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SEMIDISCRETE INTEGRABLE NONLINEAR SYSTEMS GENERATED BY THE NEW FOURTH ORDER SPECTRAL OPERATOR. SYSTEMS OF OBVERSE TYPE

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In the framework of zero-curvature representation we have proposed three distinct versions of semidiscrete integrable nonlinear systems arising due to a proper multifield augment of integrable nonlinear Schrödinger system with the background-controlled intersite resonant couplings. The specification either of these three systems is essentially based upon the lowest local conservation laws early found by means of modified recurrence procedure and consists in a proper fixation of sampling functions within the general evolution operator of obverse type. The number of actual field variables in each of obtained systems is shown to be considerably reduced due to the two natural constraints independent of sampling fixation and two additional constraints dictated by the chosen sampling.

Keywords: Zero-curvature representation; multifield integrable systems; sampling fixation; coupling parameters.

PACS number: 02.30.Ik, 11.10.Lm, 45.05.+x

1. Introduction

This article is the second part of the work dealing with semidiscrete integrable nonlinear systems generated by the new fourth order spectral operator [1]

$$L(n|z) = \begin{pmatrix} 0 & t_{12}(n) & u_{13}(n)z^{-1} & 0 \\ t_{21}(n) & r_{22}(n)z^2 + t_{22}(n) & s_{23}(n)z + u_{23}(n)z^{-1} & s_{24}(n)z \\ u_{31}(n)z^{-1} & s_{32}(n)z + u_{32}(n)z^{-1} & t_{33}(n) + v_{33}(n)z^{-2} & t_{34}(n) \\ 0 & s_{42}(n)z & t_{43}(n) & 0 \end{pmatrix}. \quad (1.1)$$

Here the prototype field amplitudes $t_{12}(n)$, $u_{13}(n)$, $t_{21}(n)$, $r_{22}(n)$, $t_{22}(n)$, $s_{23}(n)$, $u_{23}(n)$, $s_{24}(n)$, $u_{31}(n)$, $s_{32}(n)$, $u_{32}(n)$, $t_{33}(n)$, $v_{33}(n)$, $t_{34}(n)$, $s_{42}(n)$, $t_{43}(n)$ are assumed to be the functions of discrete spatial coordinate n and the continuous time τ . The auxiliary variable

z is understood as the complex-valued spectral parameter independent on time. For the sake of definiteness the coordinate n is taken to span all integers from minus to plus infinity.

In some sense the adopted spectral operator (1.1) can be considered as the generalization and simultaneous off-diagonal enlargement of the spectral operator associated with the semidiscrete integrable nonlinear Schrödinger system on a zigzag-runged ladder lattice [2, 3] whose geometrical configuration resembles that of (1,1) armchair boron nanotube [4].

In our previous paper [1] we have shown that the evolution operator $A(n|z)$ admissible by the zero-curvature equation

$$\dot{L}(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \quad (1.2)$$

should be postulated in either of two principally distinct forms referred to as the obverse ansatz and verso ansatz once the extended form (1.1) of the spectral operator $L(n|z)$ has been adopted. We have asserted that almost all matrix elements in each of admissible ansätze can be restored in the framework of zero-curvature equation (1.2) giving simultaneously rise to the respective type of semidiscrete integrable nonlinear systems. However, the explicit presentation of this statement has not been given.

In the present paper we shall partially fill in this gap and try to classify the semidiscrete integrable nonlinear systems arising due to variability of sampling fixation within the general evolution operator of obverse type. The semidiscrete integrable nonlinear systems associated with the general evolution operator of verso type will be considered in a separate paper.

According to usual practice the overdot in the left-hand side of zero-curvature equation (1.2) is reserved for the derivative with respect to time τ .

2. Obverse Evolution Operator. Ansatz and Explicit Representation

By the definition [1] the ansatz for the obverse evolution operator consistent with the proposed spectral operator (1.1) is assumed in the form

$$A(n|z) = \begin{pmatrix} 0 & c_{12}(n) & d_{13}(n)z^{-1} & 0 \\ c_{21}(n) & a_{22}(n)z^2 + c_{22}(n) & b_{23}(n)z + d_{23}(n)z^{-1} & b_{24}(n)z \\ d_{31}(n)z^{-1} & b_{32}(n)z + d_{32}(n)z^{-1} & c_{33}(n) + e_{33}(n)z^{-2} & c_{34}(n) \\ 0 & b_{42}(n)z & c_{43}(n) & 0 \end{pmatrix}. \quad (2.1)$$

Inserting the matrix-valued expressions (1.1) and (2.1) for the spectral $L(n|z)$ and evolution $A(n|z)$ operators into the zero-curvature equation (1.2) we are able both to obtain the explicit formulas for almost all matrix elements $A_{jk}(n|z)$ of obverse evolution operator (2.1) in terms of prototype field amplitudes and to recover the set of evolutionary nonlinear equations for these amplitudes.

Precisely for the constituent parts of matrix elements we have

$$c_{12}(n) = t_{12}(n-1)a_{22}/r_{22}(n-1) \quad (2.2)$$

$$d_{13}(n) = u_{13}(n-1)e_{33}/v_{33}(n-1) \quad (2.3)$$

$$c_{21}(n) = a_{22}t_{21}(n)/r_{22}(n) \quad (2.4)$$

$$a_{22}(n) = a_{22} \quad (2.5)$$

$$b_{23}(n) = a_{22}s_{23}(n)/r_{22}(n) \quad (2.6)$$

$$d_{23}(n) = u_{23}(n-1)e_{33}/v_{33}(n-1) \quad (2.7)$$

$$b_{24}(n) = a_{22}s_{24}(n)/r_{22}(n) \quad (2.8)$$

$$d_{31}(n) = e_{33}u_{31}(n)/v_{33}(n) \quad (2.9)$$

$$b_{32}(n) = s_{32}(n-1)a_{22}/r_{22}(n-1) \quad (2.10)$$

$$d_{32}(n) = e_{33}u_{32}(n)/v_{33}(n) \quad (2.11)$$

$$e_{33}(n) = e_{33} \quad (2.12)$$

$$c_{34}(n) = e_{33}t_{34}(n)/v_{33}(n) \quad (2.13)$$

$$b_{42}(n) = s_{42}(n-1)a_{22}/r_{22}(n-1) \quad (2.14)$$

$$c_{43}(n) = t_{43}(n-1)e_{33}/v_{33}(n-1). \quad (2.15)$$

Here the coordinate-independent quantities a_{22} and e_{33} can be thought as arbitrary functions of time. The only unspecified functions $c_{22}(n)$ and $c_{33}(n)$ remain to be arbitrary for the time being.

The concise form of evolutionary equations looks as follows

$$\dot{t}_{12}(n) = c_{12}(n+1)t_{22}(n) + d_{13}(n+1)s_{32}(n) - t_{12}(n)c_{22}(n) - u_{13}(n)b_{32}(n) \quad (2.16)$$

$$\dot{u}_{13}(n) = c_{12}(n+1)u_{23}(n) + d_{13}(n+1)t_{33}(n) - t_{12}(n)d_{23}(n) - u_{13}(n)c_{33}(n) \quad (2.17)$$

$$\dot{t}_{21}(n) = c_{22}(n+1)t_{21}(n) + b_{23}(n+1)u_{31}(n) - t_{22}(n)c_{21}(n) - s_{23}(n)d_{31}(n) \quad (2.18)$$

$$\begin{aligned} \dot{r}_{22}(n) = & c_{22}(n+1)r_{22}(n) + b_{23}(n+1)s_{32}(n) + b_{24}(n+1)s_{42}(n) \\ & - r_{22}(n)c_{22}(n) - s_{23}(n)b_{32}(n) - s_{24}(n)b_{42}(n) \end{aligned} \quad (2.19)$$

$$\begin{aligned} \dot{t}_{22}(n) = & c_{21}(n+1)t_{12}(n) + c_{22}(n+1)t_{22}(n) - t_{21}(n)c_{12}(n) - t_{22}(n)c_{22}(n) \\ & + b_{23}(n+1)u_{32}(n) + d_{23}(n+1)s_{32}(n) - s_{23}(n)d_{32}(n) - u_{23}(n)b_{32}(n) \end{aligned} \quad (2.20)$$

$$\begin{aligned} \dot{s}_{23}(n) = & a_{22}(n+1)u_{23}(n) + c_{22}(n+1)s_{23}(n) - r_{22}(n)d_{23}(n) - t_{22}(n)b_{23}(n) \\ & + b_{23}(n+1)t_{33}(n) + b_{24}(n+1)t_{43}(n) - s_{23}(n)c_{33}(n) - s_{24}(n)c_{43}(n) \end{aligned} \quad (2.21)$$

$$\begin{aligned} \dot{u}_{23}(n) = & c_{21}(n+1)u_{13}(n) + c_{22}(n+1)u_{23}(n) - t_{21}(n)d_{13}(n) - t_{22}(n)d_{23}(n) \\ & + b_{23}(n+1)v_{33}(n) + d_{23}(n+1)t_{33}(n) - s_{23}(n)e_{33}(n) - u_{23}(n)c_{33}(n) \end{aligned} \quad (2.22)$$

$$\dot{s}_{24}(n) = c_{22}(n+1)s_{24}(n) + b_{23}(n+1)t_{34}(n) - t_{22}(n)b_{24}(n) - s_{23}(n)c_{34}(n) \quad (2.23)$$

$$\dot{u}_{31}(n) = d_{32}(n+1)t_{21}(n) + c_{33}(n+1)u_{31}(n) - u_{32}(n)c_{21}(n) - t_{33}(n)d_{31}(n) \quad (2.24)$$

$$\begin{aligned} \dot{s}_{32}(n) = & b_{32}(n+1)t_{22}(n) + d_{32}(n+1)r_{22}(n) - s_{32}(n)c_{22}(n) - u_{32}(n)a_{22}(n) \\ & + c_{33}(n+1)s_{32}(n) + c_{34}(n+1)s_{42}(n) - t_{33}(n)b_{32}(n) - t_{34}(n)b_{42}(n) \end{aligned} \quad (2.25)$$

$$\begin{aligned} \dot{u}_{32}(n) = & d_{31}(n+1)t_{12}(n) + d_{32}(n+1)t_{22}(n) - u_{31}(n)c_{12}(n) - u_{32}(n)c_{22}(n) \\ & + c_{33}(n+1)u_{32}(n) + e_{33}(n+1)s_{32}(n) - t_{33}(n)d_{32}(n) - v_{33}(n)b_{32}(n) \end{aligned} \quad (2.26)$$

$$\begin{aligned} \dot{t}_{33}(n) = & b_{32}(n+1)u_{23}(n) + d_{32}(n+1)s_{23}(n) - s_{32}(n)d_{23}(n) - u_{32}(n)b_{23}(n) \\ & + c_{33}(n+1)t_{33}(n) + c_{34}(n+1)t_{43}(n) - t_{33}(n)c_{33}(n) - t_{34}(n)c_{43}(n) \end{aligned} \quad (2.27)$$

$$\begin{aligned} \dot{v}_{33}(n) = & d_{31}(n+1)u_{13}(n) + d_{32}(n+1)u_{23}(n) + c_{33}(n+1)v_{33}(n) \\ & - u_{31}(n)d_{13}(n) - u_{32}(n)d_{23}(n) - v_{33}(n)c_{33}(n) \end{aligned} \quad (2.28)$$

$$\dot{t}_{34}(n) = d_{32}(n+1)s_{24}(n) + c_{33}(n+1)t_{34}(n) - u_{32}(n)b_{24}(n) - t_{33}(n)c_{34}(n) \quad (2.29)$$

$$\dot{s}_{42}(n) = b_{42}(n+1)t_{22}(n) + c_{43}(n+1)s_{32}(n) - s_{42}(n)c_{22}(n) - t_{43}(n)b_{32}(n) \quad (2.30)$$

$$\dot{t}_{43}(n) = b_{42}(n+1)u_{23}(n) + c_{43}(n+1)t_{33}(n) - s_{42}(n)d_{23}(n) - t_{43}(n)c_{33}(n). \quad (2.31)$$

3. On-Cell Local Densities and the Natural Constraints

Understanding the spatial coordinate n as the discrete variable marking the lattice unit cell we introduce the term “on-cell local density” implying the local density constructed of prototype field amplitudes taken on the same cell. Thus, according to the results of our previous article [1] the on-cell local densities ought to be defined by the expressions

$$\begin{aligned} \rho_{11}(n) = & \ln[t_{12}(n)t_{21}(n)v_{33}(n) + u_{13}(n)u_{31}(n)t_{22}(n) \\ & - t_{12}(n)u_{23}(n)u_{31}(n) - u_{13}(n)u_{32}(n)t_{21}(n)] \end{aligned} \quad (3.1)$$

$$\rho_{22}(n) = \ln r_{22}(n) \quad (3.2)$$

$$\rho(n) = \ln\{[u_{13}(n)s_{42}(n) - t_{12}(n)t_{43}(n)][u_{31}(n)s_{24}(n) - t_{21}(n)t_{34}(n)]\} \quad (3.3)$$

$$\rho_{33}(n) = \ln v_{33}(n) \quad (3.4)$$

$$\begin{aligned} \rho_{44}(n) = & \ln[t_{43}(n)t_{34}(n)r_{22}(n) + s_{42}(n)s_{24}(n)t_{33}(n) \\ & - t_{43}(n)s_{32}(n)s_{24}(n) - s_{42}(n)s_{23}(n)t_{34}(n)]. \end{aligned} \quad (3.5)$$

By virtue of general evolution equations (2.16)–(2.31) concretized by the expressions (2.2)–(2.15) for the entries of obverse evolution operator the evolution of on-cell conserved densities must be governed by the equations

$$\dot{\rho}_{11}(n) = c_{22}(n+1) + c_{33}(n+1) - c_{22}(n) - c_{33}(n) \quad (3.6)$$

$$\begin{aligned} \dot{\rho}_{22}(n) = & c_{22}(n+1) + \frac{a_{22}s_{23}(n+1)s_{32}(n)}{r_{22}(n+1)r_{22}(n)} + \frac{a_{22}s_{24}(n+1)s_{42}(n)}{r_{22}(n+1)r_{22}(n)} \\ & - c_{22}(n) - \frac{a_{22}s_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{24}(n)s_{42}(n-1)}{r_{22}(n)r_{22}(n-1)} \end{aligned} \quad (3.7)$$

$$\dot{\rho}(n) = c_{22}(n+1) + c_{33}(n+1) - c_{22}(n) - c_{33}(n) \quad (3.8)$$

$$\begin{aligned} \dot{\rho}_{33}(n) = & c_{33}(n+1) + \frac{e_{33}u_{32}(n+1)u_{23}(n)}{v_{33}(n+1)v_{33}(n)} + \frac{e_{33}u_{31}(n+1)u_{13}(n)}{v_{33}(n+1)v_{33}(n)} \\ & - c_{33}(n) - \frac{e_{33}u_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n-1)} \end{aligned} \quad (3.9)$$

$$\dot{\rho}_{44}(n) = c_{22}(n+1) + c_{33}(n+1) - c_{22}(n) - c_{33}(n) \quad (3.10)$$

being the discrete-space analogs of continuity equations. The same equations (3.6)–(3.10) referred to as the lowest local conservation laws had been obtained in our previous paper [1] by the modified recurrence technique. This observation asserts to be the good indication on a reliability and adequacy of modified recurrence approach as such.

Looking at the right-hand sides of continuity equations (3.6), (3.8), (3.10) we immediately reveal two equalities

$$\dot{\rho}_{11}(n) = \dot{\rho}(n) \quad (3.11)$$

$$\dot{\rho}_{44}(n) = \dot{\rho}(n) \quad (3.12)$$

giving rise to two natural constraints

$$\mu^2(n) \exp[\rho_{11}(n)] = \sigma_{11}(n) \exp[\rho(n)] \quad (3.13)$$

$$\mu^2(n) \exp[\rho_{44}(n)] = \sigma_{44}(n) \exp[\rho(n)] \quad (3.14)$$

on the prototype field functions. Here the quantities $\sigma_{11}(n)$, $\mu(n)$, $\sigma_{44}(n)$ are independent on time τ but are permitted being arbitrary functions of space variable n .

The above arbitrariness of $\sigma_{11}(n)$, $\mu(n)$, $\sigma_{44}(n)$ appears to be useful in modelling the effects of space inhomogeneity caused e.g. by external substrat [5]. However, in the theory of integrable systems such an opportunity is usually ignored demoting the functions $\sigma_{11}(n)$, $\mu(n)$, $\sigma_{44}(n)$ to the mere constants σ_{11} , μ , σ_{44} .

In any event the obtained constraints (3.13) and (3.14) imply that the number of actual field variables is lesser by two than the number of prototype field amplitudes. The simplest way to implement this reduction is to adopt the quantities $t_{22}(n)$ and $t_{33}(n)$ as dependent on the rest of field amplitudes. Then resolving the constraint equations (3.13) and (3.14) with respect to $t_{22}(n)$ and $t_{33}(n)$ we obtain the expressions

$$t_{22}(n) = \frac{t_{12}(n)u_{23}(n)u_{31}(n) + u_{13}(n)u_{32}(n)t_{21}(n) - t_{12}(n)t_{21}(n)v_{33}(n)}{u_{13}(n)u_{31}(n)} + \frac{[u_{13}(n)s_{42}(n) - t_{12}(n)t_{43}(n)]\sigma_{11}(n)[u_{31}(n)s_{24}(n) - t_{21}(n)t_{34}(n)]}{u_{31}(n)\mu^2(n)u_{13}(n)} \quad (3.15)$$

$$t_{33}(n) = \frac{t_{43}(n)s_{32}(n)s_{24}(n) + s_{42}(n)s_{23}(n)t_{34}(n) - t_{43}(n)t_{34}(n)r_{22}(n)}{s_{42}(n)s_{24}(n)} + \frac{[s_{42}(n)u_{13}(n) - t_{43}(n)t_{12}(n)]\sigma_{44}(n)[s_{24}(n)u_{31}(n) - t_{34}(n)t_{21}(n)]}{s_{24}(n)\mu^2(n)s_{42}(n)} \quad (3.16)$$

allowing to eliminate $t_{22}(n)$ and $t_{33}(n)$ from further consideration.

4. Additional Constraints and Classification of Obverse Integrable Systems

Besides of two natural constraints (3.13) and (3.14) considered in the previous section and retaining the functions $c_{22}(n)$ and $c_{33}(n)$ as unfixed there exists another sort of constraints essentially dependent on our particular preferences. These latter constraints referring to as the additional ones cause the fixation of sampling functions $c_{22}(n)$ and $c_{33}(n)$ on the one hand and ensure further decrease in the number of actual field variables on the other.

Below we list three the most interesting variants of additional constraints given in their differential (left column) and purely algebraic (right column) forms. The first variant

$$\dot{\rho}_{22}(n) = 0 \quad r_{22}(n) = \nu_{22}(n) \quad (4.1)$$

$$\dot{\rho}_{33}(n) = 0 \quad v_{33}(n) = \nu_{33}(n). \quad (4.2)$$

The second variant

$$\dot{\rho}_{22}(n) = \dot{\rho}_{33}(n) \quad r_{22}(n)\nu_{33}(n) = v_{33}(n)\nu_{22}(n) \quad (4.3)$$

$$\dot{\rho}(n) = 0 \quad \exp[\rho(n)] = \mu^4(n). \quad (4.4)$$

The third variant

$$\dot{\rho}_{22}(n) = \dot{\rho}(n) \quad \mu^4(n)r_{22}(n) = \nu_{22}(n) \exp[\rho(n)] \quad (4.5)$$

$$\dot{\rho}_{33}(n) = \dot{\rho}(n) \quad \mu^4(n)v_{33}(n) = \nu_{33}(n) \exp[\rho(n)]. \quad (4.6)$$

Here the quantities $\nu_{22}(n)$, $\mu(n)$, $\nu_{33}(n)$ must be treated as independent on time τ otherwise being arbitrary functions of coordinate n . Naturally, the interpretation of functions $\nu_{22}(n)$, $\mu(n)$, $\nu_{33}(n)$ should alter from variant to variant.

Each particular variant of additional constraints having been applied to the evolution equations (3.7)–(3.9) for the on-cell local densities $\rho_{22}(n)$, $\rho(n)$, $\rho_{33}(n)$ yields one respective variant of sampling fixation. Sequentially we have

$$c_{22}(n) = c_{22} - \frac{a_{22}s_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{24}(n)s_{42}(n-1)}{r_{22}(n)r_{22}(n-1)} \quad (4.7)$$

$$c_{33}(n) = c_{33} - \frac{e_{33}u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n-1)} \quad (4.8)$$

for the first variant,

$$\begin{aligned} c_{22}(n) = c_{22} - \frac{a_{22}s_{23}(n)s_{32}(n-1)}{2r_{22}(n)r_{22}(n-1)} - \frac{a_{22}s_{24}(n)s_{42}(n-1)}{2r_{22}(n)r_{22}(n-1)} \\ + \frac{e_{33}u_{32}(n)u_{23}(n-1)}{2v_{33}(n)v_{33}(n-1)} + \frac{e_{33}u_{31}(n)u_{13}(n-1)}{2v_{33}(n)v_{33}(n-1)} \end{aligned} \quad (4.9)$$

$$\begin{aligned} c_{33}(n) = c_{33} - \frac{e_{33}u_{32}(n)u_{23}(n-1)}{2v_{33}(n)v_{33}(n-1)} - \frac{e_{33}u_{31}(n)u_{13}(n-1)}{2v_{33}(n)v_{33}(n-1)} \\ + \frac{a_{22}s_{23}(n)s_{32}(n-1)}{2r_{22}(n)r_{22}(n-1)} + \frac{a_{22}s_{24}(n)s_{42}(n-1)}{2r_{22}(n)r_{22}(n-1)} \end{aligned} \quad (4.10)$$

for the second variant, and

$$c_{22}(n) = c_{22} + \frac{e_{33}u_{31}(n)u_{13}(n-1)}{v_{33}(n)v_{33}(n-1)} + \frac{e_{33}u_{32}(n)u_{23}(n-1)}{v_{33}(n)v_{33}(n-1)} \quad (4.11)$$

$$c_{33}(n) = c_{33} + \frac{a_{22}s_{23}(n)s_{32}(n-1)}{r_{22}(n)r_{22}(n-1)} + \frac{a_{22}s_{24}(n)s_{42}(n-1)}{r_{22}(n)r_{22}(n-1)} \quad (4.12)$$

for the third variant. In each of three variants the quantities c_{22} and c_{33} are understood as some arbitrary functions of time τ .

We clearly see that each of above listed cases of sampling fixation is characterized by four constraints s.s. two natural and two additional. As a result there appears an opportunity to reduce sixteen original prototype field amplitudes to twelve true field functions.

Inasmuch as each proposed variant of additional constraints invokes the evolution equations for the on-cell local densities in one or another specific way we may consider this observation as the basic principle classifying feasible semidiscrete integrable nonlinear systems.

5. Reduction to the Real Fields

The general matrix structures of the spectral operator (1.1) and the obverse evolution operator (2.1) permit one to make the following mutually consistent reductions

$$t_{12}(n) = t_-(n) = t_{43}(n) \quad (5.1)$$

$$u_{13}(n) = f_-(n) = s_{42}(n) \quad (5.2)$$

$$t_{21}(n) = t_+(n) = t_{34}(n) \quad (5.3)$$

$$s_{24}(n) = f_+(n) = u_{31}(n) \quad (5.4)$$

$$r_{22}(n) = h(n) = v_{33}(n) \quad (5.5)$$

$$t_{22}(n) = t(n) = t_{33}(n) \quad (5.6)$$

$$s_{23}(n) = g_+(n) = u_{32}(n) \quad (5.7)$$

$$u_{23}(n) = g_-(n) = s_{32}(n) \quad (5.8)$$

and

$$a_{22} = k = e_{33} \quad (5.9)$$

$$c_{12}(n) = kt_-(n-1)/h(n-1) = c_{43}(n) \quad (5.10)$$

$$d_{13}(n) = kf_-(n-1)/h(n-1) = b_{42}(n) \quad (5.11)$$

$$c_{21}(n) = kt_+(n)/h(n) = c_{34}(n) \quad (5.12)$$

$$a_{22}(n) = k = e_{33}(n) \quad (5.13)$$

$$b_{23}(n) = kg_+(n)/h(n) = d_{32}(n) \quad (5.14)$$

$$d_{23}(n) = kg_-(n-1)/h(n-1) = b_{32}(n) \quad (5.15)$$

$$b_{24}(n) = kf_+(n)/h(n) = d_{31}(n) \quad (5.16)$$

$$c_{22}(n) = c(n) = c_{33}(n) \quad (5.17)$$

with $f_-(n)$, $t_-(n)$, $f_+(n)$, $t_+(n)$, $h(n)$, $t(n)$, $g_-(n)$, $g_+(n)$ and $c(n)$ being the purely real functions of spatial coordinate n and time τ , while k being the purely real function of time τ . In these reductions the model evolution equations (2.16)–(2.31) acquire the forms

$$\dot{f}_-(n) = f_-(n) \left[k \frac{t(n)}{h(n)} - c(n) \right] + kt_-(n) \left[\frac{g_-(n)}{h(n)} - \frac{g_-(n-1)}{h(n-1)} \right] \quad (5.18)$$

$$\dot{t}_-(n) = t_-(n) \left[k \frac{t(n)}{h(n)} - c(n) \right] + kf_-(n) \left[\frac{g_-(n)}{h(n)} - \frac{g_-(n-1)}{h(n-1)} \right] \quad (5.19)$$

$$\dot{f}_+(n) = f_+(n) \left[c(n+1) - k \frac{t(n)}{h(n)} \right] + kt_+(n) \left[\frac{g_+(n+1)}{h(n+1)} - \frac{g_+(n)}{h(n)} \right] \quad (5.20)$$

$$\dot{t}_+(n) = t_+(n) \left[c(n+1) - k \frac{t(n)}{h(n)} \right] + kf_+(n) \left[\frac{g_+(n+1)}{h(n+1)} - \frac{g_+(n)}{h(n)} \right] \quad (5.21)$$

$$\begin{aligned} \dot{h}(n) = & h(n)[c(n+1) - c(n)] + kg_-(n) \frac{g_+(n+1)}{h(n+1)} - kg_+(n) \frac{g_-(n-1)}{h(n-1)} \\ & + kf_-(n) \frac{f_+(n+1)}{h(n+1)} - kf_+(n) \frac{f_-(n-1)}{h(n-1)} \end{aligned} \quad (5.22)$$

$$\begin{aligned} \dot{t}(n) = & t(n)[c(n+1) - c(n)] + kg_+(n) \left[\frac{g_+(n+1)}{h(n+1)} - \frac{g_+(n)}{h(n)} \right] \\ & + kg_-(n) \left[\frac{g_-(n)}{h(n)} - \frac{g_-(n-1)}{h(n-1)} \right] + kt_-(n) \frac{t_+(n+1)}{h(n+1)} - kt_+(n) \frac{t_-(n-1)}{h(n-1)} \end{aligned} \quad (5.23)$$

$$\begin{aligned} \dot{g}_+(n) = & g_+(n)[c(n+1) - c(n)] + kh(n) \left[\frac{g_-(n)}{h(n)} - \frac{g_-(n-1)}{h(n-1)} \right] \\ & + kt(n) \left[\frac{g_+(n+1)}{h(n+1)} - \frac{g_+(n)}{h(n)} \right] + kt_-(n) \frac{f_+(n+1)}{h(n+1)} - kf_+(n) \frac{t_-(n-1)}{h(n-1)} \end{aligned} \quad (5.24)$$

$$\begin{aligned} \dot{g}_-(n) = & g_-(n)[c(n+1) - c(n)] + kh(n) \left[\frac{g_+(n+1)}{h(n+1)} - \frac{g_+(n)}{h(n)} \right] \\ & + kt(n) \left[\frac{g_-(n)}{h(n)} - \frac{g_-(n-1)}{h(n-1)} \right] + kf_-(n) \frac{t_+(n+1)}{h(n+1)} - kt_+(n) \frac{f_-(n-1)}{h(n-1)} \end{aligned} \quad (5.25)$$

So that the two original natural constraints (3.13) and (3.14) are reducible to only the single one

$$\begin{aligned} & \mu^2(n)[t_-(n)h(n)t_+(n) + f_-(n)t(n)f_+(n) - t_-(n)g_-(n)f_+(n) - f_-(n)g_+(n)t_+(n)] \\ & = \sigma(n)[f_-^2(n) - t_-^2(n)][f_+^2(n) - t_+^2(n)]. \end{aligned} \quad (5.26)$$

Here $\mu(n)$ and $\sigma(n)$ are some time-independent real functions of spatial coordinate n .

In what follows we assume the quantities $\mu(n)$, $\sigma(n)$, k as the real parameters independent both on the spatial coordinate n and time τ . Thus we can denote $\mu(n) = \mu$, $\sigma(n) = \sigma$.

Now let us consider the first variant of sampling fixation $h(n) = \nu$, where $\dot{\nu} \equiv 0$, and take into account the natural constraint (5.26). Then for the functions $c(n)$, $c(n+1)$ and $t(n)$ we obtain

$$c(n) = c - g_-(n-1)g_+(n) - f_-(n-1)f_+(n) \quad (5.27)$$

$$c(n+1) = c - g_-(n)g_+(n+1) - f_-(n)f_+(n+1) \quad (5.28)$$

and

$$\begin{aligned} t(n) = & g_-(n) \frac{t_-(n)}{f_-(n)} + g_+(n) \frac{t_+(n)}{f_+(n)} - \frac{t_-(n)t_+(n)}{f_-(n)f_+(n)} \\ & - \sigma \frac{[f_-^2(n) - t_-^2(n)][f_+^2(n) - t_+^2(n)]}{f_-(n)f_+(n)} \end{aligned} \quad (5.29)$$

where c can in principle be some real function of time. In so doing the eight evolution equations (5.18)–(5.25) give rise to only six truly independent ones

$$\dot{f}_-(n) = f_-(n)[t(n) - c(n)] + t_-(n)[g_-(n) - g_-(n-1)] \quad (5.30)$$

$$\dot{t}_-(n) = t_-(n)[t(n) - c(n)] + f_-(n)[g_-(n) - g_-(n-1)] \quad (5.31)$$

$$\dot{f}_+(n) = f_+(n)[c(n+1) - t(n)] + t_+(n)[g_+(n+1) - g_+(n)] \quad (5.32)$$

$$\dot{t}_+(n) = t_+(n)[c(n+1) - t(n)] + f_+(n)[g_+(n+1) - g_+(n)] \quad (5.33)$$

$$\begin{aligned} \dot{g}_+(n) = & g_+(n)[c(n+1) - c(n)] + g_-(n) - g_-(n-1) \\ & + t(n)[g_+(n+1) - g_+(n)] + t_-(n)f_+(n+1) - f_+(n)t_-(n-1) \end{aligned} \quad (5.34)$$

$$\begin{aligned} \dot{g}_-(n) = & g_-(n)[c(n+1) - c(n)] + g_+(n+1) - g_+(n) \\ & + t(n)[g_-(n) - g_-(n-1)] + f_-(n)t_+(n+1) - t_+(n)f_-(n-1) \end{aligned} \quad (5.35)$$

where the functions $c(n)$, $c(n+1)$ and $t(n)$ are given by early written expressions (5.27)–(5.29). In all formulas (5.27)–(5.35) of this paragraph we have tacitly adopted the following scaling $\mu^2 = 1$, $\nu = 1$, $k = 1$ which does not lead to further loss of generality.

In contrast the time independent coupling parameter σ have to be an arbitrary real number. It enters the spectral operator $L(n|z)$ through the expression (5.29) for the function $t(n)$ and can essentially regulate the structure of Jost solutions thus influencing the whole procedure of inverse scattering transform and hence the solutions to the reduced semidiscrete nonlinear system (5.30)–(5.35).

6. Concluding Remarks

In this paper we have found the zero-curvature representations for three semidiscrete integrable multifield nonlinear systems. These new systems arise due to the generalization and specific off-diagonal enlargement of spectral operator associated with the semidiscrete integrable nonlinear Schrödinger system on a zigzag-runged ladder lattice when being accompanied by the properly chosen evolution operator of obverse type. Each of these three choices suggests to fix arbitrary sampling functions appearing in general evolution operator by imposing two additional constraints onto the evolution equations for the five basic on-cell local densities and by using two so-called natural constraints. As a consequence the number of actual field variables in each of proposed systems can be considerably reduced. Instead we gain several arbitrary functions of cell variable serving as the coupling functions or at least as additional coupling parameters when the dependence on the cell variable is absent.

From the physical point of view the mere existence of several distinct coupling parameters (or coupling functions) looks as very promising fact for the model applicability to say nothing about the advantages connected with the tunability of these parameters.

Needless to underline that the prototype field amplitudes involved into the general formulation of integrable system can be segregated into several blocks of essentially distinct nature, so that we can say about the mutual influence between the fields of different types. This observation is in evident distinction with the situation for the multicomponent systems of Ablowitz–Ladik type [12–16] or other regular extensions of Ablowitz–Ladik equations where all field amplitudes are of an essentially common origin.

Of course, we can in principle satisfy the natural constraints (3.13) and (3.14) by formally equalizing the fields $t_{12}(n)$, $t_{21}(n)$, $u_{13}(n)$, $u_{31}(n)$ and $t_{43}(n)$, $t_{34}(n)$, $s_{42}(n)$, $s_{24}(n)$ to zero. This formal reduction is able to produce either the truncated version of nonlinear Schrödinger system on a zigzag-runged ladder lattice [2, 3] or the nonlinear Schrödinger system with the background controlled intersite resonant couplings [17] depending on the choice of boundary conditions imposed onto the remaining field amplitudes $r_{22}(n)$, $t_{22}(n)$, $s_{23}(n)$, $u_{23}(n)$ and $s_{32}(n)$, $u_{32}(n)$, $t_{33}(n)$, $r_{33}(n)$. However the sequence of local conservation laws for both of reduced systems turns out to be distinct from that of unreduced system. Here we come to the distinguished role of boundary conditions capable to dictate distinct admissible Hamiltonian structures even for seemingly the same set of equations [18]. In the case of our general system (2.16)–(2.31) it means that the determinant $\det L(n|z)$ of the spectral operator (1.1) coinciding with the exponent $\exp[\rho(n)]$ of the basic on-cell local density (3.3) must be adopted as essentially nonzero thus igniting the whole hierarchy of respective local conservation laws. This prerequisite is in contrast to the above mentioned reduced systems where such an on-cell local density does not appear and we should rely on an absolutely another one, originated from the spectral operator of lesser rank.

The preliminary analysis either of the limiting eigenvalue problems

$$L^\pm(z)|\chi^\pm(z)\rangle = |\chi^\pm(z)\rangle\zeta(z) \quad (6.1)$$

(where $L^\pm(z) = \lim_{n \rightarrow \pm\infty} L^\pm(n|z)$ while $|\chi^\pm(z)\rangle$ denotes four-component column matrix) shows that in general there exist four distinct eigenvalues $\zeta_j(z)$ ($j = 1, 2, 3, 4$). Thus according to Caudrey treatment [6–8] we have to invert the scattering problem of fourth order once we would intend to integrate any nonlinear system of our interest. Here we would like to stress that sometimes the integration even of simplest nonlinear systems associated with the second order scattering problems turns out to be very difficult task due to the complications inflicted by the nonvanishing boundary conditions for the field variables. As an example it is sufficient to mention the situation concerning the famous Ablowitz-Ladik system [9–11] when the solutions with nonvanishing boundary conditions [19, 20] have been found well after the solutions with the vanishing ones [10, 11]. Whether the inverse scattering theory developed by Beals and Coifman for the continuous space variable [21] or by Bhate for the discrete space variable [22] could provide one with a procedure suitable to integrate our semidiscrete systems in more simple and straightforward manner as compared with the Caudrey scheme [6–8] will be checked by time.

As to the question about continuum limits of proposed here multicomponent semidiscrete integrable nonlinear systems we would not like to speculate on this topic inasmuch as any naive step in this direction does not guarantee the integrability of resulting continual counterparts.

When this paper has already been completed we became aware of very interesting approach called as Mikhailov reduction method [23–25] which turns out to be in lines with our main task in reducing the general integrable system to a system with fewer number of fields. Unfortunately, due to lack of experience with the reduction group theory we are unable to retrace how our results could be obtained within the framework of Mikhailov approach. In this context it is sufficient to underline only the guiding observation common for the elaboration of both approaches implying that the general compatibility condition of

two auxiliary linear equations (i.e. general zero-curvature condition) provides the undetermined set of equations and thus requires some additional constraints.

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