlinear Schrödinger equations on one-dimensional or quasi-one-dimensional lattices are known to describe a number of phenomena in regular optical [1, 2], semiconducting [3, 4] and electric [5] superstructures as well as in the regular macromolecular structures of both natural [6, 7] and synthetic [8, 9] origin. In this context the integrable versions of such equations are able to give us the first and sometimes crucial hint how to handle the real physical problem both qualitatively [10] and quantitatively [11–13].

In our previous papers [14–16] we have managed to extend the standard integrable nonlinear Schrödinger system [17–19] by complicating the geometrical configuration of primary lattice from purely uniform structure with just a single site in the unit cell to the nonuniform (but regular) structure with several sites in the unit cell.

The main source for such sort of activity was the simple but important fact that even in linear limit the behavior of system embedded into the nonuniform lattice is controlled by the splitted zone structure of elementary excitations in contrast to the uniform lattice where the zone splitting is principally prohibited. Thus, it was reasonable to expect that the dynamics of nonlinear system on regular nonuniform lattice should be more rich and interesting in comparison with purely uniform case. As an example we would like to mention the soliton dynamics in nonlinear Schrödinger system on a zigzag-runged ladder lattice [14, 15]. The lattice of above geometrical configuration is typical of the $(1,1)$ armchair boron nanotube [20].

Naturally, it would be interesting to enrich the early obtained integrable systems [14–16] by some nonstandard degrees of freedom simultaneously preserving the integrability of new (enriched) ones.

In this and two forthcoming articles we will present the series of semidiscrete integrable nonlinear systems constructed in the framework of zero-curvature representation and characterized by the specific off-diagonal enlargement of the spectral operator generalized and augmented in comparison to the spectral operator for the nonlinear Schrödinger system on a zigzag-runged ladder lattice [14, 15]. We will classify the possible integrable systems proposing two principally distinct types of admissible evolution operators on the one hand and choosing the fixation of initially arbitrary sampling functions within the evolution operator of each type on the other. The most proper versions of sampling fixations will be stipulated by certain restricted collection of first local conservation laws associated with the adopted (enlarged) spectral operator. Thus, it looks reasonable that the essential part of the first article should be devoted to the extraction of these and some other local conservation laws from the infinite hierarchy.

2. Enlarged Spectral Operator and Zero-Curvature Equation

The spectral operator $L(n|z)$ giving rise to the semidiscrete integrable Schrödinger system on a zigzag-runged ladder lattice [14, 15] looks as follows

$$L(n|z) = \begin{pmatrix} z & iq_+(n) \\ ir_+(n) & z^{-1} \end{pmatrix} \begin{pmatrix} z & iq_-(n) \\ ir_-(n) & z^{-1} \end{pmatrix} \tag{2.1}$$

where the quantities $q_+(n)$, $r_+(n)$ and $q_-(n)$, $r_-(n)$ are supposed to be functions of the discrete space variable n and continuous time variable τ and might be treated as the so-called nearly probability amplitudes while the letter z stands for the time independent spectral parameter.

The most natural generalization of above spectral operator (2.1) can be written in the form

$$L(n|z) = \begin{pmatrix} r_{22}(n)z^2 + t_{22}(n) & s_{23}(n)z + u_{23}(n)z^{-1} \\ s_{32}(n)z + u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} \end{pmatrix} \tag{2.2}$$

which in the framework of zero-curvature equation [21]

$$\dot{L}(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \tag{2.3}$$

enables to produce an alternative representation for the integrable ladder system with background-controlled intersite resonant couplings [16] provided the ansatz for the evolution operator $A(n|z)$ is taken to be

$$A(n|z) = \begin{pmatrix} a_{22}(n)z^2 + c_{22}(n) & b_{23}(n)z + d_{23}(n)z^{-1} \\ b_{32}(n)z + d_{32}(n)z^{-1} & c_{33}(n) + e_{33}(n)z^{-2} \end{pmatrix} \tag{2.4}$$

and the proper parametrization for the prototype field functions $r_{22}(n)$, $t_{22}(n)$, $s_{23}(n)$, $u_{23}(n)$ and $s_{32}(n)$, $u_{32}(n)$, $t_{33}(n)$, $v_{33}(n)$ is chosen. The dot written over the spectral operator in the left-hand side of zero-curvature equation (2.3) denotes the derivative with respect to time τ.

The main question of the present work was to enlarge the generalized spectral operator (2.2) in order to enrich the previous models [14–16] by the additional degrees of freedom or even to generate principally new semidiscrete integrable nonlinear models potentially suitable for the physical applications to the multifield systems on quasi-one-dimensional lattices.

In order to guarantee the determinant of enlarged spectral operator being independent on the spectral parameter z and to achieve a rare opportunity in constructing admissible evolution operators, i.e., evolution operators consistent with the zero-curvature equation (2.3), we postulate the enlarged spectral operator as the nonsingular 4×4 matrix defined by the formula

$$L(n|z) = \begin{pmatrix} 0 & t_{12}(n) & u_{13}(n)z^{-1} & 0 \\ t_{21}(n) & r_{22}(n)z^2 + t_{22}(n) & s_{23}(n)z + u_{23}(n)z^{-1} & s_{24}(n)z \\ u_{31}(n)z^{-1} & s_{32}(n)z + u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} & t_{34}(n) \\ 0 & s_{42}(n)z & t_{43}(n) & 0 \end{pmatrix}. \tag{2.5}$$

Thus, in comparison to the initial spectral operator (2.2) given by 2×2 matrix the new spectral operator (2.5) is seen to be extended to 4×4 matrix by eight nonzero off-diagonal elements so that now we have sixteen prototype field functions $t_{12}(n)$, $u_{13}(n)$, $t_{21}(n)$, $r_{22}(n)$, $t_{22}(n)$, $s_{23}(n)$, $u_{23}(n)$, $s_{24}(n)$, $u_{31}(n)$, $s_{32}(n)$, $u_{32}(n)$, $t_{33}(n)$, $v_{33}(n)$, $t_{34}(n)$, $s_{42}(n)$, $t_{43}(n)$.

Alternatively, we may formally construct the proposed spectral operator (2.5) from the spectral operator

$$L(n|z) = \begin{pmatrix} r_{11}(n)z^2 + t_{11}(n) & r_{12}(n)z^2 + t_{12}(n) & s_{13}(n)z + u_{13}(n)z^{-1} & s_{14}(n)z + u_{14}(n)z^{-1} \\ r_{21}(n)z^2 + t_{21}(n) & r_{22}(n)z^2 + t_{22}(n) & s_{23}(n)z + u_{23}(n)z^{-1} & s_{24}(n)z + u_{24}(n)z^{-1} \\ s_{31}(n)z + u_{31}(n)z^{-1} & s_{32}(n)z + u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} & t_{34}(n) + v_{34}(n)z^{-2} \\ s_{41}(n)z + u_{41}(n)z^{-1} & s_{42}(n)z + u_{42}(n)z^{-1} & t_{43}(n) + v_{43}(n)z^{-2} & t_{44}(n) + v_{44}(n)z^{-2} \end{pmatrix} \tag{2.6}$$

by imposing rather serious constraints $r_{11}(n) = 0$, $t_{11}(n) = 0$, $r_{12}(n) = 0$, $s_{13}(n) = 0$, $s_{14}(n) = 0$, $u_{14}(n) = 0$, $r_{21}(n) = 0$, $u_{24}(n) = 0$, $s_{31}(n) = 0$, $v_{34}(n) = 0$, $s_{41}(n) = 0$, $u_{41}(n) = 0$, $u_{42}(n) = 0$, $v_{43}(n) = 0$, $t_{44}(n) = 0$, $v_{44}(n) = 0$.

Evidently both of described procedures giving rise to the new spectral operator (2.5) are in some sense the heuristical ones and practically have little to do with those based upon the matrix or tensor products of already known spectral operators (presumably of Ablowitz–Ladik type). Indeed, the spectral properties of new spectral operator (2.5) turn out to be essentially distinct from those of its small (2.2) and large (2.6) ancestors already due to a simple fact that only the determinant of new operator (2.5) does not depend on the spectral parameter z.

3. Two Types of Admissible Evolution Operators and Arbitrariness of Their Sampling Functions

Considering the general matrix structure of enlarged spectral operator (2.5) and taking into account the zero-curvature equation (2.3) we are able to propose two types of admissible evolution operators attributed to two types of ansätze, namely to obverse ansatz and verso ansatz. The etymology of terms "obverse" and "verso" will be clear when comparing the matrix structure of resepctive ansatz with the matrix structure of basic spectral operator (2.5).

Thus the matrix structure of observe evolution operator should be sought by the ansatz

$$A(n|z) = \begin{pmatrix} 0 & c_{12}(n) & d_{13}(n)z^{-1} & 0 \\ c_{21}(n) & a_{22}(n)z^2 + c_{22}(n) & b_{23}(n)z + d_{23}(n)z^{-1} & b_{24}(n)z \\ d_{31}(n)z^{-1} & b_{32}(n)z + d_{32}(n)z^{-1} & c_{33}(n) + e_{33}(n)z^{-2} & c_{34}(n) \\ 0 & b_{42}(n)z & c_{43}(n) & 0 \end{pmatrix}. \tag{3.1}$$

Conversely, the matrix structure of verso evolution operator should be sought by the ansatz

$$A(n|z) = \begin{pmatrix} a_{11}(n)z^2 + c_{11}(n) & c_{12}(n) & b_{13}(n)z & b_{14}(n)z + d_{14}(n)z^{-1} \\ c_{21}(n) & 0 & 0 & d_{24}(n)z^{-1} \\ b_{31}(n)z & 0 & 0 & c_{34}(n) \\ b_{41}(n)z + d_{41}(n)z^{-1} & d_{42}(n)z^{-1} & c_{43}(n) & c_{44}(n) + e_{44}(n)z^{-2} \end{pmatrix}. \tag{3.2}$$

Looking at the matrix elements of observe (3.1) and verso (3.2) ansätze we clearly see the crucial distinctions between two admissible types of evolution operators. As a consequence the zero-curvature equation (2.3) might inevitably produce at least two principally distinct sets of semidiscrete integrable nonlinear equations segregated according to the type of chosen evolution operator (3.1) or (3.2). Here we will not write down either of these sets but merely inform that in each of two claimed cases the evolution equations can be uniquely isolated with almost all matrix elements of respective evolution operator being specified through the field functions entering into the matrix elements of the spectral operator (2.5).

For example, the specified functions involved into the obverse evolution operator (3.1) are as follows

$$c_{12}(n) = t_{12}(n-1)a_{22}/r_{22}(n-1) \tag{3.3}$$

$$d_{13}(n) = u_{13}(n-1)e_{33}/v_{33}(n-1) \tag{3.4}$$

$$c_{21}(n) = a_{22}t_{21}(n)/r_{22}(n) \tag{3.5}$$

$$a_{22}(n) = a_{22} \tag{3.6}$$

$$b_{23}(n) = a_{22}s_{23}(n)/r_{22}(n) \tag{3.7}$$

$$d_{23}(n) = u_{23}(n-1)e_{33}/v_{33}(n-1) \tag{3.8}$$

$$b_{24}(n) = a_{22}s_{24}(n)/r_{22}(n) \tag{3.9}$$

$$d_{31}(n) = e_{33}u_{31}(n)/v_{33}(n) \tag{3.10}$$

$$b_{32}(n) = s_{32}(n-1)a_{22}/r_{22}(n-1) \tag{3.11}$$

$$d_{32}(n) = e_{33}u_{32}(n)/v_{33}(n) \tag{3.12}$$

$$e_{33}(n) = e_{33} \tag{3.13}$$

$$c_{34}(n) = e_{33}t_{34}(n)/v_{33}(n) \tag{3.14}$$

$$b_{42}(n) = s_{42}(n-1)a_{22}/r_{22}(n-1) \tag{3.15}$$

$$c_{43}(n) = t_{43}(n-1)e_{33}/v_{33}(n-1). \tag{3.16}$$

The only unspecified functions remain to be $c_{22}(n)$, $c_{33}(n)$ for the observe ansatz (3.1) and $c_{11}(n)$, $c_{44}(n)$ for the verso ansatz (3.2). We call these arbitrary functions as the sampling ones. The similar situation with an unfixed sampling arises also in other integrable models [16, 22] and can be resolved either empirically or relying upon the local conservation laws dictated by the matrix structure of proposed spectral operator. We will consider the problems of sampling fixation in our forthcoming works using the collection of lowest local conservation laws listed in the fifth section of the present paper and found by means of approach outlined in the next section.

4. Generating Functions of Local Densities and Local Currents

There are two sorts of approaches how to qualify the conservation laws associated with integrable nonlinear systems.

The first one considers the conservation laws as the integrals of motions which strictly speaking should be referred to as the global conservation laws. Usually this approach is based upon the time independence of diagonal elements of a reduced monodromy matrix and relies on the relationships of these elements with the envelope Jost functions [17, 19, 21].

The second approach deals with the local conservation laws which strictly speaking should be referred to as the continuity equations. Usually this approach is based upon the generating continuity equation and relies on some auxiliary function governed by the nonlinear equation of Riccati type with respect to the spatial coordinate variable [23–29]. The form of Riccati equation is dictated exclusively by the form of the spectral operator, while the Riccati equation itself permits to be solved recursively in powers of the spectral

parameter. Remarkably however that the auxiliary function allows one to reconstruct the generating functions both for the local densities and the local currents via the simple algebraic operations with the additional use of matrix elements of the spectral and evolution operators respectively.

In the case when the left $L^-(z) = \lim_{n\to-\infty} L(n|z)$ and the right $L^+(z) = \lim_{n\to+\infty} L(n|z)$ limiting spectral operators coincide $L^-(z) = L^+(z)$ the two above mentioned standpoints onto the treatment of conservation laws was proven to have the one-to-one correspondence [27]. On this account we are always able to eliminate all feasible discrepancies between the two approaches by means of an appropriate preliminary gauge transformation, which equalizes the left and the right limiting values of transformed spectral operator.

In this section we will modify the technique dealing with the reconstruction of local conservation laws (i.e., continuity equations) in such a way as to introduce a collection of several distinct generating equations supported by a set of auxiliary Riccati equations instead of a single generating equation and a single auxiliary Riccati equation typical of the traditional considerations [26–29].

To proceed with this plan we relinquish any attempts to concretize the relationship between the left $L^-(z)$ and the right $L^+(z)$ limiting spectral operators and remind that the zero-curvature equation (2.3) is nothing but the compatibility condition between the two linear equations

$$|\chi(n+1|z)\rangle = L(n|z)|\chi(n|z)\rangle \tag{4.1}$$

$$\frac{d}{d\tau}|\chi(n|z)\rangle = A(n|z)|\chi(n|z)\rangle \tag{4.2}$$

where $|\chi(n|z)\rangle$ is assumed to be an arbitrary column-matrix function of n and τ with the elements $\langle j|\chi(n|z)\rangle$, whose total number coincides with the rank R of chosen spectral operator $L(n|z)$. In particular, for the spectral operator given by nonsingular 4×4 matrix (2.5) the number of such elements must be four.

First of all we introduce the set of auxiliary functions

$$\Gamma_{jk}(n|z) = \frac{\langle j|\chi(n|z)\rangle}{\langle k|\chi(n|z)\rangle} \tag{4.3}$$

with the evident property

$$\Gamma_{ji}(n|z)\Gamma_{ik}(n|z) = \Gamma_{jk}(n|z) \tag{4.4}$$

saying that only $R-1$ functions among R^2-R nontrivial ones are independent. Here the term "trivial" refers to any auxiliary function with equal indices insofar as $\Gamma_{jj}(n|z) \equiv 1$. Due to their quotient structure (4.3) the functions $\Gamma_{jk}(n|z)$ unable to contain the systematic coordinate dependent factors akin to those observable, e.g. in asymptotics of Jost functions. For this reason the functions $\Gamma_{jk}(n|z)$ are expected to be sufficiently accommodated for the purposes of their serial representations in powers of one or another combination of complex spectral parameter z.

Now manipulating with the component-wise version of spectral equation (4.1) we readily derive the following set of equations

$$\Gamma_{jk}(n+1|z)M_{kk}(n|z) = M_{jj}(n|z)\Gamma_{jk}(n|z) \tag{4.5}$$

for the quantities $\Gamma_{jk}(n|z)$ where the shorthand notations

$$M_{jk}(n|z) = \sum_{i=1}^{R} L_{ji}(n|z)\Gamma_{ik}(n|z) \tag{4.6}$$

are implied while the functions $L_{jk}(n|z)$ being reserved for the matrix elements of spectral operator $L(n|z)$. By virtue of identity $\Gamma_{jj}(n|z) \equiv 1$ the obtained Eq. (4.5) are essentially nonlinear ones and can be thought as the set of Riccati-type equations allowing to be solved recursively in powers of z or $1/z$ with coefficients given in terms of prototype field functions. The time in such a procedure plays the role of implicit parameter and has no influence on the structure of desired solution.

The next step requires the proper manipulations with the component-wise version of evolution equation (4.2). As a result we come to the set of equations

$$\dot{\Gamma}_{jk}(n|z) = B_{jj}(n|z)\Gamma_{jk}(n|z) - \Gamma_{jk}(n|z)B_{kk}(n|z), \tag{4.7}$$

where the shorthand notations

$$B_{jk}(n|z) = \sum_{i=1}^{R} A_{ji}(n|z)\Gamma_{ik}(n|z) \tag{4.8}$$

are used with the functions $A_{jk}(n|z)$ being reserved for the matrix elements of evolution operator $A(n|z)$. Although the obtained Eq. (4.7) are evolutionary ones but only in combination with the zero-curvature equation (2.3) and the purely spatial Riccati equations (4.5) they are able to produce the basic continuity equations

$$\frac{d}{d\tau} \ln M_{jj}(n|z) = B_{jj}(n+1|z) - B_{jj}(n|z) \tag{4.9}$$

suitable to generate the infinite hierarchy of local conservation laws based on the serial representations of preliminary found auxiliary functions $\Gamma_{jk}(n|z)$. Here we would like to stress that the quantities $\ln M_{jj}(n|z)$ are determined exclusively by the spectral operator (through $\Gamma_{jk}(n|z)$ and $L_{jk}(n|z)$) and should be treated as the generating functions of local densities. Conversely, the quantities $-B_{jj}(n|z)$ are determined both by the spectral operator $L(n|z)$ (through $\Gamma_{jk}(n|z)$) and the evolution operator $A(n|z)$ (through $A_{jk}(n|z)$) and should be treated as the generating functions of local currents.

As the matter of fact the true sense of basic continuity equations (4.9) consists not in the finding of auxiliary functions $\Gamma_{jk}(n|z)$ (which is the prerogative of spatial Riccati equations (4.5)) but in correct combination of expansion terms into infinite collection of continuity equations referred to as the hierarchy of local conservation laws. For this reason we shall call the Eq. (4.9) as the generating ones.

We conclude this section paying attention on an alternative derivation of basic generating equations (4.9) with the use of identities

$$\frac{d}{d\tau} \ln \frac{\langle j|\chi(n+1|z)\rangle}{\langle j|\chi(n|z)\rangle} = \frac{d\langle j|\chi(n+1|z)\rangle/d\tau}{\langle j|\chi(n+1|z)\rangle} - \frac{d\langle j|\chi(n|z)\rangle/d\tau}{\langle j|\chi(n|z)\rangle} \tag{4.10}$$

as the starting point. In order to obtain the required generating equations (4.9) we must rewrite the quantities $\langle j|\chi(n+1|z)\rangle$ in the left-hand sides of identities (4.10) by invoking the

spectral equation (4.1). Simultaneously we must rewrite the quantities $d\langle j|\chi(n+1|z)\rangle/d\tau$ and $d\langle j|\chi(n|z)\rangle/d\tau$ in the right-hand sides of identities (4.10) by invoking the evolution equation (4.2). In due course of above substitutions the initial identities (4.10) acquire the status of generating equations (4.9) in as much as both of invoked formulas (4.1) and (4.2) are by no means the identities but the equations.

5. The Lowest Local Conservation Laws

The most systematic way to seek the admissible expansions for the ratios $\Gamma_{jk}(n|z)$ appears to be that relying upon the truncated Laurent-type series with respect to inverse eigenvalues $1/\zeta_i(z)$ related to either of the spectral limiting operators $L^-(z)$ or $L^+(z)$. However, such eigenvalues $\zeta_i(z)$ turn out to be essentially dependent on the peculiarities of spatial boundary conditions imposed onto the field amplitudes what inevitably leads to the technical inconveniences in overall consideration.

Another way is more empirical and assumes to look over possible realizations of expansions for the collection of ratios $\Gamma_{jk}(n|z)$ directly in powers of z or $1/z$ so that each realization must be consistent with the set of spatial Riccati equations (4.5) by ensuring the recursive recovery of involved expansion coefficients. Although this method has no guarantee in revealing all feasible realizations but the mere course of above reasoning has the right to be tested inasmuch as the spectral operator characterized by more than two distinct eigenvalues gives rise to rather sophisticated subdivision into the regularity domains of envelope Jost vectors in the plane of complex spectral parameter z [16, 30, 31] and as a result may produce several sectors of regularity with the common vertex located in the initial $|z|=0$ or the infinitely distant $|z|=\infty$ point. In this context we can expect the principal possibility to fasten the particular admissible realization for the collection of $\Gamma_{jk}(n|z)$ to the particular sector in the plane of complex spectral parameter z.

In general the enlarged spectral operator $L(n|z)$ given by formula (2.5) must be treated as the spectral operator of fourth order inasmuch as its limiting spectral operator $L^-(z)$ or $L^+(z)$ yields four distinct eigenvalues. For this reason the argumentations of previous paragraph prompt us to seek as least four distinct realizations for the collection of ratios $\Gamma_{jk}(n|z)$.

Indeed, analyzing the set of Riccati equations (4.5) for the ratios $\Gamma_{jk}(n|z)$ and assuming the spectral operator $L(n|z)$ as the spectral operator of our main interest (2.5) we have managed to select four distinct collections of ansätze supporting four distinct noncontradictive recursive procedures in the framework of above-mentioned Riccati equations (4.5).

Thus at $|z|\to 0$ we can count upon two distinct collections of ansätze, namely

$$\Gamma_{12}(n|z)=\sum_{i=0}^{\infty}x_{12}(n|i|0)z^{2i} \qquad \Gamma_{21}(n|z)=\sum_{i=0}^{\infty}y_{21}(n|i|0)z^{2i} \tag{5.1}$$

$$\Gamma_{23}(n|z)=\sum_{i=0}^{\infty}x_{23}(n|i|0)z^{2i+1} \qquad \Gamma_{32}(n|z)=\sum_{i=0}^{\infty}y_{32}(n|i|0)z^{2i-1} \tag{5.2}$$

$$\Gamma_{34}(n|z)=\sum_{i=0}^{\infty}x_{34}(n|i|0)z^{2i-2} \qquad \Gamma_{43}(n|z)=\sum_{i=0}^{\infty}y_{43}(n|i|0)z^{2i+2} \tag{5.3}$$

and

$$\Gamma_{12}(n|z) = \sum_{i=0}^{\infty} y_{12}(n|i|0)z^{2i} \qquad \Gamma_{21}(n|z) = \sum_{i=0}^{\infty} x_{21}(n|i|0)z^{2i} \tag{5.4}$$

$$\Gamma_{23}(n|z) = \sum_{i=0}^{\infty} y_{23}(n|i|0)z^{2i-1} \qquad \Gamma_{32}(n|z) = \sum_{i=0}^{\infty} x_{32}(n|i|0)z^{2i+1} \tag{5.5}$$

$$\Gamma_{34}(n|z) = \sum_{i=0}^{\infty} y_{34}(n|i|0)z^{2i+2} \qquad \Gamma_{43}(n|z) = \sum_{i=0}^{\infty} x_{43}(n|i|0)z^{2i-2}. \tag{5.6}$$

Similarly, at $|z| \to \infty$ the two distinct collections of ansätze are as follows

$$\Gamma_{21}(n|z) = \sum_{i=0}^{\infty} x_{21}(n|i|\infty)z^{-2i+2} \qquad \Gamma_{12}(n|z) = \sum_{i=0}^{\infty} y_{12}(n|i|\infty)z^{-2i-2} \tag{5.7}$$

$$\Gamma_{32}(n|z) = \sum_{i=0}^{\infty} x_{32}(n|i|\infty)z^{-2i-1} \qquad \Gamma_{23}(n|z) = \sum_{i=0}^{\infty} y_{23}(n|i|\infty)z^{-2i+1} \tag{5.8}$$

$$\Gamma_{43}(n|z) = \sum_{i=0}^{\infty} x_{43}(n|i|\infty)z^{-2i} \qquad \Gamma_{34}(n|z) = \sum_{i=0}^{\infty} y_{34}(n|i|\infty)z^{-2i} \tag{5.9}$$

and

$$\Gamma_{21}(n|z) = \sum_{i=0}^{\infty} y_{21}(n|i|\infty)z^{-2i-2} \qquad \Gamma_{12}(n|z) = \sum_{i=0}^{\infty} x_{12}(n|i|\infty)z^{-2i+2} \tag{5.10}$$

$$\Gamma_{32}(n|z) = \sum_{i=0}^{\infty} y_{32}(n|i|\infty)z^{-2i+1} \qquad \Gamma_{23}(n|z) = \sum_{i=0}^{\infty} x_{23}(n|i|\infty)z^{-2i-1} \tag{5.11}$$

$$\Gamma_{43}(n|z) = \sum_{i=0}^{\infty} y_{43}(n|i|\infty)z^{-2i} \qquad \Gamma_{34}(n|z) = \sum_{i=0}^{\infty} x_{34}(n|i|\infty)z^{-2i}. \tag{5.12}$$

According to our previous arguing each collection of ansätze must be associated with some particular sector near the initial or the infinite distant point in the plane of complex spectral parameter z. However, the strict specifications of these sectors turns out to be unessential for the purely formal recurrent calculations of expansion coefficients in either of four selected realizations (5.1)–(5.3), (5.4)–(5.6); (5.7)–(5.9), (5.10)–(5.12).

In view of the identity $\Gamma_{jk}(n|z)\Gamma_{kj}(n|z) \equiv 1$ the left column expansion and the right column expansion in each of formulas (5.1)–(5.12) can be referred to as the basic and the complementary ones. One can readily verify that the expansion coefficients in each pair of basic and complementary expansions are subjected to one of the two groups of relations

$$\sum_{l=0}^{i} x_{jk}(n|i-l|0)y_{kj}(n|l|0) = \delta_{0i} \tag{5.13}$$

or

$$\sum_{l=0}^{i} x_{jk}(n|i-l|\infty)y_{kj}(n|l|\infty) = \delta_{0i} \tag{5.14}$$

depending on whether the expansions are taken at $|z| \to 0$ or at $|z| \to \infty$, respectively. Although the above relationships (5.13) and (5.14) do facilitate the actual calculations of expansion coefficients for the ratios $\Gamma_{jk}(n|z)$ within each of four groups of ansätze (5.1)–(5.3), (5.4)–(5.6) and (5.7)–(5.9), (5.10)–(5.12), respectively, but the details of each particular recurrence procedure remain to be rather combersome and we omit them for the brevity sake.

Relying upon the spatial Riccati equation (4.5) we have found several lowest expansion terms for the ratios $\Gamma_{jk}(n|z)$ within each of their four realizations (5.1)–(5.3), (5.4)–(5.6); (5.7)–(5.9), (5.10)–(5.12) and have applied them to isolate several lowest local conservation laws

$$\dot{\rho}_{11}(n) = J_{11}(n|n-1) - J_{11}(n+1|n) \tag{5.15}$$

$$\dot{\rho}_{22}(n) = J_{22}(n|n-1) - J_{22}(n+1|n) \tag{5.16}$$

$$\dot{\rho}_{33}(n) = J_{33}(n|n-1) - J_{33}(n+1|n) \tag{5.17}$$

$$\dot{\rho}_{44}(n) = J_{44}(n|n-1) - J_{44}(n+1|n) \tag{5.18}$$

$$\dot{\rho}^{+}_{22}(n) = J^{+}_{22}(n|n-1) - J^{+}_{22}(n+1|n) \tag{5.19}$$

$$\dot{\rho}^{-}_{22}(n) = J^{-}_{22}(n|n-1) - J^{-}_{22}(n+1|n) \tag{5.20}$$

$$\dot{\rho}^{+}_{33}(n) = J^{+}_{33}(n|n-1) - J^{+}_{33}(n+1|n) \tag{5.21}$$

$$\dot{\rho}^{-}_{33}(n) = J^{-}_{33}(n|n-1) - J^{-}_{33}(n+1|n) \tag{5.22}$$

in the framework of basic generating equations (4.9) with formulas

$$M_{jj}(n|z) = \sum_{i=1}^{4} L_{ji}(n|z)\Gamma_{ij}(n|z) \tag{5.23}$$

and

$$B_{jj}(n|z) = \sum_{i=1}^{4} A_{ji}(n|z)\Gamma_{ij}(n|z) \tag{5.24}$$

having been taken into account. According to the general rule the local densities $\rho_{11}(n)$, $\rho_{22}(n)$, $\rho_{33}(n)$, $\rho_{44}(n)$ and $\rho^{+}_{22}(n)$, $\rho^{-}_{22}(n)$, $\rho^{+}_{33}(n)$, $\rho^{-}_{33}(n)$ are absolutely insensitive to the type of evolution operator (obverse (3.1) or verso (3.2)) and are given by the expressions

$$\begin{aligned}\rho_{11}(n) = \ln[&t_{12}(n)t_{21}(n)v_{33}(n) + u_{13}(n)u_{31}(n)t_{22}(n)\\ &- t_{12}(n)u_{23}(n)u_{31}(n) - u_{13}(n)u_{32}(n)t_{21}(n)]\end{aligned} \tag{5.25}$$

$$\rho_{22}(n) = \ln r_{22}(n) \tag{5.26}$$

$$\rho_{33}(n) = \ln v_{33}(n) \tag{5.27}$$

$$\begin{aligned}\rho_{44}(n) = \ln[&t_{43}(n)t_{34}(n)r_{22}(n) + s_{42}(n)s_{24}(n)t_{33}(n)\\ &- t_{43}(n)s_{32}(n)s_{24}(n) - s_{42}(n)s_{23}(n)t_{34}(n)]\end{aligned} \tag{5.28}$$

and

$$\rho^{+}_{22}(n) = \frac{t_{22}(n)}{r_{22}(n)} + \frac{s_{23}(n+1)s_{32}(n)}{r_{22}(n+1)r_{22}(n)} + \frac{s_{24}(n+1)s_{42}(n)}{r_{22}(n+1)r_{22}(n)} \tag{5.29}$$

$$\rho_{22}^{-}(n) = \frac{t_{22}(n)}{r_{22}(n)} + \frac{s_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n-1)} + \frac{s_{24}(n)s_{42}(n-1)}{r_{22}(n)r_{22}(n-1)} \tag{5.30}$$

$$\rho_{33}^{+}(n) = \frac{t_{33}(n)}{v_{33}(n)} + \frac{u_{32}(n+1|n)u_{23}(n)}{v_{33}(n+1)v_{33}(n)} + \frac{u_{31}(n+1)u_{13}(n)}{v_{33}(n+1)v_{33}(n)} \tag{5.31}$$

$$\rho_{33}^{-}(n) = \frac{t_{33}(n)}{v_{33}(n)} + \frac{u_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n-1)} + \frac{u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n-1)}. \tag{5.32}$$

In contrast the local currents $J_{11}(n|n-1)$, $J_{22}(n|n-1)$, $J_{33}(n|n-1)$, $J_{44}(n|n-1)$ and $J_{22}^{+}(n|n-1)$, $J_{22}^{-}(n+1|n)$, $J_{33}^{+}(n|n-1)$, $J_{33}^{-}(n+1|n)$ are essentially dependent on the type of evolution operator. In this paper we restrict ourself only to the case of obverse evolution operator (3.1) and obtain

$$J_{11}(n|n-1) = -c_{22}(n) - c_{33}(n) \tag{5.33}$$

$$J_{22}(n|n-1) = -c_{22}(n) - \frac{a_{22}s_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{24}(n)s_{42}(n-1)}{r_{22}(n)r_{22}(n-1)} \tag{5.34}$$

$$J_{33}(n|n-1) = -c_{33}(n) - \frac{e_{33}u_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n-1)} \tag{5.35}$$

$$J_{44}(n|n-1) = -c_{22}(n) - c_{33}(n) \tag{5.36}$$

and

$$\begin{aligned}
&J_{22}^{+}(n|n-1)\\
&\quad= \frac{a_{22}s_{23}(n+1)s_{32}(n)s_{23}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)} + \frac{a_{22}s_{24}(n+1)s_{42}(n)s_{23}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)}\\
&\qquad+ \frac{a_{22}s_{23}(n+1)s_{32}(n)s_{24}(n)s_{42}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)} + \frac{a_{22}s_{24}(n+1)s_{42}(n)s_{24}(n)s_{42}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)}\\
&\qquad- \frac{a_{22}t_{21}(n)t_{12}(n-1)}{r_{22}(n)r_{22}(n-1)} - \frac{a_{22}u_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n-1)} - \frac{s_{23}(n)e_{33}u_{32}(n)}{r_{22}(n)v_{33}(n)}\\
&\qquad- \frac{a_{22}s_{23}(n+1)t_{33}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{24}(n+1)t_{43}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n-1)}\\
&\qquad+ \frac{a_{22}t_{22}(n)s_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{23}(n+1)t_{34}(n)s_{42}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n-1)}\\
&\qquad+ \frac{a_{22}t_{22}(n)s_{24}(n)s_{42}(n-1)}{r_{22}(n)r_{22}(n)r_{22}(n-1)}
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
&J_{22}^{-}(n+1|n)\\
&\quad= \frac{a_{22}s_{23}(n+1)s_{32}(n)s_{23}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)} + \frac{a_{22}s_{24}(n+1)s_{42}(n)s_{23}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)}\\
&\qquad+ \frac{a_{22}s_{23}(n+1)s_{32}(n)s_{24}(n)s_{42}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)} + \frac{a_{22}s_{24}(n+1)s_{42}(n)s_{24}(n)s_{42}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n)r_{22}(n-1)}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{a_{22}t_{21}(n+1)t_{12}(n)}{r_{22}(n+1)r_{22}(n)} - \frac{a_{22}s_{23}(n+1)u_{32}(n)}{r_{22}(n+1)r_{22}(n)} - \frac{u_{23}(n)e_{33}s_{32}(n)}{v_{33}(n)r_{22}(n)} \\
&- \frac{a_{44}s_{23}(n+1)t_{33}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{24}(n+1)t_{43}(n)s_{32}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n-1)} \\
&+ \frac{a_{22}s_{23}(n+1)s_{32}(n)t_{22}(n)}{r_{22}(n+1)r_{22}(n)r_{22}(n)} - \frac{a_{22}s_{23}(n+1)t_{34}(n)s_{42}(n-1)}{r_{22}(n+1)r_{22}(n)r_{22}(n-1)} \\
&+ \frac{a_{22}s_{24}(n+1)s_{42}(n)t_{22}(n)}{r_{22}(n+1)r_{22}(n)r_{22}(n)}
\end{aligned}
\tag{5.38}
$$

$$
\begin{aligned}
&J_{33}^{+}(n|n-1) \\
&= \frac{e_{33}u_{32}(n+1)u_{23}(n)u_{32}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} + \frac{e_{33}u_{31}(n+1)u_{13}(n)u_{32}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} \\
&+ \frac{e_{33}u_{32}(n+1)u_{23}(n)u_{31}(n)u_{13}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} + \frac{e_{33}u_{31}(n+1)u_{13}(n)u_{31}(n)u_{13}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} \\
&- \frac{e_{33}t_{34}(n)t_{43}(n-1)}{v_{33}(n)v_{33}(n)1)} - \frac{e_{33}s_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n)1)} - \frac{u_{32}(n)a_{22}s_{23}(n)}{v_{33}(n)r_{22}(n)} \\
&- \frac{e_{33}u_{32}(n+1)t_{22}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{31}(n+1)t_{12}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n-1)} \\
&+ \frac{e_{33}t_{33}(n)u_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{32}(n+1)t_{21}(n)u_{13}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n-1)} \\
&- \frac{e_{33}t_{33}(n)u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n)v_{33}(n-1)}
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
&J_{33}^{-}(n+1|n) \\
&= \frac{e_{33}u_{32}(n+1)u_{23}(n)u_{32}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} + \frac{e_{33}u_{31}(n+1)u_{13}(n)u_{32}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} \\
&+ \frac{e_{33}u_{32}(n+1)u_{23}(n)u_{31}(n)u_{13}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} + \frac{e_{33}u_{31}(n+1)u_{13}(n)u_{31}(n)u_{13}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} \\
&- \frac{e_{33}t_{34}(n+1)t_{43}(n)}{v_{33}(n+1)v_{33}(n)} - \frac{e_{33}u_{32}(n+1)s_{23}(n)}{v_{33}(n+1)v_{33}(n)} - \frac{s_{32}(n)a_{22}u_{23}(n)}{r_{22}(n)v_{33}(n)} \\
&- \frac{e_{33}u_{32}(n+1)t_{22}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{31}(n+1)t_{12}(n)u_{23}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n)v_{33}(n-1)} \\
&+ \frac{e_{33}u_{32}(n+1)u_{23}(n)t_{33}(n)}{v_{33}(n+1)v_{33}(n)v_{33}(n)} - \frac{e_{33}u_{32}(n+1)t_{21}(n)u_{13}(n-1)}{v_{33}(n+1)v_{33}(n)v_{33}(n-1)} \\
&+ \frac{e_{33}u_{31}(n+1)u_{13}(n)t_{33}(n)}{v_{33}(n+1)v_{33}(n)v_{33}(n-1)}.
\end{aligned}
\tag{5.40}
$$

We complete this section by presenting the local conservation law

$$
\frac{d}{d\tau} \ln \left[\det L(n|z)\right] = \operatorname{Sp} A(n+1|z) - \operatorname{Sp} A(n|z) \tag{5.41}
$$

which follows directly from the zero-curvature equation (2.3) by virtue of the identity

$$\mathrm{Sp}\left[L^{-1}(n|z)\dot{L}(n|z)\right] \equiv \frac{d}{d\tau}\ln\left[\det L(n|z)\right]. \tag{5.42}$$

Using the explicit form (2.5) of spectral operator $L(n|z)$ we can define the basic on-cell local density $\rho(n)$ by the formula

$$\begin{aligned}\rho(n) &\equiv \ln[\det L(n|z)]\\ &= \ln\{[u_{13}(n)s_{42}(n) - t_{12}(n)t_{43}(n)][u_{31}(n)s_{24}(n) - t_{21}(n)t_{34}(n)]\}.\end{aligned} \tag{5.43}$$

On the other hand, choosing the evolution operator $A(n|z)$ to be the obverse one (3.1) the respective local current $J(n|n-1)$ acquires the form

$$J(n|n-1) = -c_{22}(n) - c_{33}(n) \tag{5.44}$$

in as much as

$$\mathrm{Sp}A(n+1|z) - \mathrm{Sp}A(n|z) = c_{22}(n+1) + c_{33}(n+1) - c_{22}(n) - c_{33}(n). \tag{5.45}$$

Here it is worthwhile to mention that namely the local conservation law

$$\dot{\rho}(n) = J(n|n-1) - J(n+1|n) \tag{5.46}$$

had already been used in order to write two previously listed local conservation laws (5.15) and (5.18) in their final concise forms.

6. Conclusion

In this article we have proposed the new fourth order spectral operator allowing to generate two types of semidiscrete integrable nonlinear systems in the framework of zero-curvature representation. The first type is associated with observe evolution operator whose matrix elements contain the same powers of the spectral parameter as the respective matrix elements of the spectral operator. The second type is associated with the verso evolution operator whose matrix elements contain the powers of the spectral parameter organized according to the mnemonic rule [21, 32] borrowed from the theory of Toda lattices.

The next step in concretizing the evolution operators and consequently the admissible semidiscrete integrable nonlinear systems can be made relying upon the on-cell local conservation laws. In order to implement this plan we have developed the modified recurrence procedure of finding the local conservation laws based upon a collection of four density generating functions and four current generating functions. Each generating function was shown to permit at least four distinct expansions linked to the distinct sectors in the complex plane of spectral parameter. In the framework of modified recurrence approach we have obtained a number of lowest local conservation laws some of which will be suitable for the motivated fixations of sampling functions as well as for the considerable reduction in the number of model field variables.

We intend to accomplish the problem of sampling fixation in our future publications and as a consequence to give a detailed classification of semidiscrete integrable nonlinear systems initiated by the proposed spectral operator.

Acknowledgments

The work has been supported by the National Academy of Sciences of Ukraine within the Programs No. 0110U007540 and No. 0107U000396.

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