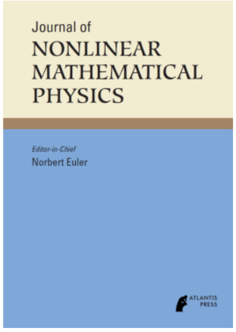

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EQUIVALENCE CLASSES OF THE SECOND ORDER ODEs WITH THE CONSTANT CARTAN INVARIANT

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Second order ordinary differential equations that possesses the constant invariant are investigated. Four basic types of these equations were found. For every type the complete list of nonequivalent equations is issued. As the examples the equivalence problem for the Painleve II equation, Painleve III equation with three zero parameters, Emden equations and for some other equations is solved.

Keywords: Invariant; equivalence problem; ordinary differential equation; point transformation; Painleve equation; Emden equation.

Mathematics Subject Classification 2000: 53A55, 34A26, 34A34, 34C14, 34C20, 34C41

1. Introduction

Let us consider the following second order ODE:

$$y'' = P(x,y) + 3\,Q(x,y)y' + 3\,R(x,y)y'^2 + S(x,y)y'^3. \tag{1.1}$$

General point transformations

$$\tilde{x} = \tilde{x}(x,y), \quad \tilde{y} = \tilde{y}(x,y) \tag{1.2}$$

preserve the form of Eq. (1.1).

Let us consider two arbitrary Eq. (1.1). The problem of existence of the point transformation (1.2) that connects these equations is called *the Equivalence Problem*. For the arbitrary Eq. (1.1) the explicit solution of the equivalence problem is rather complicated, see [23, 24]. It was effectively solved for the linear equations, see [1, 10, 11, 13, 19, 20]; for some Painleve equations, see [1, 3, 8, 12, 16, 17]; for the Emden equation, see [2] and for other equations, for example, see [14, 20].

The main approach that allows to solve the equivalence problem is based on the Invariant Theory. *Invariant* is a certain function depending on $(x,\, y)$ that is unchanged under the transformation (1.2): $I(x,y) = I(\tilde{x}(x,y), \tilde{y}(x,y))$.

Pseudoinvariant of weight m is a certain function depending on $(x,\, y)$ that is transformed under (1.2) with factor $\det T$ (the Jacobi determinant) in the degree m:

$$J(x,y) = (\det T)^m \cdot J(\tilde{x}(x,y), \tilde{y}(x,y)), \quad T = \begin{pmatrix} \partial\tilde{x}/\partial x & \partial\tilde{x}/\partial y \\ \partial\tilde{y}/\partial x & \partial\tilde{y}/\partial y \end{pmatrix}.$$

Pseudotensorial field of weight m *and valence* (r,s) is an indexed set that transforms under change of variables (1.2) by the rule

$$F^{i_1\cdots i_r}_{j_1\cdots j_s} = (\det T)^m \sum_{p_1\cdots p_r} \sum_{q_1\cdots q_s} S^{i_1}_{p_1} \cdots S^{i_r}_{p_r} T^{q_1}_{j_1} \cdots T^{q_s}_{j_s} \tilde{F}^{p_1\cdots p_r}_{q_1\cdots q_s},$$

here $S = T^{-1}$. It is easy to check that only factor $(\det T)^m$ distinguishes the pseudotensorial field from the classical tensorial field.

Invariant Theory of Eq. (1.1) goes back to the classical works of Liouville [19], Lie [18], Tresse [23, 24], Cartan [5, 22] (Late 19th and Early 20th Century) and continues in the works of [1, 4, 7, 10, 12, 13, 16, 20, 21] (Late 20th Century). It remains an active research topic in the 21th Century, see [2, 14, 17]. Background is adequately described in papers [1, 2].

In the present paper we use notations from works [7, 17, 20, 21] to calculate the invariants and pseudoinvariants of Eq. (1.1). The correlation between these (pseudo)invariants and semi-invariants from works [5, 19] (as they were presented in [2]) shows in the next chapter. The explicit formulas for their computation via known functions $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ contained in the Appendix A. Here and everywhere below notation $K_{i.j}$ denotes the partial differentiation: $K_{i.j} = \partial^{i+j}K/\partial x^i \partial y^j$.

2. Computation of Invariants

2.1. *Some geometric objects*

Step 1. From the functions P, Q, R and S, that are the coefficients of Eq. (1.1), let us organize the 3-indexes massive by the following rule:

$$\Theta_{111} = P, \quad \Theta_{121} = \Theta_{211} = \Theta_{112} = Q,$$
$$\Theta_{222} = S, \quad \Theta_{122} = \Theta_{212} = \Theta_{221} = R.$$

As the "Gramian matrixes" let us take the following couple:

$$d^{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \text{pseudotensorial field of the weight } 1,$$

$$d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad \text{pseudotensorial field of the weight } -1.$$

Step 2. Let us raise the first index

$$\Theta^k_{ij} = \sum_{r=1}^{2} d^{kr}\Theta_{rij}. \tag{2.1}$$

Under the change of variables (1.2) Θ_{ij}^k transforms "almost" as a affine connection. (The transformation rule is into the paper [7]).

Step 3. Using Θ_{ij}^k as the affine connection let us construct the "curvature tensor":

$$\Omega_{rij}^k = \frac{\partial \Theta_{jr}^k}{\partial u^i} - \frac{\partial \Theta_{ir}^k}{\partial u^j} + \sum_{q=1}^{2} \Theta_{iq}^k \Theta_{jr}^q - \sum_{q=1}^{2} \Theta_{jq}^k \Theta_{ir}^q, \quad \text{where } u^1 = x,\, u^2 = y,$$

and the "Ricci tensor" $\Omega_{rj} = \sum_{k=1}^2 \Omega_{rkj}^k$. The both objects are not the tensors. (See [7]).

Step 4. The following three indexes massive is the tensor:

$$W_{ijk} = \nabla_i \Omega_{jk} - \nabla_j \Omega_{ik}.$$

Here we use Θ_{ij}^k instead of the affine connection when made the covariant differentiation.

Step 5. Using the tensor W_{ijk} let us construct the new pseudovectorial fields:

$$\alpha_k = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} W_{ijk} d^{ij} \quad \text{pseudocovectorial field of weight 1,}$$

$$\beta_i = 3\nabla_i \alpha_k d^{kr} \alpha_r + \nabla_r \alpha_k d^{kr} \alpha_i \quad \text{pseudocovectorial field of weight 3.}$$

The coincident pseudovectorial fields are: $\alpha^j = d^{jk}\alpha_k$ of weight 2, $\beta^j = d^{ji}\beta_i$ of weight 4.

There are only three situations:

(1) Pseudovectorial field $\boldsymbol{\alpha} = \mathbf{0}$, *maximal degeneration case*, equation is equivalent to $y'' = 0$;
(2) Fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear: $3F^5 = \alpha^i \beta_i = 0$, *intermediate degeneration case*;
(3) Fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-collinear: $3F^5 = \alpha^i \beta_i \neq 0$, *general case.*

At the present paper we consider the intermediate degeneration case: $F = 0$ but $\boldsymbol{\alpha} \neq 0$.

Step 6. Let us denote the quantities φ_1 and φ_2 (their explicit formulas are in Appendix A) and organize the affine connection Γ_{ij}^k and the pseudoinvariant Ω of weight 1

$$\Gamma_{ij}^k = \Theta_{ij}^k - \frac{\varphi_k \delta_j^k + \varphi_k \delta_i^k}{3}, \quad \Omega = \frac{5}{3}\left(\frac{\partial \varphi_1}{\partial y} - \frac{\partial \varphi_2}{\partial x}\right).$$

The rule of covariant differentiation of the pseudotensorial field was presented in [7]:

$$\nabla_k F_{j_1 \cdots j_s}^{i_1 \cdots i_r} = \frac{\partial F_{j_1 \cdots j_s}^{i_1 \cdots i_r}}{\partial u^k} + \sum_{n=1}^{r} \sum_{v_n=1}^{2} \Gamma_{k v_n}^{i_n} F_{j_1 \cdots j_s}^{i_1 \cdots v_n \cdots i_r} - \sum_{n=1}^{s} \sum_{w_n=1}^{2} \Gamma_{k j_n}^{w_n} F_{j_1 \cdots w_n \cdots j_s}^{i_1 \cdots i_r} + m \varphi_k F_{j_1 \cdots j_s}^{i_1 \cdots i_r}.$$

If the pseudotensorial field F has type (r, s) and weight m, then the pseudotensorial field ∇F has type $(r, s+1)$ and weight m.

Step 7. Pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are collinear, hence exists the coefficient N, it is the pseudoinvariant of weight 2, such that: $\beta = 3N\alpha$. Then

$$\xi^i = d^{ij} \nabla_j N, \quad M = -\alpha_i \xi^i, \quad \gamma = -\xi - 2\Omega\alpha, \quad \Gamma = -\frac{d_{ij} \nabla_\xi \xi^i \xi^j}{M}.$$

Here $\boldsymbol{\xi}$ — pseudovectorial field of weight 3; M — pseudoinvariant of weight 4; $\boldsymbol{\gamma}$ — pseudovectorial field of weight 3; Γ — pseudoinvariant of weight 4. (See paper [7].)

2.2. *Invariants*

In this paper we are interested in the case $M \neq 0$, which is named as *the first case of intermediate degeneration*. By definition it means that also $N \neq 0$ and $\boldsymbol{\gamma} \neq 0$. The basic invariants are

$$I_1 = \frac{M}{N^2}, \quad I_2 = \frac{\Omega^2}{N}, \quad I_3 = \frac{\Gamma}{M}. \tag{2.2}$$

By differentiating invariants I_1, I_2, I_3 along pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ we get invariants

$$\begin{aligned} I_4 &= \frac{\nabla_{\boldsymbol{\alpha}} I_1}{N}, \quad I_5 = \frac{\nabla_{\boldsymbol{\alpha}} I_2}{N}, \quad I_6 = \frac{\nabla_{\boldsymbol{\alpha}} I_3}{N}, \\ I_7 &= \frac{(\nabla_{\boldsymbol{\gamma}} I_1)^2}{N^3}, \quad I_8 = \frac{(\nabla_{\boldsymbol{\gamma}} I_2)^2}{N^3}, \quad I_9 = \frac{(\nabla_{\boldsymbol{\gamma}} I_3)^2}{N^3}. \end{aligned} \tag{2.3}$$

Repeating this procedure more and more times, we can form an infinite sequence of invariants, adding six ones in each step.

So, to calculate the invariants we have to compute:

(1) Pseudovectorial field $\boldsymbol{\alpha} = (B, -A)^T$ of weight 2, see formula (A.1);
(2) Pseudovectorial field $\boldsymbol{\gamma}$ of weight 3, see formulas (A.8) and (A.9);
(3) Pseudoinvariant F of weight 1, see formula (A.2);
(4) Pseudoinvariant M of weight 4, see formulas (A.4) and (A.5);
(5) Pseudoinvariant N of weight 2, see formula (A.3);
(6) Pseudoinvariant Ω of weight 1, see formulas (A.6) and (A.7);
(7) Pseudoinvariant Γ of weight 4, see formula (A.10).

2.3. *Correlation between the semi-invariants*

No doubt the main part of the pseudoinvariants have been known previously.

At the work Cartan [5] adopted the following notations:

$$P = -a_4, \quad Q = -a_3, \quad R = -a_2, \quad S = -a_1, \quad A = -L_1, \quad B = -L_2.$$

At the work Liouville [19] presented the semi-invariants ν_5, w_1, i_2 and the quantity R_1 (see review in [2]). Here is a link between these quantities and pseudoinvariants F, Ω, N and quantity H:

$$F^5 = \nu_5, \quad H = L_1(L_2)_x - L_2(L_1)_x + 3R_1, \quad \Omega = -w_1 - \frac{\nu_5 a_4}{L_1^3} - 4\frac{(L_1)_x R_1}{L_1^3}, \quad N = \frac{i_2}{3}.$$

Pseudovectorial field $\boldsymbol{\gamma}$, pseudoinvariant M and pseudoinvariant Γ first appeared in the papers [7, 20, 21]. In the present paper we use the new notation in order to compute the chains of invariants (2.3).

3. The Main Problem

The main problem is to describe the equivalence classes of Eq. (1.1) from the first case of intermediate degeneration with the conditions $I_1 = \text{const} \neq 0$ and $I_2 = 0$ under the general point transformation (1.2). So, we investigate Eq. (1.1) such that

$$\boldsymbol{\alpha} \neq 0, \quad F = 0, \quad M \neq 0, \quad I_1 = \text{const} \neq 0, \quad I_2 = 0. \tag{3.1}$$

It is easy to see that two sequences of the invariants (2.3) become trivial ones: $I_4 = I_7 = \cdots = 0$ and $I_5 = I_8 = \cdots = 0$.

According to papers [1, 2], each Eq. (1.1) that satisfies the relations $F = 0$ and $I_2 = 0$ can be transformed into the form

$$y'' = P(x, y). \tag{3.2}$$

For example Eq. (3.2) holds the following relations:

$$A = P_{0.2} \neq 0, \quad B = 0, \quad M = \frac{2A_{0.1}^2}{5} - \frac{AA_{0.2}}{3} \neq 0, \quad N = -\frac{A_{0.1}}{3}, \quad I_2 = 0.$$

Let us calculate the invariant I_1:

$$I_1 = \frac{M}{N^2} = \frac{18}{5} - 3\frac{AA_{0.2}}{A_{0.1}^2} = \frac{18}{5} - 3C_1 = \text{const} \neq 0, \quad \frac{AA_{0.2}}{A_{0.1}^2} = C_1 = \text{const} \neq \frac{6}{5}. \tag{3.3}$$

4. Four Types of Equations

Theorem 4.1. *Every Eq.* (1.1) *with conditions* (3.1) *can be transformed by point transformations* (1.2) *into the form*:

$$y'' = P^*(y) + t(x)y + s(x),$$

where

$$P^*(y) = \begin{cases} e^y & \text{if } I_1 = \dfrac{3}{5}, \\ -\ln y & \text{if } I_1 = -\dfrac{9}{10}, \\ y(\ln y - 1) & \text{if } I_1 = -\dfrac{12}{5}, \\ \dfrac{y^{C+2}}{(C+1)(C+2)} & \text{if } I_1 = \dfrac{3(C+5)}{5C}, \quad C = \text{const} \neq -5, -2, -1, 0. \end{cases}$$

Proof. Let us resolve the differential equation (3.3) with respect to the function $A(x, y)$. There are two possibilities:

$$\begin{aligned} &C_1 = 1, \quad A(x, y) = b(x) \cdot e^{a(x)y}, \\ &C_1 \neq 1, \quad A(x, y) = ((a(x)y + b(x))^C, \quad C = \frac{1}{1 - C_1}, \end{aligned} \tag{4.1}$$

where $a(x)$ and $b(x)$ are arbitrary functions.

According to paper [2], the most general point transformations preserving the form (3.2) is the following transformation:

$$x = \alpha \int p^2(\tilde{x}) d(\tilde{x}) + \beta, \quad y = p(\tilde{x})\tilde{y} + h(\tilde{x}). \tag{4.2}$$

Here α, β — constants, $p(\tilde{x})$, $h(\tilde{x})$ — certain functions.

Therefore the direct and inverse transformations matrices S and T are

$$S = \begin{pmatrix} \partial x/\partial\tilde{x} & \partial x/\partial\tilde{y} \\ \partial y/\partial\tilde{x} & \partial y/\partial\tilde{y} \end{pmatrix} = \begin{pmatrix} \alpha p^2(\tilde{x}) & 0 \\ p_{1.0}(\tilde{x})\tilde{y} + h_{0.1}(\tilde{x}) & p(\tilde{x}) \end{pmatrix}, \quad \det S = \alpha p^3(\tilde{x}),$$

$$T = S^{-1} = \begin{pmatrix} \dfrac{1}{\alpha p^2(\tilde{x})} & 0 \\ \dfrac{-p_{1.0}(\tilde{x})\tilde{y} - h_{0.1}(\tilde{x})}{\alpha p^3(\tilde{x})} & \dfrac{1}{p(\tilde{x})} \end{pmatrix}, \quad \det T = \frac{1}{\alpha p^3(\tilde{x})}.$$

After the transformations (4.2) the pseudovectorial field $\boldsymbol{\alpha}$ of weight 2 changes as

$$\begin{pmatrix} \tilde{B} \\ -\tilde{A} \end{pmatrix} = \frac{T}{(\det T)^2} \begin{pmatrix} 0 \\ -A \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha p^5(\tilde{x}) A \end{pmatrix}.$$

So the transformation rules for A and B are: $\tilde{A}(\tilde{x},\tilde{y}) = \alpha^2 p^5(\tilde{x}) A(x(\tilde{x}), y(\tilde{x},\tilde{y}))$, $\tilde{B} = 0$.

Let $C_1 = 1$ and function A from (4.1), then

$$\tilde{A}(\tilde{x},\tilde{y}) = \alpha^2 p^5(\tilde{x}) b(x(\tilde{x})) \cdot e^{a(x(\tilde{x}))(p(\tilde{x})\tilde{y} + h(\tilde{x}))} = \alpha^2 p^5(\tilde{x}) b(x(\tilde{x})) e^{a(x(\tilde{x}))h(\tilde{x})} e^{a(x(\tilde{x}))p(\tilde{x})\tilde{y}}.$$

Choosing the appropriate functions $h(\tilde{x})$ and $p(\tilde{x})$ we can get $\alpha^2 p^5(\tilde{x}) b(x(\tilde{x})) e^{a(x(\tilde{x}))h(\tilde{x})} = 1$, $a(x(\tilde{x}))p(\tilde{x}) = 1$, $\tilde{A}(\tilde{x},\tilde{y}) = e^{\tilde{y}}$.

Let $C_1 \neq 1$ and function A from (4.1), then

$$\tilde{A}(\tilde{x},\tilde{y}) = \alpha^2 p^5(\tilde{x}) \left(a(x(\tilde{x}))p(\tilde{x})\tilde{y} + [a(x(\tilde{x}))h(\tilde{x}) + b(x(\tilde{x}))]\right)^C$$

Choosing the appropriate functions $h(\tilde{x})$ and $p(\tilde{x})$ we can get $a(x(\tilde{x}))h(\tilde{x}) + b(x(\tilde{x})) = 0$, $\alpha^2 p^{5+C}(\tilde{x}) a^C(x(\tilde{x})) = 1$, $\tilde{A}(\tilde{x},\tilde{y}) = \tilde{y}^C$.

So, if Eq. (1.1) with conditions (3.1) is written in terms of canonical coordinates (let these coordinates will be (x, y)), there may be two possibilities: $A(x,y) = e^y$ or $A(x,y) = y^C$. Therefore $A = P_{0.2}$ then it may be four opportunities:

$$\begin{aligned}
P(x,y) &= e^y + t(x)y + s(x), \quad C_1 = 1, \\
P(x,y) &= -\ln y + t(x)y + s(x), \quad C = -2, \\
P(x,y) &= y\ln y - y + t(x)y + s(x), \quad C = -1, \\
P(x,y) &= \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x), \quad C = \mathrm{const} \neq -5,\ -2,\ -1,\ 0.
\end{aligned}$$

Here $t(x)$, $s(x)$ are the arbitrary functions. □

Table 1. Four different types of Eq. (1.1) with conditions (3.1).

Type	Equation	A	C	C_1	I_1
I	$y'' = e^y + t(x)y + s(x)$	e^y	—	1	$\frac{3}{5}$
II	$y'' = -\ln y + t(x)y + s(x)$	$\frac{1}{y^2}$	-2	$\frac{3}{2}$	$-\frac{9}{10}$
III	$y'' = y(\ln y - 1) + t(x)y + s(x)$	$\frac{1}{y}$	-1	2	$-\frac{12}{5}$
IV	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x)$	y^C	$C \neq 0, -1, -2, -5$	$\frac{C-1}{C}$	$\frac{3(C+5)}{5C}$

Note: Here $t(x)$, $s(x)$ are the arbitrary functions.

5. Equations of Type I

Definition 5.1. Let us say that Eq. (1.1) has Type I if conditions (3.1) hold, where $I_1 = 3/5$.

According to Theorem 4.1 any Eq. (1.1) of Type I can be transformed by point transformations (1.2) into the canonical form:

$$y'' = e^y + t(x)y + s(x). \tag{5.1}$$

Lemma 5.1. *The most general point transformations that preserve the canonical form* (5.1) *are the following ones*:

$$x = \alpha\tilde{x} + \beta, \quad y = \tilde{y} - 2\ln\alpha. \tag{5.2}$$

Here α, β *are some constants. The new equation has the form*: $\tilde{y}'' = e^{\tilde{y}} + \tilde{t}(\tilde{x})\tilde{y} + \tilde{s}(\tilde{x})$, *where*

$$\tilde{t}(\tilde{x}) = \alpha^2 t(\alpha\tilde{x} + \beta), \quad \tilde{s}(\tilde{x}) = \alpha^2 s(\alpha\tilde{x} + \beta) - 2\alpha^2 \ln\alpha \cdot t(\alpha\tilde{x} + \beta).$$

The proof of Lemma 5.1 follows from the straightward calculations. We apply transformations (4.2) to the Eq. (5.1). They must preserve the form of equation. Then in these canonical coordinates the non-trivial invariants (2.2), (2.3) and (A.11) are equal:

$$I_3 = \frac{1}{15} + \frac{1}{15e^y}(t(x)y + s(x)), \quad I_6 = \frac{1}{5e^y}(t(x)y + s(x) - t(x)),$$
$$I_9 = \frac{1}{1875e^{3y}}(t'(x)y + s'(x))^2.$$

Let us introduce the additional invariants:

$$\begin{aligned} J_3 &= 15I_3 - 1 = \frac{t(x)y + s(x)}{e^y}, \quad J_6 = 5I_6 = \frac{t(x)y + s(x) - t(x)}{e^y}, \\ J &= \frac{J_3}{J_3 - J_6} = y + \frac{s(x)}{t(x)}, \quad J_1 = J + \ln\left(\frac{J_3}{J}\right) = \ln t(x) + \frac{s(x)}{t(x)}, \\ J_9 &= 1875I_9 = \frac{(t'(x)y + s'(x))^2}{e^{3y}}, \quad K = \frac{J_9}{J_3^3} = \frac{(t'(x)y + s'(x))^2}{(t(x)y + s(x))^3}. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Let Eq.* (1.1) *be an arbitrary equation of Type* I. *Then it is equivalent to some equation from the following list of nonequivalent equations of Type* I:

(1) *If* $J_3 = 0$ *from* (5.3) *then it is equivalent to* $y'' = e^y$.
(2) *If* $J_3 \neq \text{const}$, $J_3 = J_6$, $K = 0$ *from* (5.3) *then it is equivalent to* $y'' = e^y + 1$.
(3) *If* $J_3 \neq \text{const}$, $J_3 \neq J_6$, $J_6 \neq \text{const}$, $J_1 = a = \text{const}$, $K = 0$ *from* (5.3) *then it is equivalent to* $y'' = e^y + y + a$.
Two equations of Type I.3 *are equivalent if and only if invariant* J_1 *for both equations is the same and equal to constant* a.
(4) *If* $J_3 \neq \text{const}$, $J_3 = J_6$, $K = k = \text{const} \neq 0$ *from* (5.3) *then it is equivalent to* $y'' = e^y + 4/kx^2$.
Two equations of Type I.4 *are equivalent if and only if invariant* K *for both equations is the same and equal to constant* $k \neq 0$.
(5) *If* $J_3 \neq \text{const}$, $J_3 = J_6$, $K \neq \text{const}$ *from* (5.3) *then it is equivalent to* $y'' = e^y + s(x), s(x) \neq \text{const}$.
Two equations of Type I.5 *are equivalent if and only if after the transformation* $\tilde{x} = K(x, y)$, $\tilde{y} = J_3(x, y)$ *their notations become identical.*
(6) *If* $J_3 \neq \text{const}$, $J_3 \neq J_6$, $J_6 \neq \text{const}$, $J_1 \neq \text{const}$, $K \neq \text{const}$ *from* (5.3) *then it is equivalent to* $y'' = e^y + t(x)y + s(x), t(x) \neq 0$.
Two equations of Type I.6 *are equivalent if and only if after the transformation* $\tilde{x} = J_1(x, y)$, $\tilde{y} = J(x, y)$ *their notations become identical.*

In the cases I.5 *and* I.6 *functions* $t(x)$ *and* $s(x)$ *are definited up to transformations* (5.2).

Proof. Let Eq. (1.1) be an arbitrary equation of Type I. Then in the terms of canonical coordinates it has the form (5.1). Let us calculate the invariants (5.3).

(1) If $J_3 = 0$ then $(t(x)y + s(x))/e^y = 0 \Leftrightarrow t(x) \equiv 0$, $s(x) \equiv 0$. Therefore Eq. (1.1) can be reduced into the canonical form $y'' = e^y$.
(2) If $J_3 \neq \text{const}$, $J_3 = J_6$, $K = 0$, then

$$\frac{t(x)y + s(x)}{e^y} = \frac{t(x)y + s(x) - t(x)}{e^y}, \quad \frac{(t'(x)y + s'(x))^2}{(t(x)y + s(x))^3} = 0.$$

Hence $t(x) \equiv 0$, $s'(x) \equiv 0$, accordingly $s(x) = s = \text{const} \neq 0$. If $s = 0$ then $J_3 = 0$ and equation has Type I.1. So in the terms of canonical coordinates this equation has the

Table 2. Equations of Type I.

Type	J_3	J_6	J_1	K	Canonical form
I.1	0	0	—	0	$y'' = e^y$
I.2	$\neq$ const	J_3	—	0	$y'' = e^y + 1$
I.3	$\neq$ const	$\neq J_3$, $\neq$ const	const = a	0	$y'' = e^y + y + a$
I.4	$\neq$ const	J_3	—	const $= k \neq 0$	$y'' = e^y + \frac{4}{kx^2}$
I.5	$\neq$ const	J_3	—	$\neq$ const	$y'' = e^y + s(x)$, $s(x) \neq \text{const}$
I.6	$\neq$ const	$\neq J_3$, $\neq$ const	$\neq$ const	$\neq$ const	$y'' = e^y + t(x)y + s(x)$, $t(x) \neq 0$

form $y'' = e^y + s$. Let us make the transformation (5.2) then $\tilde{s} = \alpha^2 s$. Choosing the parameter α we could make $\tilde{s} = 1$. Thus the canonical form is $\tilde{y}'' = e^{\tilde{y}} + 1$.

(3) If $J_3 \neq \text{const}$, $J_3 \neq J_6$, $J_6 \neq \text{const}$, $J_1 = a = \text{const}$, $K = 0$ then

$$\ln t(x) + \frac{s(x)}{t(x)} = a, \quad \frac{(t'(x)y + s'(x))^2}{(t(x)y + s(x))^3} = 0.$$

Hence $t(x) = t = \text{const} \neq 0$, $s(x) = s = \text{const}$ and equation has the form $y'' = e^y + ty + s$. Let $Y'' = e^Y + TY + S$, $T = \text{const} \neq 0$, $S = \text{const}$ is another equation of this type. These equations are equivalent if and only if $T = \alpha^2 t$, $S = \alpha^2 s - \alpha^2 t \ln \alpha^2$ then between t, s and T, S exists the following connection

$$S = \frac{sT}{t} - T \ln\left(\frac{T}{t}\right) \Leftrightarrow \ln t + \frac{s}{t} = \ln T + \frac{S}{T} \Leftrightarrow J_1(x, y) = J_1(X, Y) = a.$$

Thus we see that two equations of Type I.3 are equivalent if and only if invariants J_1 for both equations are equal to constant a.

Now let us select the canonical form for equations of Type I.3. We make transformation (5.2) and choose the parameter α so that into the new coordinates $\tilde{t} = 1$, then invariant $J_1 = \tilde{s} = a$. Hence equation has the canonical form $\tilde{y}'' = e^{\tilde{y}} + \tilde{y} + a$.

(4) If $J_3 \neq \text{const}$, $J_3 = J_6$, $K = k = \text{const} \neq 0$, then

$$\frac{t(x)y + s(x)}{e^y} = \frac{t(x)y + s(x) - t(x)}{e^y}, \quad \frac{(t'(x)y + s'(x))^2}{(t(x)y + s(x))^3} = k.$$

Hence $t(x) \equiv 0$ and we have a differential equation on function $s(x)$: $s'^2(x) = ks^3(x)$. The solution: $s(x) = 4/(\sqrt{k} \cdot x + c)^2$, $c = \text{const}$. Now let us choose the canonical form for the equations of Type 1.4. Making the transformation (5.2) we get

$$\tilde{s}(\tilde{x}) = \alpha^2 \cdot s(\alpha\tilde{x} + \beta) = \frac{4\alpha^2}{(\alpha\sqrt{k} \cdot \tilde{x} + \sqrt{k} \cdot \beta + c)^2},$$

then select the parameter β such that $\sqrt{k} \cdot \beta + c = 0$ for any α. So equation has the canonical form $\tilde{y}'' = e^{\tilde{y}} + 4/(k\tilde{x}^2)$. Two equations of Type I.4 equivalent if and only if invariants K for both equations are equal to constant $k \neq 0$.

(5) If $J_3 \neq \text{const}$, $J_3 = J_6$, $K \neq \text{const}$ then

$$\frac{t(x)y + s(x)}{e^y} = \frac{t(x)y + s(x) - t(x)}{e^y}, \quad \text{so } t(x) \equiv 0.$$

Then in terms of canonical coordinates equation has the form $y'' = e^y + s(x)$.

Let us solve the equivalence problem for the equations of Type I.5. We have two possibilities. One way is to reduce both equations into the canonical coordinates: $y'' = e^y + s(x)$, $Y'' = e^Y + S(X)$. These equations are equivalent if and only if there exist appropriate α and β such that $S(X) = \alpha^2 s(\alpha X + \beta)$.

The other way is based on the observation that for any Eq. (1.1) of Type I.5 invariants K and J_3 are functionally independent. So, if the first equation has the form (1.1) and depends on coordinates (x, y) and the second equation has the form (1.1) and depends on coordinates (X, Y), we can make the invariant point transformation

$$\tilde{x} = K(x, y), \quad \tilde{y} = J_3(x, y) \quad \text{for the first equation},$$
$$\tilde{x} = K(X, Y), \quad \tilde{y} = J_3(X, Y) \quad \text{for the second equation}.$$

Equations are equivalent if and only if in the term of coordinates $(\tilde{x}, \tilde{y})$ their notations become identical.

(6) If $J_3 \neq \text{const}$, $J_3 \neq J_6$, $J_6 \neq \text{const}$, $J_1 \neq \text{const}$, $K \neq \text{const}$ then in terms of canonical coordinates equation has form $y'' = e^y + t(x)y + s(x), t(x) \neq 0$.

Let us solve the equivalence problem for the equations of Type I.6. As at the previous case we have two possibilities.

The first way: in terms of canonical coordinates these equations have forms

$$y'' = e^y + t(x)y + s(x), \quad Y'' = e^Y + T(X)Y + S(X).$$

They are equivalent if and only if exist constants α and β such that

$$T(X) = \alpha^2 t(\alpha X + \beta), \quad S(X) = \alpha^2 s(\alpha X + \beta) - 2\alpha^2 \ln\alpha \cdot t(\alpha X + \beta).$$

The second way: note that for any Eq. (1.1) of Type I.6 invariants J_1 and J are functionally independent. So, we can make the invariant point transformation

$$\tilde{x} = J_1(x, y), \quad \tilde{y} = J(x, y) \quad \text{for the first equation},$$
$$\tilde{x} = J_1(X, Y), \quad \tilde{y} = J(X, Y) \quad \text{for the second equation}.$$

Equations are equivalent if and only if in the term of coordinates $(\tilde{x}, \tilde{y})$ their notations become identical.

□

6. Equations of Type II

Definition 6.1. Let us say that Eq. (1.1) has Type II if conditions (3.1) hold, where $I_1 = -9/10$.

According to Theorem 4.1 any Eq. (1.1) of Type II can be reduced by point transformations (1.2) into the canonical form:

$$y'' = -\ln y + t(x)y + s(x). \tag{6.1}$$

Lemma 6.1. *The most general point transformations that preserve the canonical form* (6.1) *are the following ones*:

$$x = \alpha^{-\frac{1}{3}}\tilde{x} + \beta, \quad y = \alpha^{-\frac{2}{3}}\tilde{y}. \tag{6.2}$$

Here α, β *are some constants. In the new coordinates this equation has the following form*: $\tilde{y}'' = -\ln\tilde{y} + \tilde{t}(\tilde{x})\tilde{y} + \tilde{s}(\tilde{x})$, *where*

$$\tilde{t}(\tilde{x}) = \alpha^{-\frac{2}{3}} t(\alpha^{-\frac{1}{3}}\tilde{x} + \beta), \quad \tilde{s}(\tilde{x}) = s(\alpha^{-\frac{1}{3}}\tilde{x} + \beta) + \frac{2}{3}\ln\alpha.$$

Then in terms of canonical coordinates the non-trivial invariants (2.2), (2.3) and (A.11):

$$I_3 = \frac{2}{5}\ln y - \frac{2}{5}(t(x)y + s(x)), \quad I_6 = \frac{3}{5} - \frac{3}{5}yt(x), \quad I_9 = -\frac{54}{625}y(t'(x)y + s'(x))^2.$$

Let us introduce the additional invariants

$$\begin{aligned} J_3 &= \frac{5I_3}{2} = \ln y - t(x)y - s(x), \quad J_6 = \frac{3-5I_6}{3} = t(x)y, \\ J_9 &= -\frac{625 I_9}{54} = y(t'(x)y + s'(x))^2, \quad J = \ln(J_6) - J_3 - J_6 = s(x) + \ln t(x). \end{aligned} \tag{6.3}$$

Theorem 6.1. *Let Eq.* (1.1) *be an arbitrary equation of Type* II. *Then it is equivalent to some equation from the following list of nonequivalent equations of Type* II:

(1) *If* $J_6 = 0$, $J_9 = 0$ *from* (6.3) *then it is equivalent to* $y'' = -\ln y$.
(2) *If* $J_6 \neq \text{const}$, $J_9 = 0$, $J = a$ *from* (6.3) *then it is equivalent to* $y'' = -\ln y + y + a$. *Two equations of Type* II.2 *are equivalent if and only if invariant* J *for both equations is the same and equals to constant* a.
(3) *If* $J_6 = 0$, $J_9 \neq \text{const}$ *from* (6.3), *then the equation is equivalent to* $y'' = -\ln y + s(x)$, $s(x) \neq \text{const}$.
Two equations of Type II.3 *are equivalent if and only if after the transformation* $\tilde{x} = J_3(x, y)$, $\tilde{y} = J_9(x, y)$ *their notations become identical.*
(4) *If* $J_6 \neq \text{const}$, $J_9 \neq \text{const}$ *from* (6.3) *then it is equivalent to* $y'' = -\ln y + t(x)y + s(x)$, $t(x) \neq 0$.
Two equations of Type II.4 *are equivalent if and only if after the transformation* $\tilde{x} = J(x, y)$, $\tilde{y} = J_6(x, y)$ *their notations become identical.*

In the cases II.2, II.4 *functions* $t(x)$, $s(x)$ *are definited up to transformations* (6.2).

Proof. Let us have the certain Eq. (1.1) of Type II. Then in terms of canonical coordinates it has form (6.1).

(1) If $J_6 = 0$ and $J_9 = 0$ then $t(x)y = 0$, $y(t'(x)y + s'(x))^2 = 0 \Leftrightarrow t(x) \equiv 0$, $s(x) = s = \text{const}$ and equation may be reduced into the form $y'' = -\ln y + s$. Let us make transformations (6.2). Then $\tilde{s} = s + 2/3\ln\alpha$. Choosing the appropriate α we can make $\tilde{s} = 0$ then equation will be $\tilde{y}'' = -\ln\tilde{y}$.
(2) If $J_6 \neq \text{const}$, $J_9 = 0$, $J = a$, then $y(t'(x)y + s'(x))^2 = 0 \Leftrightarrow t'(x) \equiv 0$, $s'(x) \equiv 0$. Accordingly $t(x) = t = \text{const}$, $s(x) = s = \text{const}$ and equation will be $y'' = -\ln y + ty + s, t \neq 0$. Let we have another equation of Type II.2 such that in terms of canonical coordinates

Table 3. Equations of Type II.

Type	J_6	J_9	J	Canonical form
II.1	0	0	—	$y'' = -\ln y$
II.2	$\neq$ const	0	$a = \text{const} \neq 0$	$y'' = -\ln y + y + a$
II.3	0	$\neq$ const,	—	$y'' = -\ln y + s(x)$, $s(x) \neq \text{const}$
II.4	$\neq$ const	$\neq$ const	$\neq$ const	$y'' = -\ln y + t(x)y + s(x)$, $t(x) \neq 0$

it has the form $Y'' = -\ln Y + TY + S$, $T = \text{const}$, $T \neq 0$, $S = \text{const}$. These equations equivalent if and only if

$$T = \alpha^{-\frac{2}{3}} t, \quad S = s + \frac{2}{3}\ln\alpha \Leftrightarrow S + \ln T = s + \ln t \Leftrightarrow J(x,y) = J(X;Y) = a.$$

So we see that two equations of Type II.2 are equivalent if and only if invariants J for both equations are equal to constant a.

Now let us find the canonical form for the equations of Type II.2. After the transformations (6.2) $\tilde{t} = \alpha^{-\frac{2}{3}} t$, $\tilde{s} = s + 2/3\ln\alpha$. Choosing the appropriate α we can get $\tilde{t} = 1$. Consequently the canonical form is $\tilde{y}'' = -\ln\tilde{y} + \tilde{y} + a$.

(3) If $J_6 = 0$, $J_9 \neq \text{const}$, then $t(x) \equiv 0$, $s'(x) \neq 0$. Therefore in terms of canonical coordinates equation has form $y'' = -\ln y + s(x)$, $s(x) \neq \text{const}$.

As at the previous case for the equations of Type I we have two possibilities. The first way: in terms of canonical coordinates two equations have forms

$$y'' = -\ln y + s(x), \quad Y'' = -\ln Y + S(X).$$

They equivalent if and only if exist constants α and β such that $S(X) = s(\alpha^{-\frac{1}{3}} X + \beta) + 2\ln\alpha/3$. The second way: note that for any Eq. (1.1) of the Type II.3 invariants J_3 and J_9 are functionally independent. So, we can make the invariant point transformation

$$\tilde{x} = J_3(x,y), \quad \tilde{y} = J_9(x,y) \quad \text{for the first equation,}$$
$$\tilde{x} = J_3(X,Y), \quad \tilde{y} = J_9(X,Y) \quad \text{for the second equation.}$$

These equations are equivalent if and only if in terms of new coordinates $(\tilde{x}, \tilde{y})$ their notations become identical.

(4) If $J_6 \neq \text{const}$, $J_9 \neq \text{const}$ then equation is equivalent to $y'' = -\ln y + t(x)y + s(x)$, $t(x) \neq 0$. The first way: in terms of canonical coordinates two equations have forms

$$y'' = e^y + t(x)y + s(x), \quad Y'' = e^Y + T(X)Y + S(X).$$

They equivalent if and only if exist the constants α and β such that

$$T(X) = \alpha^{-\frac{2}{3}} t(\alpha^{-\frac{1}{3}} X + \beta), \quad S(X) = s(\alpha^{-\frac{1}{3}} X + \beta) + \frac{2}{3}\ln\alpha.$$

The second way: note that for any Eq. (1.1) of the Type II.4 invariants J and J_6 are functionally independent. So, we can make the invariant point transformation

$$\tilde{x} = J(x,y), \quad \tilde{y} = J_6(x,y), \quad \tilde{x} = J(X,Y), \quad \tilde{y} = J_6(X,Y).$$

These equations are equivalent if and only if in terms of new coordinates $(\tilde{x}, \tilde{y})$ their notations become identical. □

7. Equations of Type III

Definition 7.1. Let us say that Eq. (1.1) has Type III if conditions (3.1) hold, where $I_1 = -12/5$.

According to Theorem 4.1 any Eq. (1.1) of the Type III can be reduced by point transformations (1.2) into the canonical form:

$$y'' = y(\ln y - 1) + t(x)y + s(x). \tag{7.1}$$

Lemma 7.1. *The most general point transformations that preserve the canonical form* (7.1) *are the following ones*:

$$x = \pm\tilde{x} + \beta, \quad y = \frac{\tilde{y}}{\sqrt{\alpha}}. \tag{7.2}$$

Here α, β *are certain constants. In term of new coordinates equation has form*: $\tilde{y}'' = \tilde{y}(\ln\tilde{y} - 1) + \tilde{t}(\tilde{x})\tilde{y} + \tilde{s}(\tilde{x})$, *where*

$$\tilde{t}(\tilde{x}) = t(\pm\tilde{x} + \beta) - \frac{\ln\alpha}{2}, \quad \tilde{s}(\tilde{x}) = \sqrt{\alpha}\cdot s(\pm\tilde{x} + \beta).$$

Then in term of canonical coordinates the non-trivial invariants (2.2), (2.3) and (A.11):

$$I_3 = \frac{4}{15}(1 - \ln y) - \frac{4}{15y}(t(x)y + s(x)), \quad I_6 = -\frac{4}{5} - \frac{4s(x)}{5y},$$
$$I_9 = -\frac{256}{1875y^2}(t'(x)y + s'(x))^2.$$

Let us denote the additional invariants

$$\begin{aligned} J_3 &= \frac{4 - 15I_3}{4} = \ln y + t(x) + \frac{s(x)}{y}, \quad J_6 = \frac{1 + 5I_6}{4} = \frac{s(x)}{y}, \\ J &= J_6 e^{J_3 - J_6} = s(x)e^{t(x)}, \quad J_9 = -\frac{1875 I_9}{256} = \frac{(t'(x)y + s'(x))^2}{y^2}. \end{aligned} \tag{7.3}$$

Theorem 7.1. *Let Eq.* (1.1) *be an arbitrary equation of Type* III. *Then it is equivalent to some equation from the following list of nonequivalent equations of Type* III:

(1) *If* $J_6 = 0$, $J_9 = 0$ *from* (7.3) *then it is equivalent to* $y'' = y(\ln y - 1)$.

(2) *If* $J_6 = 0$, $J_9 = b^2 = \text{const} \neq 0$ *from* (7.3) *then it is equivalent to* $y'' = y(\ln y - 1) \pm bxy$. *Two equations of Type* III.2 *are equivalent if and only if invariant* J_9 (7.3) *for both equations is the same and equals to the constant* b^2.

(3) *If* $J_6 \neq \text{const}$, $J_9 = 0$, $J = a$ *from* (7.3) *then it is equivalent to* $y'' = y(\ln y - 1) + a$. *Two equations of Type* III.3 *are equivalent if and only if invariant* J (7.3) *for both equations is the same and equals to constant* a.

(4) *If* $J_6 \neq \text{const}$, $J_9 = b^2 = \text{const} \neq 0$ *from* (7.3) *then it is equivalent to* $y'' = y(\ln y - 1) \pm bxy + 1$.
Two equations of Type III.4 *are equivalent if and only if invariant* J_9 (7.3) *for both equations is the same and equals to constant* b^2.

(5) *If* $J_6 \neq \text{const}$, $J_9 \neq \text{const}$ *from* (7.3) *then it is equivalent to* $y'' = y(\ln y - 1) + t(x)y + s(x)$, $s(x) \neq 0$.
Two equations of Type III.4 *are equivalent if and only if after the transformation* $\tilde{x} = J(x, y)$, $\tilde{y} = J_6(x, y)$ *their notations become identical.*

Table 4. Equations of Type III.

Type	J_6	J_9	J	Canonical form
III.1	0	0	0	$y'' = y(\ln y - 1)$
III.2	0	$b^2 = \text{const} \neq 0$	0	$y'' = y(\ln y - 1) \pm bxy$
III.3	$\neq$ const	0	$a = \text{const} \neq 0$	$y'' = y(\ln y - 1) + a$
III.4	$\neq$ const	$b^2 = \text{const} \neq 0$	$\neq$ const	$y'' = y(\ln y - 1) \pm bxy + 1$
III.5	$\neq$ const	$\neq$ const	$\neq$ const	$y'' = y(\ln y - 1) + t(x)y + s(x)$, $s(x) \neq 0$

In the case III.5 *functions* $t(x)$ *and* $s(x)$ *are definited up to transformations* (7.2).

Proof. Let us have the certain Eq. (1.1) of Type III. Then in terms of canonical coordinates it has the form (7.1).

(1) If $J_6 = 0$ and $J_9 = 0$, then $s(x) \equiv 0$, $t'(x) \equiv 0$. Therefore $t(x) = t = \text{const}$. Choosing the parameter α from the point transformation (7.2) we can get $\tilde{t} = t - \ln \alpha/2 = 0$. Thus the canonical form is $\tilde{y}'' = \tilde{y}(\ln \tilde{y} - 1)$.

(2) If $J_6 = 0$ and $J_9 = b^2 = \text{const}$ then $s(x) \equiv 0$, $t'(x) = \pm b = \text{const}$. So $t(x) = \pm bx + t$, $t = \text{const}$. Let us make the point transformation (7.2): $\tilde{t}(\tilde{x}) = \pm b(\pm\tilde{x} + \beta) + t - \ln \alpha/2 = \pm b\tilde{x} + (\pm b\beta + t - \ln \alpha/2)$. Choosing the parameters α and β we get $\tilde{t}(\tilde{x}) = \pm b\tilde{x}$.

It is easy to check that two equation of Type III.2 are equivalent if and only if invariants J_9 (7.3) for both equations are equal to the constant b^2.

(3) If $J_6 \neq \text{const}$, $J_9 = 0$, then $t'(x) \equiv$ and $s'(x) \equiv 0$, $s(x) \neq 0$. Hence $t(x) = t = \text{const}$, $s(x) = s = \text{const} \neq 0$. Let us make the point transformation (7.2): $\tilde{t} = t - \ln \alpha/2$, $\tilde{s} = \sqrt{\alpha} \cdot s$. Choosing the parameter α we can make $\tilde{t} = 0$. At the new coordinates $J = \tilde{s} = a$. Thus two equations of Type III.3 are equivalent if and only if invariants J for both equations are equal to the constant a.

(4) If $J_9 = \text{const} \neq 0$ then $s'(x) \equiv 0$ and $t'(x) = \pm b = \text{const} \neq 0$. So $t(x) = \pm bx + t$, $t = \text{const}$, $s(x) = s = \text{const} \neq 0$. Hence $\tilde{t}(\tilde{x}) = \pm b(\pm\tilde{x} + \beta) + t - \ln \alpha/2$, $\tilde{s} = \sqrt{\alpha} \cdot s$. Choosing the parameters α and β we can make $\tilde{t} = \pm b\tilde{x}$, $\tilde{s} = 1$. So in terms of new coordinates equation has form $\tilde{y}'' = \tilde{y}(\ln \tilde{y} - 1) \pm b\tilde{x}\tilde{y} + 1$. It is easy to check that two equation of Type III.4 are equivalent if and only if invariants J_9 (7.3) for both equations are equal to the constant b^2.

(5) If $J_3 \neq \text{const}$, $J_9 \neq \text{const}$ then equation has form $y'' = y(\ln y - 1) + t(x)y + s(x)$, $s(x) \neq 0$. The first way: two equations of Type III.5 are equivalent if and only if in terms of canonical coordinates condition $t(\pm X + \beta) + \ln s(\pm X + \beta) = T(X) + \ln S(X)$ holds, where the second equation in terms of canonical coordinates has form

$$Y'' = Y(\ln Y - 1) + T(X)Y + S(X), \quad S(X) \neq 0.$$

The second way: note that for any Eq. (1.1) of Type III.5 invariants J and J_6 are functionally independent. Hence we can make the invariant point transformation:

$$\tilde{x} = J(x, y), \quad \tilde{y} = J_6(x, y), \quad \tilde{x} = J(X, Y), \quad \tilde{y} = J_6(X, Y).$$

Equations are equivalent if and only if in terms of coordinates $(\tilde{x}, \tilde{y})$ their notations become identical. □

8. Equations of Type IV

Definition 8.1. Let us say that Eq. (1.1) has Type IV if conditions (3.1) hold, where $I_1 = 3(C+5)/5C$, $C = \text{const}$, $C \neq 0, -1, -2, -5$.

According to Theorem 4.1 any Eq. (1.1) of Type IV can be reduced by point transformations (1.2) into the canonical form:

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x). \tag{8.1}$$

Lemma 8.1. *The most general point transformations that preserve the canonical form* (8.1) *are the following ones*:

$$x = \alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta, \quad y = \alpha^{\frac{-2C}{C+5}}\tilde{y}. \tag{8.2}$$

Here α, β are some constants. In terms of new coordinates equation has the form: $\tilde{y}'' = \tilde{y}^{C+2}/((C+1)(C+2)) + \tilde{t}(\tilde{x})\tilde{y} + \tilde{s}(\tilde{x})$, *where*

$$\tilde{t}(\tilde{x}) = \alpha^{\frac{2(C+1)}{C+5}} \cdot t(\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta), \quad \tilde{s}(\tilde{x}) = \alpha^{\frac{2(C+2)}{C+5}} \cdot s(\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta).$$

The basic invariants (2.2), (2.3) and (A.11) are

$$I_3 = \frac{C(C+5)}{15(C+1)(C+2)} \cdot \frac{y^{C+2} + (t(x)y + s(x))(C+1)(C+2)}{y^{C+2}},$$

$$I_6 = \frac{(C+5)}{5} \cdot \frac{t(x)y(C+1) + s(x)(C+2)}{y^{C+2}},$$

$$I_9 = \frac{C(C+5)^4}{1875} \cdot \frac{(t'(x)y + s'(x))^2}{y^{3C+5}},$$

$$I_{21} = \frac{(\nabla_\gamma I_9)^2}{N^3} = \frac{4C(C+5)^{10}}{29296875} \cdot \frac{(t'(x)y + s'(x))^2(t''(x)y + s''(x))^2}{y^{7C+11}}.$$

The additional invariants

$$\begin{aligned}
J_3 &= \frac{15(C+1)(C+2)I_3 - C(C+5)}{C(C+1)(C+2)(C+5)} = \frac{t(x)y + s(x)}{y^{C+2}},\\
J_6 &= \frac{5I_6}{C+5} = \frac{t(x)y(C+1) + s(x)(C+2)}{y^{C+2}}, \quad J_9 = \frac{1875}{C(C+5)^4} = \frac{(t'(x)y + s'(x))^2}{y^{3C+5}},\\
J_{21} &= \frac{29296875 I_{21}}{4C(C+5)^{10}} = \frac{(t'(x)y + s'(x))^2(t''(x)y + s''(x))^2}{y^{7C+11}},\\
J_1 &= (C+2)J_3 - J_6 = \frac{t(x)}{y^{C+1}}, \quad J_2 = J_6 - (C+1)J_3 = \frac{s(x)}{y^{C+2}},\\
J &= \frac{J_2}{J_1^{\frac{C+2}{C+1}}} = \frac{s(x)}{t(x)^{\frac{C+2}{C+1}}}, \quad K = \frac{J_9}{J_1^3} = \frac{(t'(x)y + s'(x))^2}{y^2 t^3(x)},\\
K_1 &= \sqrt{\frac{J_{21}}{J_9}} = \frac{t''(x)y + s''(x)}{y^{2C+3}}, \quad K_2 = \frac{K_1}{J_1^2} = \frac{t''(x)y + s''(x)}{y t^2(x)}.
\end{aligned} \tag{8.3}$$

Theorem 8.1. *Let Eq.* (1.1) *be an arbitrary equation of Type* IV*. Then it is equivalent to some equation from the following list of nonequivalent equations of Type* IV:

(1) *If* $J_1 = 0,\ J_2 = 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)}$.
(2) *If* $J_1 = 0,\ J_2 \neq 0,\ J_9 = 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + 1$.
(3) *If* $J_1 = 0,\ J_2 \neq 0,\ J_9 \neq 0,\ K_1 = 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + x$.
(4) *If* $J_1 = 0,\ J_2 \neq 0,\ K_1 \neq 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + s(x)$, $s''(x) \neq 0$.
Two equations of Type IV.4 *with the same parameter* C *are equivalent if and only if after the transformation* $\tilde{x} = J_2(x, y)$, $\tilde{y} = J_9(x, y)$ *their notations become identical.*
(5) *If* $J_1 \neq 0,\ J_2 = 0,\ J_9 \neq 0,\ K_1 = 0$ *from* (8.3), *it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + xy$.
(6) *If* $J_1 \neq 0,\ J_2 = 0,\ K = k = \text{const} \neq 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + \frac{4y}{kx^2}$.
Two equations of Type IV.6 *with the same parameter* C *are equivalent if and only if invariant* K (8.3) *for both equations is the same and equal to the constant* $k \neq 0$.
(7) *If* $J_1 \neq 0,\ J_2 \neq 0,\ K = k = \text{const} \neq 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + \frac{4y}{kx^2} + 1$.
Two equations of Type IV.7 *with the same parameter* C *are equivalent if and only if invariant* K (8.3) *for both equations is the same and equal to the constant* $k \neq 0$.
(8) *If* $J_1 \neq 0,\ J_9 = 0,\ K = 0,\ J = a$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + y + a$.
Two equations of Type IV.8 *with the same parameter* C *are equivalent if and only if invariant* J (8.3) *for both equations is the same and equal to the constant* a.
(9) *If* $J_1 \neq 0,\ J_9 \neq 0,\ J = a = \text{const}$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + at^{(C+2)/(C+1)}(x)$.
Two equations of Type IV.9 *with the same parameter* C *and invariant* $J = a = \text{const}$ *are equivalent if and only if after the transformation* $\tilde{x} = J_1(x, y)$, $\tilde{y} = J_9(x, y)$ *their notations become identical.*
(10) *If* $J_1 \neq 0,\ J_2 \neq 0,\ J_9 \neq 0,\ K_1 = 0$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + xy + mx + n$.
Two equations of Type IV.10 *with the equal parameter* C *are equivalent if and only if in terms of canonical coordinates their constants* m *and* n *are the same.*
(11) *If* $J_1 \neq 0,\ J_2 \neq 0,\ J_9 \neq 0,\ K_1 \neq 0,\ K \neq \text{const},\ J \neq \text{const}$ *from* (8.3) *then it is equivalent to* $y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x)$.
Two equations of Type IV.11 *with the same parameter* C *are equivalent if and only if after the transformation* $\tilde{x} = J(x, y)$, $\tilde{y} = J_1(x, y)$ *their notations become identical.*

In the cases IV.4, IV.9 *and* IV.11 $t(x)$ *and* $s(x)$ *are definited up to transformations* (8.2).

Proof. Let us have the certain Eq. (1.1) of Type IV. Then in terms of canonical coordinates it has the form (8.1).

Table 5. Equations of Type IV.

Type	J_1	J_2	K	J	J_9	K_1	Canonical form
IV.1	0	0	—	—	0	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)}$
IV.2	0	$\neq 0$	—	—	0	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + 1$
IV.3	0	$\neq 0$	—	—	$\neq 0$	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + x$
IV.4	0	$\neq 0$	—	—	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + s(x)$
IV.5	$\neq 0$	0	$\neq$ const	0	$\neq 0$	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + xy$
IV.6	$\neq 0$	0	$k =$ const $\neq 0$	0	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + \frac{4y}{kx^2}$
IV.7	$\neq 0$	$\neq 0$	$k =$ const $\neq 0$	$\neq$ const	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + \frac{4y}{kx^2} + 1$
IV.8	$\neq 0$	$\forall$	0	$a =$ const	0	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + y + a$
IV.9	$\neq 0$	$\forall$	$\neq$ const	$a =$ const	$\neq 0$	$\forall$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + at(x)^{\frac{C+2}{C+1}}$
IV.10	$\neq 0$	$\neq 0$	$\neq$ const	$\neq$ const	$\neq 0$	0	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + xy + mx + n$
IV.11	$\neq 0$	$\neq 0$	$\neq$ const	$\neq$ const	$\neq 0$	$\neq 0$	$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x)$

(1) If $J_1 = 0$ and $J_2 = 0$ then $t(x) \equiv 0$ and $s(x) \equiv 0$ and equation has the form

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)}.$$

(2) If $J_1 = 0$, $J_2 \neq 0$, $J_9 = 0$, then $t(x) \equiv 0$ and $s'(x) \equiv 0$. So $s(x) = s = \text{const} \neq 0$. Let us make the transformation (8.2). Then $\tilde{s} = \alpha^{2(C+2)/(C+5)} \cdot s$. Choosing the appropriate α we can make $\tilde{s} = 1$, therefore:

$$\tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + 1.$$

(3) If $J_1 = 0$, $J_2 \neq 0$, $J_9 \neq 0$, $K_1 = 0$, then $t(x) \equiv 0$, $s''(x) \equiv 0$. So $s(x) = s_1 x + s_2$, $s_1 = \text{const} \neq 0$, $s_2 = \text{const}$. Let us make the transformation (8.2), then

$$\tilde{s}(\tilde{x}) = \alpha^{\frac{2(C+2)}{C+5}} \cdot (s_1(\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta) + s_2) = \alpha^{\frac{3C+5}{C+5}} s_1 \tilde{x} + \alpha^{\frac{2(C+2)}{C+5}} \cdot (s_1\beta + s_2).$$

Choosing the parameters α and β we can make $\tilde{s}(\tilde{x}) = \tilde{x}$, then equation will be

$$\tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + \tilde{x}.$$

(4) If $J_1 = 0$, $J_2 \neq 0$, $J_9 \neq 0$ then $t(x) \equiv 0$, $s''(x) \neq 0$. Hence in terms of special coordinates:

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + s(x).$$

Let we have two equations of Type IV.4 with the same parameter C. The first way: we reduce both equations in the canonical forms

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + s(x), \quad Y'' = \frac{Y^{C+2}}{(C+1)(C+2)} + S(X).$$

They are equivalent if and only if there exist the constants α and β such that

$$S(X) = \alpha^{\frac{2(C+2)}{C+5}} \cdot s(\alpha^{\frac{C+1}{C+5}} X + \beta).$$

The second way: note that for any Eq. (1.1) of the Type IV.4 invariants J_2 and J_9 are functionally independent. So, we can make the invariant point transformation

$$\tilde{x} = J_2(x, y), \quad \tilde{y} = J_9(x, y), \quad \tilde{x} = J_2(X, Y), \quad \tilde{y} = J_9(X, Y)$$

for both equations. Equations are equivalent if and only if in terms of new coordinates $(\tilde{x}, \tilde{y})$ their notations become identical.

(5) If $J_1 \neq \text{const}$, $J_2 = 0$, $J_9 \neq \text{const}$, $K_1 = 0$, then $s(x) \equiv 0$, $t''(x) \equiv 0$, so $t(x) = t_1 x + t_2$, $t_1 = \text{const} \neq 0$, $t_2 = \text{const}$. After the transformation (8.2):

$$\tilde{t}(\tilde{x}) = \alpha^{\frac{2(C+1)}{C+5}} \cdot (t_1(\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta) + t_2) = \alpha^{\frac{3(C+1)}{C+5}} t_1 \tilde{x} + \alpha^{\frac{2(C+1)}{C+5}} \cdot (t_1\beta + t_2).$$

Choosing parameters α and β we can make $\tilde{t}(\tilde{x}) = \tilde{x}$. Hence equation will be

$$\tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + \tilde{x}\tilde{y}.$$

(6) If $J_1 \neq \text{const}$, $J_2 = 0$, $J = 0$, $K = k = \text{const} \neq 0$, hence $s(x) \equiv 0$. In this case

$$\frac{t'^2(x)}{t^3(x)} = k, \quad t(x) = \frac{4}{(\sqrt{k} \cdot x + c_0)^2}, \quad c_0 = \text{const}.$$

After the transformation (8.2):

$$\tilde{t}(\tilde{x}) = \frac{4\alpha^{\frac{2(C+1)}{C+5}}}{(\sqrt{k} \cdot (\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta) + c_0)^2} = \frac{4\alpha^{\frac{2(C+1)}{C+5}}}{(\alpha^{\frac{C+1}{C+5}}\sqrt{k} \cdot \tilde{x} + \beta\sqrt{k} + c_0)^2}.$$

So we can take the appropriate parameters α and β so that $\tilde{s} = 1$ and $\beta\sqrt{k} + c_0 = 0$:

$$\tilde{t}(\tilde{x}) = \frac{4}{k\tilde{x}^2}, \quad \tilde{s} = 1, \quad \tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + \frac{4\tilde{y}}{k\tilde{x}^2}.$$

Two equations of Type IV.6 with the same parameter C are equivalent if and only if invariants K for both equations are the same and equal to constant $k \neq 0$.

(7) If $J_1 \neq \text{const}$, $J_2 \neq \text{const}$, $K = k = \text{const} \neq 0$, then $s'(x) \equiv 0$ and $s(x) = s = \text{const} \neq 0$:

$$\frac{t'^2(x)}{t^3(x)} = k, \quad t(x) = \frac{4}{(\sqrt{k} \cdot x + c_0)^2}, \quad c_0 = \text{const}.$$

After the transformation (8.2):

$$\tilde{t}(\tilde{x}) = \frac{4\alpha^{\frac{2(C+1)}{C+5}}}{(\sqrt{k} \cdot (\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta) + c_0)^2} = \frac{4\alpha^{\frac{2(C+1)}{C+5}}}{(\alpha^{\frac{C+1}{C+5}}\sqrt{k} \cdot \tilde{x} + \beta\sqrt{k} + c_0)^2}, \quad \tilde{s} = \alpha^{\frac{2(C+2)}{C+5}} \cdot s.$$

So we can take the appropriate parameters α and β so that $\tilde{s} = 1$ and $\beta\sqrt{k} + c_0 = 0$:

$$\tilde{t}(\tilde{x}) = \frac{4}{k\tilde{x}^2}, \quad \tilde{s} = 1, \quad \tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + \frac{4\tilde{y}}{k\tilde{x}^2} + 1.$$

Two equations of Type IV.7 with the same parameter C are equivalent if and only if invariants K for both equations are the same and equal to the constant $k \neq 0$.

(8) If $J_1 \neq \text{const}$, $J_9 = 0$, then $t'(x) \equiv 0$ and $s'(x) \equiv 0$, so $t(x) = \text{const} \neq 0$, $s(x) = \text{const} \neq 0$. As $J = a = \text{const}$, then $s = at^{\frac{C+2}{C+1}}$. Let us make the transformation (8.2), then

$$\tilde{t} = \alpha^{\frac{2(C+1)}{C+5}} \cdot t, \quad \tilde{s} = \alpha^{\frac{2(C+2)}{C+5}} \cdot s.$$

Choosing the appropriate α we can make $\tilde{t} = 1$, then $\tilde{s} = a$. In terms of new coordinates equation has the form

$$\tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + \tilde{y} + a.$$

Two equations of Type IV.8 with the same parameter C are equivalent if and only if invariants J for both equations are the same and equal to constant a.

(9) If $J_1 \neq 0$, $J_9 \neq 0$, $J = a = \text{const}$, then $s(x) = at(x)^{\frac{C+2}{C+1}}$, $t(x) \neq \text{const}$. Therefore in terms of special coordinates equation has the form

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + a \cdot t(x)^{\frac{C+2}{C+1}}.$$

Let we have two equations of Type IV.9 with the same parameter C and invariant $J = a$. To solve the equivalence problem we can

(1) reduce equations into the canonical forms

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + a \cdot t(x)^{\frac{C+2}{C+1}},$$

$$Y'' = \frac{Y^{C+2}}{(C+1)(C+2)} + T(X)Y + a \cdot T(X)^{\frac{C+2}{C+1}}.$$

They are equivalent if and only if there exist the constants α and β such that

$$T(X) = \alpha^{\frac{2(C+1)}{C+5}} \cdot t(\alpha^{\frac{C+1}{C+5}}X + \beta).$$

(2) make the invariant point transformation.

$$\tilde{x} = J_1(x,\, y), \quad \tilde{y} = J_9(x,\, y), \quad \tilde{x} = J_1(X,\, Y), \quad \tilde{y} = J_9(X,\, Y).$$

Note that for any Eq. (1.1) of Type IV.9 invariants J_1 and J_9 (8.3) are functionally independent. Equations are equivalent if and only if in terms of new coordinates $(\tilde{x}, \tilde{y})$ their notations become identical.

(10) If $J_1 \neq 0$, $J_2 \neq 0$, $J_9 \neq 0$, $K_1 = 0$, then $t''(x) \equiv 0$ and $s''(x) \equiv 0$. So $t(x) = t_1 x + t_2$, $t_1 = \text{const} \neq 0$, $t_2 = \text{const}$; $s(x) = s_1 x + s_2$, $s_1 = \text{const} \neq 0$, $s_2 = \text{const}$. Let us make the transformation (8.2), then

$$\tilde{t}(\tilde{x}) = \alpha^{\frac{2(C+1)}{C+5}} \cdot (t_1(\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta) + t_2) = \alpha^{\frac{3(C+1)}{C+5}} t_1 \tilde{x} + \alpha^{\frac{2(C+1)}{C+5}} \cdot (t_1\beta + t_2),$$

$$\tilde{s}(\tilde{x}) = \alpha^{\frac{2(C+2)}{C+5}} \cdot (s_1(\alpha^{\frac{C+1}{C+5}}\tilde{x} + \beta) + s_2) = \alpha^{\frac{3C+5}{C+5}} s_1 \tilde{x} + \alpha^{\frac{2(C+2)}{C+5}} \cdot (s_1\beta + s_2).$$

Choosing the parameters α and β we can make $\tilde{t}(\tilde{x}) = \tilde{x}$, but in this case $\tilde{s}(\tilde{x}) = m\tilde{x} + n$, $m = \text{const} \neq 0$, $n = \text{const}$. Thus equation has the form

$$\tilde{y}'' = \frac{\tilde{y}^{C+2}}{(C+1)(C+2)} + \tilde{x}\tilde{y} + m\tilde{x} + n.$$

Two equations of Type IV.9 with the equal parameter C are equivalent if and only if in terms of canonical coordinates their constants m and n are identical.

(11) If $J_1 \neq 0$, $J_2 \neq 0$, $J_9 \neq 0$, $K \neq \text{const}$, $J \neq \text{const}$, $K_1 \neq 0$, then in terms of special coordinates equation has the general form

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x).$$

Let we have two equations of Type IV.11 with the same parameter C. To solve the equivalence problem we could

(1) reduce both equations into the canonical forms

$$y'' = \frac{y^{C+2}}{(C+1)(C+2)} + t(x)y + s(x), \quad Y'' = \frac{Y^{C+2}}{(C+1)(C+2)} + T(X)Y + S(X).$$

They are equivalent if and only if there exist the constants α and β such that

$$T(X) = \alpha^{\frac{2(C+1)}{C+5}} \cdot t(\alpha^{\frac{C+1}{C+5}} X + \beta), \quad S(X) = \alpha^{\frac{2(C+2)}{C+5}} \cdot s(\alpha^{\frac{C+1}{C+5}} X + \beta).$$

(2) make the invariant point transformation.

$$\tilde{x} = J(x,\, y), \quad \tilde{y} = J_1(x,\, y), \quad \tilde{x} = J(X,\, Y), \quad \tilde{y} = J_1(X,\, Y).$$

Note that for any Eq. (1.1) of Type IV.11 invariants J and J_1 (8.3) are functionally independent. Equations are equivalent if and only if in terms of new coordinates $(\tilde{x}, \tilde{y})$ their notations become identical. □

8.1. *Examples*

8.1.1. *Equation Painleve* III *with* 3 *zero parameters*

The Painleve III equation depends on four parameters (a, b, c, d)

$$PIII(a,b,c,d): \quad y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{(ay^2+b)}{x} + cy^3 + \frac{d}{y}.$$

By the paper [12] all equations Painleve III with 3 zero parameters are equivalent

$$PIII(0,b,0,0) \overset{(1)}{\to} PIII(-b,0,0,0) \overset{(2)}{\to} PIII(0,0,-b,0) \overset{(3)}{\to} PIII(0,0,0,b),$$

where the point transformations (1) and (3) are: $x = \tilde{x}$, $y = 1/\tilde{y}$ and (2) is: $x = \tilde{x}^2/2$, $y = \tilde{y}^2$.

In this way suppose that $a \neq 0$, $b = c = d = 0$. For the equation $PIII(a, 0, 0, 0)$ conditions (3.1) hold and invariants (2.2) are equal to:

$$I_1 = \frac{3}{5}, \quad I_2 = 0, \quad I_3 = \frac{1}{15}.$$

As $J_3 = 0$, then according to Theorem 5.1, equation $PIII(a, 0, 0, 0)$ has Type I.1. Let us find the corresponding change of variables. The first point transformation (see [1]) takes equation $PIII(a, 0, 0, 0)$ into the following form: $x = e^t$, $y = e^z$, $z'' = ae^{t+z}$.

The second transformation reduces it in the canonical form: $t = \tilde{x}/\sqrt{a}$, $z = \tilde{y} - \tilde{x}/\sqrt{a}$, $\tilde{y}'' = e^{\tilde{y}}$.

8.1.2. *Equation from the handbook Kamke No. 6.172*

Equation from the handbook Kamke [15] No. 172

$$y'' = \frac{y'^2}{y} - \frac{ay'}{x} - by^2, \quad b \neq 0$$

has Type I and invariants (2.2), (2.3):

$$I_1 = \frac{3}{5}, \quad I_2 = 0, \quad I_3 = \frac{1}{15} + \frac{2a}{15(a-1)x^2e^y},$$

$$I_6 = \frac{2a}{5(a-1)x^2e^y}, \quad I_9 = \frac{16a^2}{1875(a-1)^2x^6e^{3y}}.$$

Let us calculate the additional invariants (5.3):

$$J_3 = J_6 = -\frac{2a(a-1)}{ybx^2}, \quad K = \frac{2(a-1)}{a}.$$

According to Theorem 5.1 this equation has Type I.4. with $k = 2(a-1)/a$ if $a \neq 0$ and $a \neq 1$. In the special cases $a = 0$ or $a = 1$ equation has Type I.1.

Let us find corresponding change of variables. The first transformation (as $b \neq 0$):

$$x = \frac{t}{\sqrt{-b}}, \quad y = z, \quad \tilde{z}'' = \frac{z'^2}{z} - \frac{az'}{t} + z^2.$$

Then, as $a \neq 1$, the second transformation

$$t = (1-a)^{\frac{1}{1-a}} X^{\frac{1}{1-a}}, \quad z = e^Y, \quad Y'' = (1-a)^{\frac{2a}{1-a}} X^{\frac{2a}{1-a}} e^Y.$$

In the end, as $a \neq 0$, the third transformation:

$$X = \tilde{x}, \quad Y = \tilde{y} + \frac{2a}{a-1}\ln((1-a)\tilde{x}), \quad \tilde{y}'' = \frac{2a}{(a-1)\tilde{x}^2} + e^{\tilde{y}}.$$

If $a = 1$, then the second and the third transformations:

$$t = e^X, \quad z = e^Y, \quad Y'' = be^{Y+2X}; \quad X = \tilde{x}, \quad Y = \tilde{y} - 2\tilde{x}, \quad \tilde{y}'' = e^{\tilde{y}}.$$

8.1.3. *Equation Painleve* II

The Painleve II equation depends on one parameter a

$$PII(a)\colon \quad y'' = 2y^3 + xy + a.$$

It also has form (3.2) and satisfies to the conditions (3.1) with invariant $I_1 = 18/5$. According to Theorem 8.1 it has Type IV, case $C = 1$. Let us calculate the additional invariants

$$J_1 = \frac{x}{12y^2}, \quad J_2 = \frac{a}{12y^3}, \quad J = \frac{2a\sqrt{3}}{x\sqrt{x}}, \quad K = \frac{1}{x^3}, \quad J_9 = \frac{1}{1728y^6}, \quad K_1 = 0.$$

We see that the equation PII(a) has Type IV.10 if $a \neq 0$ and Type IV.5 if $a = 0$.

Let us calculate x, y and a via invariants: $x = 1/\sqrt[3]{K}$, $y = 1/(2\sqrt{3}\sqrt[6]{J_9})$, $a = J_2/2\sqrt{3}\sqrt{J_9}$.

Theorem 8.2. *Equation* (1.1) *of Type* IV *is equivalent to Painleve* II *equation with parameter* $\pm a$ *if and only if*

$$C = 1, \quad \frac{J_2}{2\sqrt{3}\sqrt{J_9}} = a = \text{const}, \quad K_1 = 0. \tag{8.4}$$

The explicit point transformation is $\tilde{x} = 1/\sqrt[3]{K(x,y)}$, $\tilde{y} = 1/(2\sqrt{3}\sqrt[6]{J_9(x,y)})$.

Proof. At first we write the PII(a) equation in the canonical form:

$$x = X, \quad y = \frac{Y}{2\sqrt{3}}, \quad Y'' = \frac{Y^3}{6} + XY + 2\sqrt{3}a.$$

Let $a = 0$, then equation PII(0) is already of Type IV.5.

Let $a \neq 0$. If for the certain equation of Type IV.10 conditions (8.4) hold, then in terms of canonical coordinates

$$\frac{s(x)y^{\frac{C+1}{2}}}{t'(x)y + s'(x)} = 2\sqrt{3}a.$$

It is true if and only if $C = 1$ and $s'(x) \equiv 0$. So $s(x) = s = \text{const} \neq 0$. Then $t(x) = sx/(2\sqrt{3}a) + t$, $t = \text{const}$, therefore $K_1 = 0$.

Let us make the point transformation (8.2):

$$\tilde{t}(\tilde{x}) = \alpha^{\frac{2}{3}} \left(\frac{s(\alpha^{\frac{1}{3}}\tilde{x} + \beta)}{2\sqrt{3}a} + t \right), \quad \tilde{s} = \alpha \cdot s.$$

If we take $\alpha = 2\sqrt{3}a/s$, $\beta = -ts^{\frac{2}{3}}(2\sqrt{3}a)^{\frac{1}{3}}$, then $\tilde{t}(\tilde{x}) = \tilde{x}$, $\tilde{s} = 2\sqrt{3}a$ and we get the Painleve II equation that is written in terms of the canonical coordinates $(\tilde{x}, \tilde{y})$. □

8.1.4. *Equation from the handbook Kamke No. 6.24*

Equation from the handbook Kamke [15] No. 6.24

$$y'' = -3ay' + 2y^3 - 2a^2 y$$

has Type IV and invariants $I_1 = 18/5$, $I_3 = 1/15$, $J_1 = 0$, $J_2 = 0$. According to Theorem 8.1 this equation has Type IV.1 with $C = 1$. Let us make the transformations:

$$x = -\frac{\ln(-3aX)}{3a}, \quad y = Y, \quad Y'' = -\frac{2Y(a^2 - Y^2)}{9a^2X^2};$$

$$X = \frac{a^2\tilde{x}^3}{36}, \quad Y = \frac{a\tilde{x}\tilde{y}}{2\sqrt{3}}, \quad \tilde{y}'' = \frac{\tilde{y}^3}{6}.$$

8.1.5. *Equation from the handbook Kamke No. 6.140*

Equation from the handbook Kamke [15] No. 6.140

$$y'' = \frac{y'^2}{2y} + 4y^2$$

has Type IV and invariants $I_1 = 18/5$, $I_3 = 1/15$, $J_1 = 0$, $J_2 = 0$. Thus it has Type IV.1 with $C = 1$. By the transformations $x = \tilde{x}/\sqrt{3}$, $y = \tilde{y}^2/4$ it is reduced into the canonical form: $\tilde{y}'' = \tilde{y}^3/6$.

So equations from the handbook Kamke Nos. 6.24 and 6.140 are equivalent.

8.1.6. *Equation from the handbook Kamke No. 6.141*

Equation from the handbook Kamke [15] No. 6.141

$$y'' = \frac{y'^2}{2y} + 4y^2 + 2y$$

has Type IV and invariants $I_1 = 18/5$, $J_1 = 1/(12y)$, $J_2 = J = K = J_9 = 0$. Hence it has Type IV.8 with $a = 0$. Thus it can be reduced into the canonical form:

$$x = \tilde{x}, \quad y = \frac{\tilde{y}^2}{12}, \quad \tilde{y}'' = \frac{\tilde{y}^3}{6} + \tilde{y}.$$

8.1.7. *Emden equation*

The Emden equation in the form described in paper [2]:

$$y'' = \lambda(x)y^n, \quad n \neq -3,\, 0,\, 1,\, 2, \quad \lambda(x) \neq 0 \tag{8.5}$$

has $I_1 = 3(n+3)/(5(n-2))$, $I_2 = 0$, $J_2 = 0$ hence this equation has Type IV with parameter $C = n-2$. Possible cases IV.1, IV.5, IV.6, IV.8 and IV.9.

Invariants (8.3) are as follows:

$$J_1 = \frac{\lambda\lambda''(n+3) - \lambda'^2(n+4)}{(n+3)^2(n-1)n\lambda^3 y^{n-1}},$$

$$J_9 = \frac{((n+3)^2\lambda'''\lambda^2 - 3(n+3)(n+5)\lambda'\lambda'' + 2(n+4)(n+5)\lambda'^3)^2}{(n+3)^6(n-1)^3n^3\lambda(x)^9y^{3(n-1)}},$$

$$K = \frac{J_9}{J_1^3} = \frac{((n+3)^2\lambda'''\lambda^2 - 3(n+3)(n+5)\lambda'\lambda'' + 2(n+4)(n+5)\lambda'^3)^2}{(\lambda(x)\lambda''(x)(n+3) - \lambda(x)'^2(n+4))^3},$$

$$K_1 = \frac{K_1^*}{(n+3)^4(n-1)^2n^2\lambda(x)^6y^{2(n-1)}},$$

$$\begin{aligned} K_1^* = {} & (n+3)^3\lambda^3\lambda'''' - 4(n+3)^2(n+6)\lambda^2\lambda'\lambda''' - 3(n+3)^2(n+5)\lambda^2\lambda''^2 \\ & + 12(n+3)(n+5)^2\lambda\lambda'^2\lambda'' - 6(n+4)(n+5)^2\lambda'^4 = 0. \end{aligned}$$

Type IV.1. The Emden Eq. (8.5) has Type IV.1 if and only if invariant $J_1 = 0$.

In this case function $\lambda(x)$ is the solution of following differential equation:

$$\lambda\lambda''(n+3) = \lambda'^2(n+4). \tag{8.6}$$

Therefore $\lambda(x) = c_2/(c_1x+1)^{n+3}$, c_1, $c_2 = \text{const}$, $c_2 \neq 0$. In this case the Emden equation could be transformed into the canonical form

$$\tilde{y}'' = \frac{\tilde{y}^n}{(n-1)n}, \quad x = -\frac{\sqrt[n+1]{n(n-1)c_2}}{\tilde{x}c_1\sqrt[n+1]{c_1^2}} - \frac{1}{c_1}, \quad y = -\frac{\tilde{y}}{\tilde{x}}, \quad c_1 \neq 0,$$

$$x = \frac{\tilde{x}}{\sqrt{c_2n(n-1)}}, \quad y = \tilde{y}, \quad c_1 = 0.$$

Type IV.8. The Emden Eq. (8.5) has Type IV.8 if and only if invariants $J_1 \neq 0$, $J_9 = 0$. Then $\lambda(x)$ satisfies to the following differential equation

$$(n+3)^2\lambda'''\lambda^2 - 3(n+3)(n+5)\lambda'\lambda'' + 2(n+4)(n+5)\lambda'^3 = 0 \tag{8.7}$$

and does not satisfies to (8.6). Then Eq. (8.5) could be transformed into

$$\tilde{y}'' = \frac{\tilde{y}^n}{(n-1)n} + \tilde{y}.$$

Type IV.5. The Emden equation (8.5) has Type IV.5 if and only if invariants $J_1 \neq 0$, $J_9 \neq 0$, $K \neq \text{const}$, $K_1 = 0$. Then $\lambda(x)$ satisfies to the following differential equation

$$\begin{aligned}(n+3)^3\lambda^3\lambda'''' - 4(n+3)^2(n+6)\lambda^2\lambda'\lambda''' - 3(n+3)^2(n+5)\lambda^2\lambda''^2 \\ + 12(n+3)(n+5)^2\lambda\lambda'^2\lambda'' - 6(n+4)(n+5)^2\lambda'^4 = 0\end{aligned} \tag{8.8}$$

and does not satisfies to (8.6), (8.7), (8.9). Equation (8.5) could be transformed into

$$\tilde{y}'' = \frac{\tilde{y}^n}{(n-1)n} + \tilde{x}\tilde{y}.$$

In particular, this equations is equivalent to the Painleve II with the parameter $a = 0$ if and only if $n = 3$ and the function $\lambda(x)$ satisfies to the following equation

$$9\lambda^3\lambda'''' - 54\lambda^2\lambda'\lambda''' - 36\lambda^2\lambda''^2 + 192\lambda\lambda'^2\lambda'' - 112\lambda'^4 = 0.$$

Type IV.6. The Emden equation (8.5) has Type IV.6 if and only if invariants $J_1 \neq 0$, $J_9 \neq 0$, $K = k = \text{const} \neq 0$, $K_1 \neq 0$. Then $\lambda(x)$ satisfies to the following equation

$$\begin{aligned}\lambda'''^2\lambda^4(n+3)^4 - 2\lambda'''\lambda'\lambda^2(n+5)(n+3)^2(3\lambda''\lambda(n+3) - 2\lambda'^2(n+4)) \\ - \lambda''^3\lambda^3k(n+3)^3 + 3\lambda''^2\lambda'^2\lambda^2(n+3)^2(3n^2+30n+kn+75+4k) \\ - 3\lambda''\lambda'^4\lambda(n+4)(n+3)(4n^2+40n+kn+100+4k) \\ + \lambda'^6(n+4)^2(4n^2+40n+kn+100+4k) = 0\end{aligned} \tag{8.9}$$

and does not satisfies to (8.6)–(8.8). Equation (8.5) could be transformed into

$$\tilde{y}'' = \frac{\tilde{y}^n}{(n-1)n} + \frac{4\tilde{y}}{k\tilde{x}^2}.$$

Type IV.9. At the other cases the Emden equation (8.5) has Type IV.9. Then exists the special coordinate system, so that in terms of these coordinates equation has form

$$\tilde{y}'' = \frac{\tilde{y}^n}{(n-1)n} + \tilde{t}(\tilde{x})\tilde{y}.$$

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Appendix A. Explicit Formulas for the Components of the Pseudovectorial Fields, Pseudoinvariants and Invariants

Here and everywhere the notation $K_{i.j}$ denotes partial differentiation: $K_{i.j} = \partial^{i+j}K/\partial x^i \partial y^j$.

The coordinates of the pseudovectorial field $\boldsymbol{\alpha}$ are $\alpha^1 = B$, $\alpha^2 = -A$, where

$$\begin{aligned} A &= P_{0.2} - 2Q_{1.1} + R_{2.0} + 2PS_{1.0} + SP_{1.0} - 3PR_{0.1} - 3RP_{0.1} - 3QR_{1.0} + 6QQ_{0.1}, \\ B &= S_{2.0} - 2R_{1.1} + Q_{0.2} - 2SP_{0.1} - PS_{0.1} + 3SQ_{1.0} + 3QS_{1.0} + 3RQ_{0.1} - 6RR_{1.0}. \end{aligned} \tag{A.1}$$

The pseudoinvariant F of weight 5 is:

$$3F^5 = AG + BH, \quad \text{where} \tag{A.2}$$

$$\begin{aligned} G &= -BB_{1.0} - 3AB_{0.1} + 4BA_{0.1} + 3SA^2 - 6RBA + 3QB^2, \\ H &= -AA_{0.1} - 3BA_{1.0} + 4AB_{1.0} - 3PB^2 + 6QAB - 3RA^2. \end{aligned}$$

The pseudoinvariant N in the cases $A \neq 0$ and $B \neq 0$, respectively, is:

$$N = -\frac{H}{3A}, \quad N = \frac{G}{3B}. \tag{A.3}$$

The pseudoinvariant M in the case $A \neq 0$:

$$M = -\frac{12BN(BP + A_{1.0})}{5A} + BN_{1.0} + \frac{24}{5}BNQ + \frac{6}{5}NB_{1.0} + \frac{6}{5}NA_{0.1} - AN_{0.1} - \frac{12}{5}ANR. \tag{A.4}$$

And in the case $B \neq 0$ is:

$$M = -\frac{12AN(AS - B_{0.1})}{5B} - AN_{0.1} + \frac{24}{5}ANR - \frac{6}{5}NA_{0.1} - \frac{6}{5}NB_{1.0} + BN_{1.0} - \frac{12}{5}BNQ. \tag{A.5}$$

The pseudoinvariant Ω in the case $A \neq 0$:

$$\begin{aligned} \Omega = {} & \frac{2BA_{1.0}(BP + A_{1.0})}{A^3} - \frac{(2B_{1.0} + 3BQ)A_{1.0}}{A^2} + \frac{(A_{0.1} - 2B_{1.0})BP}{A^2} \\ & - \frac{BA_{2.0} + B^2P_{1.0}}{A^2} + \frac{B_{2.0}}{A} + \frac{3B_{1.0}Q + 3BQ_{1.0} - B_{0.1}P - BP_{0.1}}{A} + Q_{0.1} - 2R_{1.0}. \end{aligned} \tag{A.6}$$

The pseudoinvariant Ω in the case $B \neq 0$:

$$\begin{aligned} \Omega = {} & \frac{2AB_{0.1}(AS - B_{0.1})}{B^3} - \frac{(2A_{0.1} - 3AR)B_{0.1}}{B^2} + \frac{(B_{1.0} - 2A_{0.1})AS}{B^2} \\ & + \frac{AB_{0.2} - A^2S_{0.1}}{B^2} - \frac{A_{0.2}}{B} + \frac{3A_{0.1}R + 3AR_{0.1} - A_{1.0}S - AS_{1.0}}{B} + R_{1.0} - 2Q_{0.1}. \end{aligned} \tag{A.7}$$

In the case $A \neq 0$ the field $\boldsymbol{\gamma}$ is:

$$\begin{aligned}\gamma^1 &= -\frac{6BN(BP + A_{1.0})}{5A^2} + \frac{18NBQ}{5A} + \frac{6N(B_{1.0} + A_{0.1})}{5A} - N_{0.1} - \frac{12}{5}NR - 2\Omega B,\\ \gamma^2 &= -\frac{6N(BP + A_{1.0})}{5A} + N_{1.0} + \frac{6}{5}NQ + 2\Omega A.\end{aligned} \tag{A.8}$$

In the case $B \neq 0$ the field $\boldsymbol{\gamma}$ is:

$$\begin{aligned}\gamma^1 &= -\frac{6N(AN - B_{0.1})}{5B} - N_{0.1} + \frac{6}{5}NR - 2\Omega B,\\ \gamma^2 &= -\frac{6AN(AS - B_{0.1})}{5B^2} + \frac{18NAR}{5B} - \frac{6N(A_{0.1} + B_{1.0})}{5B} + N_{1.0} - \frac{12}{5}NQ + 2\Omega A.\end{aligned} \tag{A.9}$$

The pseudoinvariant Γ is:

$$\begin{aligned}\Gamma = {} & \frac{\gamma^1\gamma^2(\gamma^1_{1.0} - \gamma^2_{0.1})}{M} + \frac{(\gamma^2)^2\gamma^1_{0.1} - (\gamma^1)^2\gamma^2_{1.0}}{M}\\ & + \frac{P(\gamma^1)^3 + 3Q(\gamma^1)^2\gamma^2 + 3R\gamma^1(\gamma^2)^2 + S(\gamma^2)^3}{M}.\end{aligned} \tag{A.10}$$

Explicit formulas for the invariants I_6, I_9, I_{21}:

$$I_6 = \frac{(I_3)'_x B - (I_3)'_y A}{N}, \quad I_9 = \frac{((I_3)'_x\gamma^1 + (I_3)'_y\gamma^2)^2}{N^3}, \quad I_{21} = \frac{((I_9)'_x\gamma^1 + (I_9)'_y\gamma^2)^2}{N^3}. \tag{A.11}$$

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