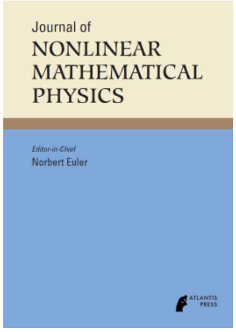

Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852 **ISSN (Print): 1402-9251**
Journal Home Page: https://www.atlantis-press.com/journals/jnmp

New Integrable Multicomponent Nonlinear Partial Differential-Difference Equations

R. Sahadevan, S. Balakrishnan

Article

World Scientific
www.worldscientific.com

ATLANTIS PRESS

Journal of Nonlinear Mathematical Physics, Vol. 18, No. 4 (2011) 501–518

DOI: 10.1142/S140292511100174X

NEW INTEGRABLE MULTICOMPONENT NONLINEAR PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS

R. SAHADEVAN* and S. BALAKRISHNAN†

Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chepauk, Chennai 600 005
Tamilnadu, India
**ramajayamsaha@yahoo.co.in*
†s_krishnan1982@yahoo.co.in

Received 6 January 2011
Accepted 18 May 2011

We report a new three and four coupled nonlinear partial differential-difference equations each admits Lax representation, possess infinitely many generalized (nonpoint) symmetries, conserved quantities and a recursion operator. Hence they are completely integrable both in the sense of Lax and Liouville.

Keywords: Integrable equations; nonlinear partial differential-difference equations; soliton equations.

1. Introduction

In recent years, searching for new integrable discrete systems governed by nonlinear partial differential-difference equations (PDΔEs) is an important and interesting task in nonlinear systems[3, 4, 14, 16, 19, 22, 23, 28, 30, 31]. A variety of analytical techniques have been devised toward this goal both for nonlinear partial differential equations (PDEs) and PDΔEs [1, 8, 15, 18, 20, 25, 27, 29]. As a result, considerable number of completely integrable nonlinear scalar PDΔEs with polynomial forms with (1+1) dimensions have been identified. More often these integrable equations exhibit rich mathematical structures such as Lax representation[1, 5, 9, 16, 22, 23, 31], an infinitely many generalized symmetries, conserved densities [10–13, 15, 25] and master symmetries[8, 11, 20], etc.[7, 17, 21, 32] which are common properties of completely integrable systems. However only a limited number of integrable coupled nonlinear PDΔEs with (1+1) dimensions exist in the literature. Also, if one introduces more components to a known scalar nonlinear PDΔEs possessing mathematical structures related with integrability, the resulting equation may not preserve all the characteristics of original equation and hence it is important to investigate further towards their integrability. Thus it is interesting to identify integrable coupled nonlinear PDΔEs. With this aim, in this article we report a new integrable three and four coupled nonlinear partial differential-difference equations (PDΔEs).

More specifically we consider a 3- and 4-coupled PDΔEs, respectively given by

$$\frac{\partial u_n}{\partial t} = \frac{1}{v_n} - \frac{p_n}{u_{n+1}\, v_n}, \tag{1.1a}$$

$$\frac{\partial v_n}{\partial t} = \frac{v_n p_{n-1}}{u_n u_{n-1} v_{n-1}} - \frac{1}{u_n}, \tag{1.1b}$$

$$\frac{\partial p_n}{\partial t} = \frac{p_n}{u_n v_n} - \frac{p_n^2}{u_n v_n u_{n+1}} \tag{1.1c}$$

and

$$\frac{\partial u_n}{\partial t} = \frac{1}{v_n} - \frac{q_n}{u_{n+1} v_n} - \frac{u_n p_n}{u_{n+1} v_n}, \tag{1.2a}$$

$$\frac{\partial v_n}{\partial t} = \frac{v_n q_{n-1}}{u_n u_{n-1} v_{n-1}} - \frac{1}{u_n} + \frac{p_n}{u_{n+1}}, \tag{1.2b}$$

$$\frac{\partial p_n}{\partial t} = \frac{p_n q_{n-1}}{v_{n-1} u_{n-1} u_n} - \frac{p_n q_n}{v_n u_n u_{n+1}}, \tag{1.2c}$$

$$\frac{\partial q_n}{\partial t} = \frac{q_n}{u_n v_n} - \frac{q_n^2}{u_n v_n u_{n+1}} - \frac{p_n q_{n-1}}{u_{n-1} v_{n-1}}, \tag{1.2d}$$

where $u_n = u(n,t), v_n = v(n,t), p_n = p(n,t), q_n = q(n,t), u_{n-1} = u(n-1,t), u_{n+1} = u(n+1,t)$ and show that both of them are Hamiltonian ones and admit Lax representation with (2×2) Lax matrices and possess infinitely many generalized (nonpoint) symmetries, conserved quantities and recursion operator. Hence (1.1) and (1.2) are completely integrable in the sense of Lax and Liouville.

Different research groups have been engaged in this direction and reported a limited number of integrable coupled equations with both polynomial and rational terms with (1+1) dimensions [26, 30, 31]. It is appropriate to mention that (1.1) and (1.2) reduce into the following integrable 2-coupled nonlinear PDΔE [26]

$$\frac{\partial u_n}{\partial t} = \frac{1}{v_n} - \frac{u_n}{u_{n+1} v_n}, \tag{1.3a}$$

$$\frac{\partial v_n}{\partial t} = \frac{v_n}{u_n v_{n-1}} - \frac{1}{u_n} \tag{1.3b}$$

when $p_n = u_n$ in (1.1) and $p_n = 0$ and $q_n = u_n$ in (1.2). We would like to mention that Blaszak and Marciniak [6] have reported three and four component nonlinear integrable PDΔEs. Furthermore the authors of reference [33, 34] have shown that the Blaszak and Marciniak system arises from the Lax representation with (3×3) and (4×4) Lax matrices. Also (1.1) and (1.2) cannot be deduced from the Blaszak and Marciniak systems [6] and hence to the best of our knowledge (1.1) and (1.2) are new multicomponent integrable equations.

The plan of the paper is as follows: In Sec. 2, we show that (1.1) and (1.2) admit Lax representation indicating that they are integrable in the sense of Lax. In Sec. 3, we establish that both (1.1) and (1.2) are Hamiltonian ones. In Sec. 4, we show explicitly that both 3-coupled and 4-coupled nonlinear PDΔEs possess an infinitely many generalized (nonpoint) symmetries and conserved quantities and hence they are integrable in the sense of Liouville. In Sec. 5, we derive the recursion operator for coupled system (1.1) and (1.2) separately. In Sec. 6, we give summary of our results.

2. Lax Representation of Differential-Difference Equations

A nonlinear (autonomous) PDΔE with two independent variables (one continuous + one discrete) is an equation of the form

$$\frac{\partial \mathbf{u}_n}{\partial t} = \mathbf{F}(\ldots, \mathbf{u}_{n-1}, \mathbf{u}_n, \mathbf{u}_{n+1}, \ldots), \tag{2.1}$$

where $\mathbf{u}_n$ and $\mathbf{F}$ are vector valued functions. Consider a linear system

$$\mathbf{\Phi}_{n+1}(t,\lambda) = \mathbf{L}_n(t,\lambda)\mathbf{\Phi}_n(t,\lambda), \quad \frac{d}{dt}\mathbf{\Phi}_n(t,\lambda) = \mathbf{M}_n(t,\lambda)\mathbf{\Phi}_n(t,\lambda), \tag{2.2}$$

where $\mathbf{L}_n(t,\lambda)$ and $\mathbf{M}_n(t,\lambda)$ are nonsingular square matrices. When the Lax matrices $\mathbf{L}_n(= \mathbf{L}_n(t,\lambda))$ and $\mathbf{M}_n(= \mathbf{M}_n(t,\lambda))$ are (2×2) square matrices then (2.2) reads

$$\begin{bmatrix} \phi_{1n+1}(t,\lambda) \\ \phi_{2n+1}(t,\lambda) \end{bmatrix} = \begin{bmatrix} L_{11n}(t,\lambda) & L_{12n}(t,\lambda) \\ L_{21n}(t,\lambda) & L_{22n}(t,\lambda) \end{bmatrix} \begin{bmatrix} \phi_{1n}(t,\lambda) \\ \phi_{2n}(t,\lambda) \end{bmatrix}, \tag{2.3a}$$

$$\begin{bmatrix} \frac{d}{dt}\phi_{1n}(t,\lambda) \\ \frac{d}{dt}\phi_{2n}(t,\lambda) \end{bmatrix} = \begin{bmatrix} A_n(t,\lambda) & B_n(t,\lambda) \\ C_n(t,\lambda) & D_n(t,\lambda) \end{bmatrix} \begin{bmatrix} \phi_{1n}(t,\lambda) \\ \phi_{2n}(t,\lambda) \end{bmatrix}, \tag{2.3b}$$

where λ is the spectral parameter and $L_{ijn}(t,\lambda), A_n(t,\lambda), B_n(t,\lambda), C_n(t,\lambda)$ and $D_n(t,\lambda)$ are functions of $\mathbf{u}_n$ and their shifts. The compatibility of the linear system, (2.2) gives

$$\frac{d}{dt}\mathbf{L_n} + \mathbf{L}_n\mathbf{M}_n - \mathbf{M}_{n+1}\mathbf{L}_n = \mathbf{0} \tag{2.4}$$

and the compatibility of (2.3) yields

$$\frac{d}{dt}L_{11n} + L_{11n}A_n + L_{12n}C_n - A_{n+1}L_{11n} - B_{n+1}L_{21n} = 0, \tag{2.5a}$$

$$\frac{d}{dt}L_{12n} + L_{11n}B_n + L_{12n}D_n - A_{n+1}L_{12n} - B_{n+1}L_{22n} = 0, \tag{2.5b}$$

$$\frac{d}{dt}L_{21n} + L_{21n}A_n + L_{22n}C_n - C_{n+1}L_{11n} - D_{n+1}L_{21n} = 0, \tag{2.5c}$$

$$\frac{d}{dt}L_{22n} + L_{21n}B_n + L_{22n}D_n - C_{n+1}L_{12n} - D_{n+1}L_{22n} = 0. \tag{2.5d}$$

The explicit form of the Lax matrices $\mathbf{L}_n$ and $\mathbf{M}_n$ can be derived by extending a well known procedure devised by Ablowitz, Kaup, Newell and Segur (AKNS) for nonlinear partial differential equations[2]. More precisely for a given suitable matrix $\mathbf{L}_n$ the entries of the matrix $\mathbf{M}_n$ can be derived by expanding its entries as a polynomial in the spectral parameter λ satisfying the Lax Eq. (2.4).

2.1. *Lax representation of* (1.1) *and* (1.2)

To start with, we consider the discrete spectral problem (2.2) for (1.1) with $\mathbf{L}_n$ and $\mathbf{M}_n$ as

$$\mathbf{L}_n = \begin{bmatrix} 0 & \lambda p_n \\ -\lambda v_n & 1+\lambda^2 u_n v_n \end{bmatrix}, \quad \mathbf{M}_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}. \tag{2.6}$$

Then the compatibility condition (2.4) gives the following

$$\begin{aligned} C_n p_n + B_{n+1} v_n &= 0, \\ u_{nt} = u_n A_n - \lambda\, {u_n}^2 C_n - u_n D_n + \frac{B_n}{\lambda} - \frac{C_n u_n}{\lambda\, v_n} &+ \frac{C_{n+1} p_n}{\lambda\, v_n} + \frac{D_{n+1}}{\lambda^2 v_n} - \frac{D_n}{\lambda^2 v_n}, \\ v_{nt} = D_{n+1} v_n - v_n A_n &+ \frac{C_n}{\lambda} + \lambda\, C_n u_n v_n, \\ p_{nt} = A_{n+1} p_n - p_n D_n &+ \frac{B_{n+1}}{\lambda} + \lambda\, B_{n+1} u_n v_n. \end{aligned} \tag{2.7}$$

In order to find the entries of the associated matrix $\mathbf{M}_n$ we expand each of them as a quadratic polynomial in the spectral parameter λ, that is

$$A_n = \sum_{l=0}^{2} a_n^{(l)} \lambda^l, \quad B_n = \sum_{l=0}^{2} b_n^{(l)} \lambda^l, \quad C_n = \sum_{l=0}^{2} c_n^{(l)} \lambda^l, \quad D_n = \sum_{l=0}^{2} d_n^{(l)} \lambda^l,$$

where $a_n^{(l)}$, $b_n^{(l)}$, $c_n^{(l)}$ and $d_n^{(l)}$ are unknown functions to be determined. Substituting the above expansions into (2.7) and then equating the like powers of λ to zero we obtain a system of equations along with evolution equations and solving them consistently yields the explicit form of A_n, B_n, C_n and D_n. As a result the matrix $\mathbf{M}_n$ for (1.1) reads

$$\mathbf{M}_n = \begin{bmatrix} -\dfrac{\lambda^2}{2} - \dfrac{p_{n-1}}{u_n u_{n-1} v_{n-1}} & \dfrac{\lambda\, p_{n-1}}{u_{n-1} v_{n-1}} \\ -\dfrac{\lambda}{u_n} & \dfrac{\lambda^2}{2} \end{bmatrix}. \tag{2.8}$$

Proceeding in a similar manner we find that (1.2) arises from the compatibility condition (2.4) with Lax matrices $\mathbf{L}_n$ and $\mathbf{M}_n$ as

$$\mathbf{L}_n = \begin{bmatrix} p_n & \lambda q_n \\ -\lambda v_n & 1+\lambda^2 u_n v_n \end{bmatrix}, \quad \mathbf{M}_n = \begin{bmatrix} -\dfrac{\lambda^2}{2} - \dfrac{q_{n-1}}{u_n u_{n-1} v_{n-1}} & \dfrac{q_{n-1}}{u_{n-1} v_{n-1}}\lambda \\ -\dfrac{1}{u_n}\lambda & \dfrac{\lambda^2}{2} \end{bmatrix}$$

satisfying (2.4). Thus 3- and 4-coupled systems given in (1.1) and (1.2) are integrable in the sense of Lax.

3. Hamiltonian Structure of (1.1) and (1.2)

Let us recall some of the basics related with Hamiltonian system governed by nonlinear partial differential and differential-difference equations [20]. Let $H : \mathcal{L}^q \to \mathcal{L}^q$ be a linear operator and V_H be a formal evolutionary vector field with characteristic is the q-tuple,

$$(H\theta)_\alpha = \sum_{\beta=1}^{q} H_{\alpha\beta}\theta^\beta \tag{3.1}$$

of vertical uni-vector. Then the prolongation of the vector field is given by

$$\operatorname{Pr} V_{H\theta} = \sum_{\alpha,J} E^J \left(\sum_\beta H_{\alpha\beta}\theta^\beta\right) \frac{\partial}{\partial E^J \mathbf{u}_n^\alpha}, \tag{3.2}$$

where E is a shift operator defined by $Ef(n) = f(n+1)$.

Definition 3.1. A linear operator H is said to be a Hamiltonian operator of (2.1) if it is skew symmetric and satisfies Jacobi's identity [21].

Definition 3.2. A system of coupled nonlinear PDΔEs is said to be a Hamiltonian system if it can be written as

$$\frac{\partial \mathbf{u}_n}{\partial t} = H\left(\frac{\delta\mathcal{H}}{\delta\,\mathbf{u}_n}\right), \tag{3.3}$$

where H is a Hamiltonian operator and $\mathcal{H}$ is the appropriate Hamiltonian functional.

In order to prove that the skew symmetric operator H be Hamiltonian it remains to prove that it satisfies the Jacobi's identity. For clarity, we mention the following theorem for a system of nonlinear partial differential equations $\frac{\partial \mathbf{u}}{\partial t} = \mathbf{K}(\mathbf{u})$ due to Olver[21].

Theorem. *Let D be a skew-adjoint $q\times q$ matrix differential operator of the system of partial differential equations, $\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{K}(\boldsymbol{u})$ and $\Theta = \frac{1}{2}\int\{\Theta\wedge\mathcal{D}\Theta\}dx$, the corresponding functional bi-vector. Then D is Hamiltonian if and only if*

$$\operatorname{Pr} V_{\mathcal{D}\theta}(\Theta) = 0. \tag{3.4}$$

Here $\theta = \theta(x,t,\boldsymbol{u})$.

Recent investigations by Sanders and Wang[24] suggest that the above result holds for nonlinear PDΔEs as well. For nonlinear PDΔEs, the prolongation of the vector field takes the form given in (3.2).

We now establish the Hamiltonian structure of (1.1) and (1.2) through the definitions and theorems stated above. The 3-coupled system (1.1) can be written as

$$\begin{pmatrix} u_{nt} \\ v_{nt} \\ p_{nt} \end{pmatrix} = H \begin{pmatrix} \dfrac{\delta\mathcal{H}}{\delta u_n} \\ \dfrac{\delta\mathcal{H}}{\delta v_n} \\ \dfrac{\delta\mathcal{H}}{\delta p_n} \end{pmatrix} = \begin{pmatrix} 0 & -u_n\,v_n & 0 \\ u_n\,v_n & 0 & v_n\,p_n \\ 0 & -v_n\,p_n & 0 \end{pmatrix} \begin{pmatrix} \dfrac{p_n}{u_n^2 v_n v_{n+1}} + \dfrac{p_{n-1}}{u_n^2 u_{n-1} v_{n-1}} - \dfrac{1}{u_n^2 v_n} \\ \dfrac{p_n}{u_n v_n^2 u_{n+1}} - \dfrac{1}{u_n v_n^2} \\ -\dfrac{1}{u_n v_n u_{n+1}} \end{pmatrix}. \tag{3.5}$$

Similarly 4-coupled system (1.2) can be written as

$$\begin{pmatrix} u_{nt} \\ v_{nt} \\ p_{nt} \\ q_{nt} \end{pmatrix} = J \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta u_n} \\ \frac{\delta \mathcal{H}}{\delta v_n} \\ \frac{\delta \mathcal{H}}{\delta p_n} \\ \frac{\delta \mathcal{H}}{\delta q_n} \end{pmatrix} = \begin{pmatrix} 0 & -u_n\, v_n & -u_n\, p_n & u_n^2\, p_n \\ u_n\, v_n & 0 & v_n\, p_n & -u_n\, v_n\, p_n + v_n\, q_n \\ u_n\, p_n & -v_n\, p_n & 0 & p_n\, q_n \\ -u_n^2\, p_n & u_n\, v_n\, p_n - v_n\, q_n & -p_n\, q_n & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{q_n}{u_n^2 v_n v_{n+1}} + \frac{q_{n-1}}{u_n^2 u_{n-1} v_{n-1}} - \frac{1}{u_n^2 v_n} \\ \frac{q_n}{u_n v_n^2 u_{n+1}} - \frac{1}{u_n v_n^2} \\ 0 \\ -\frac{1}{u_n v_n u_{n+1}} \end{pmatrix}. \tag{3.6}$$

Theorem 3.1. *The operator H given in (3.5) is a Hamiltonian operator for the 3-coupled system (1.1).*

Proof. Let $\theta = (\theta_1, \theta_2, \theta_3)^T$. Then

$$H\theta = H \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -u_n\, v_n \theta_2 \\ u_n\, v_n \theta_1 + v_n\, p_n \theta_3 \\ -v_n\, p_n \theta_2 \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}. \tag{3.7}$$

Next, define a bi-vector Θ of H by

$$\begin{aligned} \Theta_H &= \frac{1}{2} \sum [\theta_1 \wedge \Phi_1 + \theta_2 \wedge \Phi_2 + \theta_3 \wedge \Phi_3], \\ &= \frac{1}{2} \sum [-u_n v_n \theta_1 \wedge \theta_2 + u_n v_n \theta_2 \wedge \theta_1 + v_n p_n \theta_2 \wedge \theta_3 - v_n p_n \theta_3 \wedge \theta_2]. \end{aligned}$$

Using the property of wedge product $\theta_1 \wedge \theta_1 = 0, \theta_2 \wedge \theta_2 = 0, \theta_3 \wedge \theta_3 = 0, \theta_2 \wedge \theta_1 = -\theta_1 \wedge \theta_2, \theta_3 \wedge \theta_1 = -\theta_1 \wedge \theta_3, \theta_3 \wedge \theta_2 = -\theta_2 \wedge \theta_3$, we have,

$$\Theta_H = \sum [-u_n v_n \theta_1 \wedge \theta_2 + v_n p_n \theta_2 \wedge \theta_3] \tag{3.8}$$

and further calculation shows that

$$\mathrm{Pr}\, V_{H\theta}(\Theta_H) = 0 \tag{3.9}$$

and hence the skew symmetric operator H is Hamiltonian of (1.1). □

Theorem 3.2. *The operator J given in (3.6) is a Hamiltonian operator for the 4-coupled system (1.2).*

Proof. The proof is similar to that of Theorem 3.1 and hence omitted. □

Thus the 3-coupled system (1.1) and 4-coupled system (1.2) are Hamiltonian systems.

4. Generalized Symmetries and Conserved Densities

4.1. *Generalized symmetries: 3-coupled system* (1.1)

In this subsection, we present the computational details of the derivation of generalized symmetries for the 3-coupled system (1.1). Obviously (1.1) is invariant under the scaling (dilation) symmetry

$$(t, u_n, v_n, p_n) \to (s^{-2}t, s^{-1}u_n, s^{-1}v_n, s^{-1}p_n), \tag{4.1}$$

where s is an arbitrary parameter. Let us assume that (1.1) is invariant under a continuous non-point transformations

$$\begin{aligned} &n^* = n, \quad t^* = t, \quad u_n^* = u_n + \epsilon G_i^{(1)}(n) + O(\epsilon^2), \quad v_n^* = v_n + \epsilon G_i^{(2)}(n) + O(\epsilon^2), \\ &p_n^* = p_n + \epsilon G_i^{(3)}(n) + O(\epsilon^2), \quad i = 1, 2, \ldots, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} G_i^{(1)}(n) &= G_i^{(1)}(\ldots, p_{n-1}, v_{n-1}, u_{n-1}, u_n, v_n, p_n, u_{n+1}, v_{n+1}, p_{n+1}, \ldots), \\ G_i^{(2)}(n) &= G_i^{(2)}(\ldots, p_{n-1}, v_{n-1}, u_{n-1}, u_n, v_n, p_n, u_{n+1}, v_{n+1}, p_{n+1}, \ldots), \\ G_i^{(3)}(n) &= G_i^{(3)}(\ldots, p_{n-1}, v_{n-1}, u_{n-1}, u_n, v_n, p_n, u_{n+1}, v_{n+1}, p_{n+1}, \ldots), \end{aligned}$$

provided u_n, v_n and p_n satisfy (1.1). For clarity, we denote $\mathbf{G}_i(n) = (G_i^{(1)}(n), G_i^{(2)}(n), G_i^{(3)}(n))^T$ and the subscript i represents the ith order generalized symmetry. Consequently, we obtain the following invariant equations

$$\frac{\partial G_i^{(1)}(n)}{\partial t} = \frac{v_n p_n G_i^{(1)}(n+1) + u_{n+1} p_n G_i^{(2)}(n) - u_{n+1} v_n G_i^{(3)}(n)}{u_{n+1}^2 v_n^2} - \frac{G_i^{(2)}(n)}{v_n^2}, \tag{4.3a}$$

$$\begin{aligned} \frac{\partial G_i^{(2)}(n)}{\partial t} = {} & \frac{G_i^{(1)}(n)}{u_n^2} + \frac{G_i^{(2)}(n) p_{n-1} + G_i^{(3)}(n-1) v_n}{u_n u_{n-1} v_{n-1}} \\ & - \frac{v_n p_{n-1}(u_{n-1} v_{n-1} G_i^{(1)}(n) + u_n v_{n-1} G_i^{(1)}(n-1) + u_n u_{n-1} G_i^{(2)}(n-1))}{u_n^2 u_{n-1}^2 v_{n-1}^2}, \end{aligned} \tag{4.3b}$$

$$\begin{aligned} \frac{\partial G_i^{(3)}(n)}{\partial t} = {} & \frac{u_n v_n G_i^{(3)}(n) - u_n p_n G_i^{(2)}(n) - v_n p_n G_i^{(1)}(n)}{u_n^2 v_n^2} - \frac{2 p_n G_i^{(3)}(n)}{u_n u_{n+1} v_n} \\ & + \frac{v_n p_n^2 u_{n+1} G_i^{(1)}(n) + u_n v_n p_n^2 G_i^{(1)}(n+1) + u_n p_n^2 u_{n+1} G_i^{(2)}(n)}{u_n^2 v_n^2 u_{n+1}^2}. \end{aligned} \tag{4.3c}$$

The invariant equations (4.3) can be solved for the generalized symmetry $\mathbf{G}_i(n) = (G_i^{(1)}(n), G_i^{(2)}(n), G_i^{(3)}(n))^{\mathrm{T}}$ in more than one ways[7, 15, 18, 21, 25]. We show below how to derive the generalized symmetries of (1.1) through the algorithmic procedure developed by Hereman and his co-workers[15]. Basically, Hereman's algorithmic procedure is based on the concept of weights and ranks. To start with, we briefly explain the concept of weights and ranks. The weight, w, of a variable is defined as the exponent in the scaling parameter s which multiplies the variable. Similarly the rank of a monomial is defined as the total weight of the monomial. An expression is uniform in rank if all its terms have the same rank.

We set $w(\frac{d}{dt}) = 2$. From (1.1) we see

$$\begin{aligned} &w\left(\frac{d}{dt}\right) + w(u_n) = -w(v_n) = w(p_n) - w(u_{n+1}) - w(v_n), \\ &w\left(\frac{d}{dt}\right) + w(v_n) = w(v_n) + w(p_{n-1}) - w(u_n) - w(u_{n-1}) - w(v_{n-1}) = -w(u_n), \\ &w\left(\frac{d}{dt}\right) + w(p_n) = w(p_n) - w(u_n) - w(v_n) = 2w(p_n) - w(u_n) - w(u_{n+1}) - w(v_n) \end{aligned}$$

and so

$$w(u_n) = -1, \quad w(v_n) = -1, \quad w(p_n) = -1,$$

and hence (1.1a)–(1.1c) are of rank 1, 1 and 1, respectively. We wish to mention that Hereman and his collaborators[12, 13, 15] have developed a Mathematical software package (known as INVARIANTSSYMMETRIES.M) in Mathematica for finding higher-order symmetries and conservation laws for nonlinear PDEs and nonlinear PDΔEs with two independent variables provided the weight of the dependent variable is positive. Since the weights of the dependent variables associated with (1.1) are negative, the software package is not applicable. However we demonstrate that one can derive higher-order generalized symmetries for (1.1) and (1.2) by exploiting their ideas. In this article, we have computed the generalized symmetries and conserved densities manually. Hereafter, we use the more compact notation

$$\begin{aligned} &u_n = u, \quad v_n = v, \quad p_n = p, \quad u_{n-1} = \underline{u}, \quad v_{n-1} = \underline{v}, \quad p_{n-1} = \underline{p}, \\ &u_{n-2} = \underline{\underline{u}}, \quad v_{n-2} = \underline{\underline{v}}, \quad p_{n-2} = \underline{\underline{p}}, \quad u_{n+1} = \overline{u}, \quad v_{n+1} = \overline{v}, \\ &p_{n+1} = \overline{p}, \quad u_{n+2} = \overline{\overline{u}}, \quad v_{n+2} = \overline{\overline{v}}, \quad p_{n+2} = \overline{\overline{p}}, \text{ etc.} \end{aligned}$$

Note that the trivial generalized symmetry is of rank (1, 1, 1), then the next nontrivial generalized symmetry $\mathbf{G}_2(n) = (G_2^{(1)}(n), G_2^{(2)}(n), G_2^{(3)}(n))^{\mathrm{T}}$ must have rank (3, 3, 3). With this in mind, we first form monomial in u, v and p of rank (3, 3, 3) that leads to a set $\mathcal{L} = \{u, v, p, \frac{1}{u}, \frac{1}{v}, \frac{1}{p}, \frac{1}{u^2}, \frac{1}{v^2}, \frac{1}{p^2}, \frac{1}{u\,v}, \frac{1}{u\,p}, \frac{1}{v\,p}, \frac{1}{u^3}, \frac{1}{v^3}, \frac{1}{p^3}, \frac{1}{u\,v^2}, \frac{1}{u\,p^2}, \frac{1}{u^2\,v}, \frac{1}{u^2\,p}, \frac{1}{v\,p^2}, \frac{1}{v^2\,p}, \frac{1}{u\,v\,p}\}$. Then the necessary partial derivatives with respect to t in each monomial of $\mathcal{L}$ along with (1.1) and leads to an another set $\mathcal{M}$ involving u, v, p and its backward and forward shifts. Note that each monomial in $\mathcal{M}$ is of rank 3. The linear combination of the monomials in $\mathcal{M}$ gives the most general form of the nontrivial generalized symmetry $\mathbf{G}_2(n) = (G_2^{(1)}(n), G_2^{(2)}(n), G_2^{(3)}(n))^{\mathrm{T}}$. Substituting the above linear combination in the invariant equations along with (1.1), leads to a system of linear equations and solving them consistently gives the first nontrivial generalized symmetry with rank (3,3,3) as

$$G_2^{(1)}(n) = \frac{p}{v\,\overline{u}^{2}\overline{v}} - \frac{1}{u\,v^{2}} + \frac{\underline{p}}{u\,v\,\underline{u}\,\underline{v}} - \frac{p\,\underline{p}}{u\,v\,\overline{u}\,\underline{u}\,\underline{v}} - \frac{p\,\overline{p}}{v\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}} - \frac{p^{2}}{u\,v^{2}\overline{u}^{2}} + \frac{2p}{u\,v^{2}\overline{u}}, \tag{4.4a}$$

$$G_2^{(2)}(n) = \frac{1}{u^{2}v} - \frac{2\underline{p}}{u^{2}\underline{u}\,\underline{v}} + \frac{p\,\underline{p}}{u^{2}\,\underline{u}\,\underline{v}\,\overline{u}} - \frac{v\,\underline{p}}{u\,\underline{u}^{2}\underline{v}^{2}} + \frac{v\,\underline{p}^{2}}{u^{2}\,\underline{u}^{2}\underline{v}^{2}} + \frac{v\,\underline{p}\,\underline{\underline{p}}}{u\,\underline{u}^{2}\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{p}{u^{2}v\,\overline{u}}, \tag{4.4b}$$

$$G_2^{(3)}(n) = \frac{p^{2}}{u\,v\,\overline{u}^{2}\overline{v}} - \frac{p^{2}\overline{p}}{u\,v\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}} + \frac{2p^{2}}{u^{2}v^{2}\overline{u}} - \frac{p^{3}}{u^{2}v^{2}\overline{u}^{2}} - \frac{p^{2}\,\underline{p}}{u^{2}\,v\,\underline{u}\,\underline{v}\,\overline{u}} + \frac{p\,\underline{p}}{u^{2}v\,\underline{u}\,\underline{v}} - \frac{p}{u^{2}v^{2}}. \tag{4.4c}$$

Proceeding as above, we obtain the next generalized symmetry $\mathbf{G}_3(n) = (G_3^{(1)}(n), G_3^{(2)}(n), G_3^{(3)}(n))^{\mathrm{T}}$ with rank (5, 5, 5), where

$$\begin{aligned} G_3^{(1)}(n) = {} & \frac{1}{u^{2}v^{3}} + \frac{\underline{p}^{2}}{u^{2}\,v\,\underline{u}^{2}\underline{v}^{2}} - \frac{p^{3}}{u^{2}v^{3}\overline{u}^{3}} + \frac{p\,\underline{p}}{u\,v\,\overline{u}^{2}\,\overline{v}\,\underline{u}\,\underline{v}} - \frac{p\,\underline{p}\,\underline{\underline{p}}}{u\,v\,\overline{u}\,\underline{u}^{2}\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{\underline{p}}{u\,v\,\underline{u}^{2}\underline{v}^{2}} \\ & - \frac{p}{v\,\overline{u}^{3}\overline{v}^{2}} - \frac{p\,\overline{p}\,\overline{\overline{p}}}{v\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}^{2}\overline{\overline{v}}\,\overline{\overline{\overline{u}}}} + \frac{p\,\underline{p}}{u\,v\,\underline{u}^{2}\underline{v}^{2}\,\overline{u}} - \frac{p\,\overline{p}^{2}}{v\,\overline{u}^{3}\overline{v}^{2}\overline{\overline{u}}^{2}} - \frac{p\,\underline{p}\,\overline{p}}{u\,v\,\underline{u}\,\underline{v}\,\overline{u}^{2}\,\overline{v}\,\overline{\overline{u}}} \\ & + \frac{\underline{p}\,\underline{\underline{p}}}{u\,v\,\underline{u}^{2}\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{p\,\overline{p}}{v\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}^{2}\overline{\overline{v}}} - \frac{p\,\underline{p}^{2}}{u^{2}\,v\,\underline{u}^{2}\underline{v}^{2}\overline{u}} + \frac{2\,p\,\overline{p}}{u\,v^{2}\overline{u}^{2}\overline{v}\,\overline{\overline{u}}} + \frac{2\,p\,\overline{p}}{v\,\overline{u}^{3}\overline{v}^{2}\overline{\overline{u}}} \\ & - \frac{2\,p^{2}\overline{p}}{u\,v^{2}\overline{u}^{3}\overline{v}\,\overline{\overline{u}}} - \frac{2\,p}{u\,v^{2}\overline{u}^{2}\overline{v}} - \frac{2\,p^{2}\,\underline{p}}{u^{2}v^{2}\overline{u}^{2}\underline{u}\,\underline{v}} + \frac{2\,p^{2}}{u\,v^{2}\overline{u}^{3}\overline{v}} - \frac{2\,\underline{p}}{u^{2}v^{2}\underline{u}\,\underline{v}} + \frac{3\,p^{2}}{u^{2}v^{3}\overline{u}^{2}} \\ & - \frac{3\,p}{u^{2}v^{3}\overline{u}} + \frac{4\,p\,\underline{p}}{u^{2}v^{2}\overline{u}\,\underline{u}\,\underline{v}}, \end{aligned} \tag{4.5a}$$

$$\begin{aligned} G_3^{(2)}(n) = {} & \frac{p}{u^{2}v\,\overline{u}^{2}\overline{v}} - \frac{1}{u^{3}v^{2}} + \frac{p^{2}\,\underline{p}}{u^{3}\,v\,\underline{u}\,\underline{v}\,\overline{u}^{2}} + \frac{v\,\underline{p}^{3}}{u^{3}\underline{u}^{3}\underline{v}^{3}} - \frac{p\,\underline{p}}{u^{2}\underline{u}\,\underline{v}\,\overline{u}^{2}\overline{v}} + \frac{v\,\underline{p}}{u\,\underline{u}^{3}\underline{v}^{3}} \\ & + \frac{v\,\underline{p}\,\underline{\underline{p}}\,\underline{\underline{\underline{p}}}}{u\,\underline{u}^{2}\,\underline{v}\,\underline{\underline{u}}^{2}\,\underline{\underline{v}}\,\underline{\underline{\underline{u}}}\,\underline{\underline{\underline{v}}}} - \frac{p^{2}}{u^{3}v^{2}\overline{u}^{2}} + \frac{p\,\underline{p}\,\underline{\underline{p}}}{u^{2}\underline{u}^{2}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}\,\overline{u}} - \frac{p\,\underline{p}}{u^{2}\underline{u}^{2}\underline{v}^{2}\overline{u}} + \frac{v\,\underline{p}\,\underline{\underline{p}}^{2}}{u\,\underline{u}^{3}\underline{v}\,\underline{\underline{u}}^{2}\underline{\underline{v}}^{2}} \\ & - \frac{v\,\underline{p}\,\underline{\underline{p}}}{u\,\underline{u}^{2}\,\underline{v}\,\underline{\underline{u}}^{2}\underline{\underline{v}}^{2}} + \frac{p\,\underline{p}\,\overline{p}}{u^{2}\underline{u}\,\underline{v}\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}} - \frac{p\,\overline{p}}{u^{2}v\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}} + \frac{2\,\underline{p}}{u^{2}\,\underline{u}^{2}\underline{v}^{2}} + \frac{2\,p}{u^{3}v^{2}\overline{u}} \\ & + \frac{2\,v\,\underline{p}^{2}\,\underline{\underline{p}}}{u^{2}\,\underline{u}^{3}\,\underline{v}^{2}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{2\,v\,\underline{p}\,\underline{\underline{p}}}{u\,\underline{u}^{3}\,\underline{v}^{2}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{2\,p\,\underline{p}^{2}}{u^{3}\underline{u}^{2}\underline{v}^{2}\overline{u}} - \frac{2\,\underline{p}\,\underline{\underline{p}}}{u^{2}\underline{u}^{2}\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{2\,v\,\underline{p}^{2}}{u^{2}\underline{u}^{3}\underline{v}^{3}} \\ & + \frac{3\,\underline{p}}{u^{3}v\,\underline{u}\,\underline{v}} - \frac{3\,\underline{p}^{2}}{u^{3}\underline{u}^{2}\underline{v}^{2}} - \frac{4\,p\,\underline{p}}{u^{3}v\,\overline{u}\,\underline{u}\,\underline{v}}, \end{aligned} \tag{4.5b}$$

$$\begin{aligned} G_3^{(3)}(n) = {} & \frac{p}{u^{3}v^{3}} + \frac{p^{2}\,\underline{p}}{u^{2}\,v\,\underline{u}^{2}\underline{v}^{2}\,\overline{u}} + \frac{p^{2}\,\underline{p}}{u^{2}\,v\,\underline{u}\,\underline{v}\,\overline{u}^{2}\overline{v}} + \frac{p^{2}\overline{p}}{u\,v\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}^{2}\overline{\overline{v}}} - \frac{p^{4}}{u^{3}v^{3}\,\overline{u}^{3}} - \frac{p\,\underline{p}}{u^{2}v\,\underline{u}^{2}\underline{v}^{2}} \\ & + \frac{p\,\underline{p}^{2}}{u^{3}v\,\underline{u}^{2}\underline{v}^{2}} + \frac{p\,\underline{p}\,\underline{\underline{p}}}{u^{2}v\,\underline{u}^{2}\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{p^{2}\,\underline{p}^{2}}{u^{3}\,v\,\underline{u}^{2}\underline{v}^{2}\overline{u}} - \frac{p^{2}\,\underline{p}\,\overline{p}}{u^{2}\,v\,\underline{u}\,\underline{v}\,\overline{u}^{2}\overline{v}\,\overline{\overline{u}}} - \frac{p^{2}\,\underline{p}\,\underline{\underline{p}}}{u^{2}v\,\overline{u}\,\underline{u}^{2}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} \end{aligned}$$

$$
\begin{aligned}
&-\frac{p^2\overline{p}^2}{u\,v\,\overline{u}^3\overline{v}^2\overline{\overline{u}}^2}-\frac{p^2}{u\,v\,\overline{u}^3\overline{v}^2}-\frac{p^2\overline{p}\,\overline{\overline{p}}}{u\,v\,\overline{u}^2\overline{v}\,\overline{\overline{u}}^2\overline{\overline{v}}\,\overline{\overline{\overline{u}}}}+\frac{2p^3}{u^2v^2\overline{u}^3\,\overline{v}}+\frac{2p^2\overline{p}}{u\,v\,\overline{u}^3\overline{v}^2\overline{\overline{u}}}\\
&+\frac{2p^2\overline{p}}{u^2v^2\,\overline{u}^2\,\overline{v}\,\overline{\overline{u}}}-\frac{2p^3\overline{p}}{u^2v^2\overline{u}^3\,\overline{v}\,\overline{\overline{u}}}-\frac{2p^3\,\underline{p}}{u^3\,v^2\,\underline{u}\,\underline{v}\,\overline{u}^2}-\frac{2p\,\underline{p}}{u^3v^2\underline{u}\,\underline{v}}-\frac{2p^2}{u^2v^2\,\overline{u}^2\overline{v}}\\
&+\frac{3p^3}{u^3v^3\,\overline{u}^2}-\frac{3p^2}{u^3v^3\,\overline{u}}+\frac{4p^2\,\underline{p}}{u^3\,v^2\,\underline{u}\,\underline{v}\,\overline{u}}.
\end{aligned}
\tag{4.5c}
$$

In a similar manner, we have checked that 3-coupled system (1.1) admits a sequence of higher-order generalized symmetries $\mathbf{G}_i(n)$ with ranks $(2i-1, 2i-1, 2i-1), i = 4, 5, \ldots$ which involve a huge number of terms and hence we refrain from presenting it here. We have checked that the obtained generalized symmetries are commutable, that is, the generalized symmetries satisfies the following relations

$$
[\mathbf{G}_l(n), \mathbf{G}_k(n)] = \mathbf{G}_k(n)'[\mathbf{G}_l(n)] - \mathbf{G}_l(n)'[\mathbf{G}_k(n)] = 0, \quad l, k = 1, 2, \ldots,
$$

where $\mathbf{G}_k(n)'[\mathbf{G}_l(n)] = \frac{\partial}{\partial\epsilon}\mathbf{G}_k(\mathbf{u}_n + \epsilon\mathbf{G}_l(n))\big|_{\epsilon=0}$ is the Frechet derivative of $\mathbf{G}_k(n)$ along the direction of $\mathbf{G}_l(n)$.

4.2. *Generalized symmetries: 4-coupled system* (1.2)

Proceeding as above, we have checked that 4-coupled system (1.2) admits a sequence of generalized symmetries $\mathbf{G}_i(n)$ with rank $(2i-1, 2i-1, 2i, 2i-1)$, $i = 1, 2, \ldots$. The first two members of the sequence of generalized symmetries are as follows:

$$
\mathbf{G}_1(n) = \begin{pmatrix} G_1^{(1)}(n) \\ G_1^{(2)}(n) \\ G_1^{(3)}(n) \\ G_1^{(4)}(n) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{v} - \dfrac{q}{\overline{u}\,v} - \dfrac{u\,p}{\overline{u}\,v} \\ \dfrac{v\,q}{u\,\underline{u}\,\underline{v}} - \dfrac{1}{u} + \dfrac{p}{\overline{u}} \\ \dfrac{p\,q}{u\,\underline{u}\,\underline{v}} - \dfrac{p\,q}{u\,v\,\overline{u}} \\ \dfrac{q}{u\,v} - \dfrac{q^2}{u\,v\,\overline{u}} - \dfrac{p\,q}{\underline{u}\,\underline{v}} \end{pmatrix}, \quad \mathbf{G}_2(n) = \begin{pmatrix} G_2^{(1)}(n) \\ G_2^{(2)}(n) \\ G_2^{(3)}(n) \\ G_2^{(4)}(n) \end{pmatrix},
\tag{4.6}
$$

where

$$
\begin{aligned}
G_2^{(1)}(n) ={}& \frac{p}{v^2\overline{u}} - \frac{1}{u\,v^2} + \frac{2q}{u\,v^2\overline{u}} - \frac{q^2}{u\,v^2\overline{u}^2} - \frac{q\,\underline{q}}{u\,v\,\underline{u}\,\underline{v}\,\overline{u}} - \frac{p\,\underline{q}}{v\,\underline{u}\,\underline{v}\,\overline{u}} + \frac{q}{v\,\overline{u}^2\overline{v}} - \frac{q\,\overline{p}}{v\,\overline{u}\,\overline{v}\,\overline{\overline{u}}}\\
&- \frac{q\,\overline{q}}{v\,\overline{u}^2\overline{v}\,\overline{\overline{u}}} + \frac{\underline{q}}{u\,v\,\underline{u}\,\underline{v}} + \frac{u\,p}{v\,\overline{u}^2\overline{v}} - \frac{u\,p\,\overline{q}}{v\,\overline{u}^2\overline{v}\,\overline{\overline{u}}} - \frac{u\,p\,\overline{p}}{v\,\overline{u}\,\overline{v}\,\overline{\overline{u}}} - \frac{p\,q}{v^2\overline{u}^2},\\
G_2^{(2)}(n) ={}& \frac{1}{u^2v} - \frac{q}{u^2v\,\overline{u}} - \frac{p}{u\,v\,\overline{u}} - \frac{2\underline{q}}{u^2\underline{u}\,\underline{v}} + \frac{q\,\underline{q}}{u^2\underline{u}\,\underline{v}\,\overline{u}} + \frac{v\,\underline{q}^2}{u^2\underline{u}^2\underline{v}^2} + \frac{p\,\underline{q}}{u\,\underline{u}\,\underline{v}\,\overline{u}}\\
&- \frac{v\,\underline{q}}{u\,\underline{u}^2\underline{v}^2} - \frac{p}{\overline{u}^2\overline{v}} + \frac{p\,\overline{q}}{\overline{u}^2\overline{v}\,\overline{\overline{u}}} + \frac{p\,\overline{p}}{\overline{u}\,\overline{v}\,\overline{\overline{u}}} + \frac{p\,q}{u\,v\,\overline{u}^2} + \frac{v\,\underline{q}\,\underline{\underline{q}}}{u\,\underline{u}^2\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{v\,\underline{p}\,\underline{\underline{q}}}{u\,\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}},
\end{aligned}
$$

$$G_2^{(3)}(n) = p\left[\frac{q}{u\,v\,\overline{u}^{\,2}\overline{v}} - \frac{q\,\overline{p}}{u\,v\,\overline{u}\,\overline{v}\,\overline{\overline{u}}} - \frac{q\,\overline{q}}{u\,v\,\overline{u}^{\,2}\overline{v}\,\overline{\overline{u}}} - \frac{q^{\,2}}{u^{\,2}v^{\,2}\,\overline{u}^{\,2}} + \frac{q}{u^{\,2}v^{\,2}\,\overline{u}} - \frac{\underline{q}}{u^{\,2}v\,\underline{u}\,\underline{v}}\right.$$
$$\left. + \frac{\underline{q}^{\,2}}{u^{\,2}\underline{u}^{\,2}\underline{v}^{\,2}} - \frac{\underline{q}}{u\,\underline{u}^{\,2}\underline{v}^{\,2}} + \frac{\underline{q}\,\underline{\underline{q}}}{u\,\underline{u}^{2}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{\underline{p}\,\underline{\underline{q}}}{u\,\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}}\right],$$

$$G_2^{(4)}(n) = \frac{q\,\underline{q}}{u^{\,2}v\,\underline{u}\,\underline{v}} - \frac{q}{u^{\,2}v^{\,2}} + \frac{2\,q^{\,2}}{u^{\,2}v^{\,2}\,\overline{u}} + \frac{p\,\underline{q}}{u\,v\,\underline{u}\,\underline{v}} + \frac{q^{\,2}}{u\,v\,\overline{u}^{2}\,\overline{v}} - \frac{p\,\underline{q}^{\,2}}{u\,\underline{u}^{\,2}\underline{v}^{\,2}} - \frac{q^{\,3}}{u^{\,2}v^{\,2}\,\overline{u}^{2}}$$
$$- \frac{p\,\underline{p}\,\underline{\underline{q}}}{\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{p\,\underline{q}}{\underline{u}^{\,2}\underline{v}^{\,2}} - \frac{q^{\,2}\,\overline{p}}{u\,v\,\overline{u}\,\overline{v}\,\overline{\overline{u}}} - \frac{q^{\,2}\overline{q}}{u\,v\,\overline{u}^{2}\,\overline{v}\,\overline{\overline{u}}} - \frac{p\,q\,\underline{q}}{u\,v\,\overline{u}\,\underline{u}\,\underline{v}} - \frac{q^{\,2}\,\underline{q}}{u^{\,2}v\,\overline{u}\,\underline{u}\,\underline{v}} - \frac{p\,\underline{q}\,\underline{\underline{q}}}{\underline{u}^{\,2}\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}}.$$

Here again we have checked that the obtained generalized symmetries $\{\mathbf{G}_i(n)\}$ are commutable.

4.3. *Conserved densities and fluxes: 3-coupled system* (1.1)

A scalar function $\rho_n(\mathbf{u}_n)$ is a conserved density of (2.1) if there exists a scalar function $J_n(\mathbf{u}_n)$ called the flux, such that

$$\frac{\partial \rho_n}{\partial t} + \Delta J_n = 0 \tag{4.7}$$

is satisfied on the solutions of (2.1). Here $\Delta J_n = (E - I)J_n = J_{n+1} - J_n$.

To derive a conserved density with different ranks, we use the algorithmic procedure of Hereman and his co-workers [12, 13]. For rank 2 as usual we form monomials of u, v and p which give a list $\mathcal{L}_1 = \{u, v, p, \frac{1}{u}, \frac{1}{v}, \frac{1}{p}, \frac{1}{u^2}, \frac{1}{v^2}, \frac{1}{p^2}, \frac{1}{u\,v}, \frac{1}{u\,p}, \frac{1}{v\,p}\}$. Introducing then the necessary t derivatives in each monomial of $\mathcal{L}_1$ leads to an another set $\mathcal{M}_1$ involving u, v, p and its backward and forward shifts. Note that each monomial in $\mathcal{M}_1$ is of rank 2. The linear combination of the monomials in $\mathcal{M}_1$ gives the most general form of the conserved density $\rho_n^{(1)}$ of rank 2. Substituting the above linear combination in the (4.7) along with (1.1), leads to a system of linear equations and solving them consistently gives the conserved density $\rho_n^{(1)}$ with rank 2 as

$$\rho_n^{(1)} = \frac{1}{u\,v} - \frac{\underline{p}}{u\,\underline{u}\,\underline{v}} \tag{4.8}$$

and the associated fluxes $J_n^{(1)}$ is

$$J_n^{(1)} = \frac{p\,\underline{\underline{p}}}{u\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{\underline{p}}{u\,\underline{u}^2\,\underline{v}^2}. \tag{4.9}$$

Proceeding as above, we obtain the next conserved densities $\rho_n^{(2)}$ with rank 4

$$\rho_n^{(2)} = \frac{1}{2\,u^2\,v^2} + \frac{\underline{p}^2}{2\,u^2\,\underline{u}^2\,\underline{v}^2} + \frac{p\,\underline{p}}{u^2\,v\,\underline{u}\,\underline{v}\,\overline{u}} - \frac{p}{u^2\,v^2\,\overline{u}} - \frac{\underline{p}}{u^2\,v\,\underline{u}\,\underline{v}} \tag{4.10}$$

and the associated fluxes $J_n^{(2)}$ are

$$J_n^{(2)} = \frac{\underline{p}^2}{u^2\,\underline{u}^3\,\underline{v}^3} + \frac{p\,\underline{p}}{u^2\,v\,\underline{u}^2\,\underline{v}^2\,\overline{u}} + \frac{\underline{p}\,\underline{\underline{p}}}{u^2\,v\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{\underline{p}}{u^2\,v\,\underline{u}^2\,\underline{v}^2} - \frac{\underline{p}^2\,\underline{\underline{p}}}{u^2\,\underline{u}^3\,\underline{v}^2\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{p\,\underline{p}\,\underline{\underline{p}}}{u^2\,v\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}\,\overline{u}}. \tag{4.11}$$

In a similar manner we have checked that the 3-coupled system (1.1) admits a sequence of conserved densities $\rho_n^{(i)}$ with rank $2i, i = 3, 4, 5, \ldots$ along with the flux $J_n^{(i)}$ of rank $2(i+1), i = 3, 4, 5, \ldots$, respectively, which involves lengthy expressions and so the details are omitted here.

4.4. *Conserved densities and fluxes: 4-coupled system* (1.2)

Proceeding as above, we have checked that 4-coupled system (1.2) admits a sequence of conserved densities $\rho_n^{(i)}$ with rank $2i, i = 1, 2, \ldots$ and fluxes $J_n^{(i)}$ with rank $2(i+1), i = 1, 2, \ldots$. The first two members of the sequence of conserved quantities are as follows:

$$\rho_n^{(1)} = \frac{1}{u\,v} - \frac{\underline{q}}{u\,\underline{u}\,\underline{v}},$$

$$J_n^{(1)} = \frac{\underline{q}\,\underline{\underline{q}}}{u\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{\underline{p}\,\underline{\underline{q}}}{u\,\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{\underline{q}}{u\,\underline{u}^2\,\underline{v}^2}$$

and

$$\rho_n^{(2)} = \frac{1}{2\,u^2\,v^2} + \frac{\underline{q}^2}{2\,u^2\,\underline{u}^2\,\underline{v}^2} + \frac{q\,\underline{q}}{u^2\,v\,\underline{u}\,\underline{v}\,\overline{u}} + \frac{p\,\underline{q}}{u\,v\,\underline{u}\,\underline{v}\,\overline{u}} - \frac{q}{u^2\,v^2\,\overline{u}} - \frac{\underline{q}}{u^2\,v\,\underline{u}\,\underline{v}},$$

$$\begin{aligned} J_n^{(2)} = {} & \frac{\underline{q}^2}{u^2\,\underline{u}^3\,\underline{v}^3} + \frac{q\,\underline{q}}{u^2\,v\,\underline{u}^2\,\underline{v}^2\,\overline{u}} + \frac{\underline{q}\,\underline{\underline{q}}}{u^2\,v\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} + \frac{p\,\underline{q}}{u\,v\,\underline{u}^2\,\underline{v}^2\,\overline{v}} + \frac{\underline{p}\,\underline{\underline{q}}}{u^2\,v\,\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{\underline{q}}{u^2\,v\,\underline{u}^2\,\underline{v}^2} \\ & - \frac{\underline{q}^2\,\underline{\underline{q}}}{u^2\,\underline{u}^3\,\underline{v}^2\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{q\,\underline{q}\,\underline{\underline{q}}}{u^2\,v\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}\,\overline{u}} - \frac{p\,\underline{p}\,\underline{\underline{q}}}{u\,v\,\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}\,\overline{u}} - \frac{\underline{p}\,\underline{q}\,\underline{\underline{q}}}{u^2\,\underline{u}^2\,\underline{v}^2\,\underline{\underline{u}}\,\underline{\underline{v}}} - \frac{p\,\underline{q}\,\underline{\underline{q}}}{u\,v\,\underline{u}^2\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}\,\overline{u}} \\ & - \frac{\underline{p}\,q\,\underline{\underline{q}}}{u^2\,v\,\underline{u}\,\underline{v}\,\underline{\underline{u}}\,\underline{\underline{v}}\,\overline{u}}. \end{aligned}$$

5. Recursion Operator: PDΔEs

An operator valued function $\mathcal{R}$ is said to be a *recursion operator* of Eq. (2.1) if it connects symmetries into symmetries, that is,

$$\mathbf{G}_{k+1}(n) = \mathcal{R}\,\mathbf{G}_k(n) \quad \forall\, k, \tag{5.1}$$

where $\mathbf{G}_k(n)$ and $\mathbf{G}_{k+1}(n)$ are consecutive generalized symmetries. Note that there exist different methods to construct recursion operator $\mathcal{R}$ for PDΔE [15, 20, 24, 25]. We show below how to derive $\mathcal{R}$ through the algorithmic procedure developed by Hereman and his collaborators. For 3-component systems, (5.1) becomes

$$\begin{bmatrix} G_{k+1}^{(1)}(n) \\ G_{k+1}^{(2)}(n) \\ G_{k+1}^{(3)}(n) \end{bmatrix} = \mathcal{R} \begin{bmatrix} G_{k}^{(1)}(n) \\ G_{k}^{(2)}(n) \\ G_{k}^{(3)}(n) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} G_{k}^{(1)}(n) \\ G_{k}^{(2)}(n) \\ G_{k}^{(3)}(n) \end{bmatrix}, \tag{5.2}$$

where $\mathbf{G}_k(n) = (G_k^{(1)}(n), G_k^{(2)}(n), G_k^{(3)}(n))^{\mathrm{T}}$ and $\mathbf{G}_{k+1}(n) = (G_{k+1}^{(1)}(n), G_{k+1}^{(2)}(n), G_{k+1}^{(3)}(n))^{\mathrm{T}}$ are the generalized symmetries. The entries R_{ij} of $\mathcal{R}$ involve dependent variables along with their shifts and its inverse, difference operators and inverse difference operators, that is,[15]

$$R_{ij} = U_{ij}(\mathbf{u}_n)\mathcal{O}(\triangle^{-1}, E^{-1}, I, E)V_{ij}(\mathbf{u}_n),$$

where $\mathbf{u}_n = (u_n, v_n, p_n), U_{ij}$ and V_{ij} are functions of the potentials u_n, v_n, p_n and their shifts, $E^{-1}f(n) = f(n-1)$, $If(n) = f(n)$, $Ef(n) = f(n+1)$ and $\triangle$ is difference operator defined by

$$\triangle f(n) = (E - I)f(n) = f(n+1) - f(n)$$

and $\triangle^{-1}$ is inverse difference operator defined as

$$\triangle^{-1} f(n) = \frac{1}{2}\left[\sum_{k=-\infty}^{-1}[f(n+1+2k) - f(n+2k)] - \sum_{k=1}^{\infty}[f(n-1+2k) - f(n-2+2k)]\right].$$

5.1. *Recursion operator: 3-coupled system* (1.1)

The construction of the recursion operator $\mathcal{R}$ for the 3-coupled system (1.1) is as follows: For $k = 2$, (5.2) becomes

$$\begin{bmatrix} G_{3}^{(1)}(n) \\ G_{3}^{(2)}(n) \\ G_{3}^{(3)}(n) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} G_{2}^{(1)}(n) \\ G_{2}^{(2)}(n) \\ G_{2}^{(3)}(n) \end{bmatrix}, \tag{5.3}$$

where $(G_2^{(1)}(n), G_2^{(2)}(n), G_2^{(3)}(n))^{\mathrm{T}}$ and $(G_3^{(1)}(n), G_3^{(2)}(n), G_3^{(3)}(n))^{\mathrm{T}}$ are consecutive generalized symmetries of rank (3, 3, 3) and (5, 5, 5) given in (4.4) and (4.5), respectively. From (5.3) it is clear that the entries $R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}$ andR_{33} of the matrix operator $\mathcal{R}$ must be of rank 2 which can be determined from the following relations,

$$\begin{aligned} \operatorname{rank} G_3^{(1)}(n) &= \operatorname{rank} R_{11} + \operatorname{rank} G_2^{(1)}(n) = \operatorname{rank} R_{12} + \operatorname{rank} G_2^{(2)}(n) \\ &= \operatorname{rank} R_{13} + \operatorname{rank} G_2^{(3)}(n), \\ \operatorname{rank} G_3^{(2)}(n) &= \operatorname{rank} R_{21} + \operatorname{rank} G_2^{(1)}(n) = \operatorname{rank} R_{22} + \operatorname{rank} G_2^{(2)}(n) \\ &= \operatorname{rank} R_{23} + \operatorname{rank} G_2^{(3)}(n), \\ \operatorname{rank} G_3^{(3)}(n) &= \operatorname{rank} R_{31} + \operatorname{rank} G_2^{(1)}(n) = \operatorname{rank} R_{32} + \operatorname{rank} G_2^{(2)}(n) \\ &= \operatorname{rank} R_{33} + \operatorname{rank} G_2^{(3)}(n). \end{aligned} \tag{5.4}$$

With this goal, we expand $R_{ij},\ i,j=1,2,3$ as the functions of dependent variable along with their shifts, difference operator and inverse difference operator, with rank 2, that is,

$$R_{ij} = U_{ij}(\mathbf{u}_n)\mathcal{O}(\triangle^{-1},\, E^{-1},\, I,\, E)\; V_{ij}(\mathbf{u}_n) \tag{5.5}$$

with the following relations

$$\operatorname{rank} R_{ij} = \operatorname{rank} U_{ij}(\mathbf{u}_n) \; + \; \operatorname{rank} V_{ij}(\mathbf{u}_n). \tag{5.6}$$

After a tedious calculation, we find that (5.3) along with (5.4) satisfies for the following forms of $R_{ij},\ i,j=1,2,3$ reads as

$$R_{11} = \left(\frac{p}{uv\overline{u}} + \frac{\underline{p}}{u\,\underline{u}\,\underline{v}} - \frac{2}{u\,v}\right) I + \frac{p}{\overline{u}^2 v}E + \left(\frac{p}{\overline{u}v} - \frac{1}{v}\right)\triangle^{-1}\frac{1}{u}$$
$$+\,u\triangle^{-1}\left(\frac{p}{u^2 v\overline{u}} + \frac{\underline{p}}{u^2\,\underline{u}\,\underline{v}} - \frac{1}{u^2\,v}\right), \tag{5.7a}$$

$$R_{12} = \left(\frac{p}{\overline{u}v^2} - \frac{1}{v^2}\right) I + \left(\frac{p}{\overline{u}v} - \frac{1}{v}\right)\triangle^{-1}\frac{1}{v} + u\triangle^{-1}\left(\frac{p}{uv^2\overline{u}} - \frac{1}{u\,v^2}\right), \tag{5.7b}$$

$$R_{13} = -u\triangle^{-1}\frac{1}{uv\overline{u}}, \tag{5.7c}$$

$$R_{21} = \left(\frac{1}{u^2} - \frac{v\underline{p}}{u^2\underline{u}\,\underline{v}}\right) I + \frac{v\underline{p}}{u\underline{u}^2\,\underline{v}}E^{-1} + \left(\frac{1}{u} - \frac{v\underline{p}}{u\underline{u}\,\underline{v}}\right)\triangle^{-1}\frac{1}{u}$$
$$-\,v\triangle^{-1}\left(\frac{p}{u^2 v\overline{u}} + \frac{\underline{p}}{u^2\underline{u}\,\underline{v}} - \frac{1}{u^2 v}\right), \tag{5.7d}$$

$$R_{22} = \frac{v\underline{p}}{u\underline{u}\,\underline{v}^2}E^{-1} + \left(\frac{1}{u} - \frac{v\underline{p}}{u\underline{u}\,\underline{v}}\right)\triangle^{-1}\frac{1}{v} + v\triangle^{-1}\left(\frac{1}{uv^2} - \frac{p}{uv^2\overline{u}}\right), \tag{5.7e}$$

$$R_{23} = -\frac{v}{u\underline{u}\,\underline{v}} + v\triangle^{-1}\frac{1}{uv\overline{u}}, \tag{5.7f}$$

$$R_{31} = \left(\frac{p^2}{u^2 v\overline{u}} - \frac{p\underline{p}}{u^2\underline{u}\,\underline{v}} - \frac{p}{u^2 v}\right) I + \frac{p^2}{uv\overline{u}^2}E + \left(\frac{p^2}{uv\overline{u}} - \frac{p}{uv}\right)\triangle^{-1}\frac{1}{u}$$
$$+\,p\triangle^{-1}\left(\frac{p}{u^2 v\overline{u}} + \frac{\underline{p}}{u^2\underline{uv}} - \frac{1}{u^2 v}\right), \tag{5.7g}$$

$$R_{32} = \left(\frac{p^2}{uv^2\overline{u}} - \frac{p}{uv^2}\right) I + \left(\frac{p^2}{uv\overline{u}} - \frac{p}{uv}\right)\triangle^{-1}\frac{1}{v} + p\triangle^{-1}\left(\frac{p}{uv^2\overline{u}} - \frac{1}{uv^2}\right), \tag{5.7h}$$

$$R_{33} = -\frac{1}{uv}I - p\triangle^{-1}\frac{1}{uv\overline{u}}. \tag{5.7i}$$

Proceeding as above, we have checked that (5.2) holds for $k=3,4,\ldots$ with the recursion operator given above. Thus we conclude that $\mathcal{R}$ with the entries $R_{ij},\ i,j=1,2,3$ given in (5.7) is a recursion operator for 3-coupled system (1.1).

5.2. *Recursion operator: 4-coupled system* (1.2)

For 4-component systems, (5.1) becomes

$$\begin{bmatrix} G_{k+1}^{(1)}(n) \\ G_{k+1}^{(2)}(n) \\ G_{k+1}^{(3)}(n) \\ G_{k+1}^{(4)}(n) \end{bmatrix} = \mathcal{R} \begin{bmatrix} G_{k}^{(1)}(n) \\ G_{k}^{(2)}(n) \\ G_{k}^{(3)}(n) \\ G_{k}^{(4)}(n) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix} \begin{bmatrix} G_{k}^{(1)}(n) \\ G_{k}^{(2)}(n) \\ G_{k}^{(3)}(n) \\ G_{k}^{(4)}(n) \end{bmatrix}, \tag{5.8}$$

where $\mathbf{G}_k(n) = (G_k^{(1)}(n), G_k^{(2)}(n), G_k^{(3)}(n), G_k^{(4)}(n))^{\mathrm{T}}$ and $\mathbf{G}_{k+1}(n) = (G_{k+1}^{(1)}(n), G_{k+1}^{(2)}(n), G_{k+1}^{(3)}(n), G_{k+1}^{(4)}(n))^{\mathrm{T}}$ are the generalized symmetries. For $k = 2$, (5.8) becomes

$$\begin{bmatrix} G_{3}^{(1)}(n) \\ G_{3}^{(2)}(n) \\ G_{3}^{(3)}(n) \\ G_{3}^{(4)}(n) \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix} \begin{bmatrix} G_{2}^{(1)}(n) \\ G_{2}^{(2)}(n) \\ G_{2}^{(3)}(n) \\ G_{2}^{(4)}(n) \end{bmatrix}. \tag{5.9}$$

Note that the rank of generalized symmetries $(G_2^{(1)}(n), G_2^{(2)}(n), G_2^{(3)}(n), G_2^{(4)}(n))^{\mathrm{T}}$ is (3,3,4,3) while the rank of $(G_3^{(1)}(n), G_3^{(2)}(n), G_3^{(3)}(n), G_3^{(4)}(n))^{\mathrm{T}}$ is (5,5,6,5). Thus it is clear that, for uniformity in rank the entries $R_{ij},\ i,j = 1,2,3,4$ must be of rank 2. Proceeding as above, after a tedious calculation, we find that (5.9) satisfies for the following forms of R_{ij},

$$\begin{aligned} R_{11} = {} & \left(\frac{q}{uv\overline{u}} + \frac{\underline{q}}{u\,\underline{u}\,\underline{v}} - \frac{2}{u\,v} + \frac{p}{\overline{u}v}\right) I + \left(\frac{up}{\overline{u}^2 v} + \frac{q}{\overline{u}^2 v}\right) E + \left(\frac{q}{\overline{u}v} + \frac{up}{\overline{u}v} - \frac{1}{v}\right) \triangle^{-1} \frac{1}{u} \\ & + u\triangle^{-1}\left(\frac{q}{u^2 v\overline{u}} + \frac{\underline{q}}{u^2\,\underline{u}\,\underline{v}} - \frac{1}{u^2\,v}\right), \end{aligned} \tag{5.10a}$$

$$R_{12} = \left(\frac{q}{\overline{u}v^2} + \frac{up}{\overline{u}v^2} - \frac{1}{v^2}\right) I + \left(\frac{q}{\overline{u}v} + \frac{up}{\overline{u}v} - \frac{1}{v}\right) \triangle^{-1}\frac{1}{v} + u\triangle^{-1}\left(\frac{q}{uv^2\overline{u}} - \frac{1}{u\,v^2}\right), \tag{5.10b}$$

$$R_{13} = 0, \tag{5.10c}$$

$$R_{14} = -u\triangle^{-1}\frac{1}{uv\overline{u}}, \tag{5.10d}$$

$$\begin{aligned} R_{21} = {} & \left(\frac{1}{u^2} - \frac{v\underline{q}}{u^2\underline{u}\,\underline{v}} - \frac{p}{\overline{u}u}\right) I + \frac{v\underline{q}}{u\underline{u}^2\,\underline{v}} E^{-1} - \frac{p}{\overline{u}^2} E + \left(\frac{1}{u} - \frac{v\underline{q}}{u\underline{u}\,\underline{v}} - \frac{p}{\overline{u}}\right) \triangle^{-1}\frac{1}{u} \\ & - v\triangle^{-1}\left(\frac{q}{u^2 v\overline{u}} + \frac{\underline{q}}{u^2\underline{u}\,\underline{v}} - \frac{1}{u^2 v}\right), \end{aligned} \tag{5.10e}$$

$$R_{22} = -\frac{p}{\overline{u}v} I + \frac{v\underline{q}}{u\underline{u}\,\underline{v}^2} E^{-1} + \left(\frac{1}{u} - \frac{v\underline{q}}{u\underline{u}\,\underline{v}} - \frac{p}{\overline{u}}\right) \triangle^{-1}\frac{1}{v} + v\triangle^{-1}\left(\frac{1}{uv^2} - \frac{q}{uv^2\overline{u}}\right), \tag{5.10f}$$

$$R_{23} = 0, \tag{5.10g}$$

$$R_{24} = -\frac{v}{u\underline{u}\,\underline{v}} + v\triangle^{-1}\frac{1}{uv\overline{u}}, \tag{5.10h}$$

$$R_{31} = \frac{pq}{u^2v\overline{u}}I + \frac{pq}{uv\overline{u}^2}E + \frac{p\underline{q}}{u\underline{u}^2\,\underline{v}}E^{-1} + \left(\frac{pq}{uv\overline{u}} - \frac{p\underline{q}}{u\overline{u}\,\underline{v}}\right)\triangle^{-1}\frac{1}{u}, \tag{5.10i}$$

$$R_{32} = \frac{pq}{uv^2\overline{u}}I + \frac{p\underline{q}}{u\underline{u}\,\underline{v}^2}E^{-1} + \left(\frac{pq}{uv\overline{u}} - \frac{p\underline{q}}{u\overline{u}\,\underline{v}}\right)\triangle^{-1}\frac{1}{v}, \tag{5.10j}$$

$$R_{33} = -\frac{1}{uv}I, \tag{5.10k}$$

$$R_{34} = -\frac{p}{u\underline{u}\,\underline{v}}E^{-1}, \tag{5.10l}$$

$$\begin{aligned} R_{41} = &\left(\frac{q^2}{u^2v\overline{u}} - \frac{q\underline{q}}{u^2\underline{u}\,\underline{v}} - \frac{q}{u^2v}\right)I - \frac{p\underline{q}}{\underline{u}^2\,\underline{v}}E^{-1} + \frac{q^2}{uv\overline{u}^2}E \\ &+ \left(\frac{q^2}{uv\overline{u}} + \frac{p\underline{q}}{\underline{u}\,\underline{v}} - \frac{q}{uv}\right)\triangle^{-1}\frac{1}{u} + q\triangle^{-1}\left(\frac{q}{u^2v\overline{u}} + \frac{\underline{q}}{u^2\underline{uv}} - \frac{1}{u^2v}\right), \end{aligned} \tag{5.10m}$$

$$\begin{aligned} R_{42} = &\left(\frac{q^2}{uv^2\overline{u}} - \frac{q}{uv^2}\right)I - \frac{p\underline{q}}{\underline{u}\,\underline{v}^2}E^{-1} + \left(\frac{q^2}{uv\overline{u}} + \frac{p\underline{q}}{\underline{u}\,\underline{v}} - \frac{q}{uv}\right)\triangle^{-1}\frac{1}{v} \\ &+ q\triangle^{-1}\left(\frac{q}{uv^2\overline{u}} - \frac{1}{uv^2}\right), \end{aligned} \tag{5.10n}$$

$$R_{43} = 0, \tag{5.10o}$$

$$R_{44} = -\frac{1}{uv}I + \frac{p}{\underline{u}\,\underline{v}} - q\triangle^{-1}\frac{1}{uv\overline{u}}. \tag{5.10p}$$

Also we have checked that (5.8) holds for $k = 3, 4, \ldots$ with the recursion operator given above. Thus we conclude that $\mathcal{R}$ with the entries $R_{ij},\ i, j = 1, 2, 3$ given in (5.10) is a recursion operator for 4-coupled system (1.2).

6. Summary

In this article, we report a new multicomponent nonlinear PDΔEs which are Hamiltonian ones admitting Lax representation, possessing infinitely many generalized symmetries, conserved quantities and recursion operator. Hence both of them are integrable in the sense of Lax and Liouville. One of the characteristics of integrable nonlinear PDΔEs with two independent variables is the existence of recursion operator which connects the consecutive members of the sequence of generalized symmetries [10].

Acknowledgments

The authors wish to thank the anonymous referee for his helpful and critical comments. One of the authors (S.B) would like to thank the Council of Scientific and Industrial Research (CSIR), Government of India, New Delhi for providing financial support in the form of Senior Research Fellow.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).
[2] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.* **53** (1974) 249.
[3] M. J. Ablowitz and J. F. Ladik, A nonlinear difference scheme and inverse scattering, *Stud. Appl. Math.* **55** (1976) 213.
[4] M. J. Ablowitz and J. F. Ladik, Nonlinear differential-difference equations and Fourier analysis, *J. Math. Phys.* **17** (1976) 1011.
[5] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM, Philadelphia, 1981).
[6] M. Blaszak and K. Marciniak, R-matrix approach to lattice integrable systems, *J. Math. Phys.* **35** (1994) 4661.
[7] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations* (Springer, Berlin, 1989).
[8] I. Yu. Cherdantsev and R. I. Yamilov, Master symmetries from differential-difference equations of the Volterra type, *Physica D* **87** (1995) 140.
[9] P. A. Damianou, Symmetries of Toda equations, *J. Phys. A* **26** (1993) 3791.
[10] A. S. Fokas, Symmetries and integrability, *Stud. Appl. Math.* **77** (1987) 253.
[11] B. Fuchssteiner, Master symmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Progr. Theoret. Phys.* **70** (1983) 1508.
[12] Ü. Göktaş and W. Hereman, Computation of conserved densities for nonlinear lattices, *Physica D* **123** (1998) 425.
[13] Ü. Göktaş, W. Hereman and G. Erdmann, Computation of conserved densities for systems of nonlinear differential-difference equations, *Phys. Lett. A* **236** (1997) 30.
[14] B. Grammaticos, Y. Kosmann Schwarzbach and K. M. Tamizhmani, *Discrete Integrable Systems*, Lecture Notes in Physics, Vol. 644 (Springer, New York, 2004).
[15] W. Hereman, J. A. Sanders, J. Sayers and J. P. Wang, in Group Theory and Numerical Analysis, eds. P. Winternitz *et al.*, CRM Proceedings and Lecture Series, Vol. 39 (American Mathematical Society, Providence, RI, 2005), pp. 267–282.
[16] N. Joshi, D. Burtonclay and R. G. Halburd, Nonlinear non-autonomous discrete dynamical system from a general discrete isomonodromy problem, *Lett. Math. Phys.* **26** (1992) 123.
[17] M. Lakshmanan and S. Rajasekar, *Nonlinear Dynamics: Integrability, Chaos, and Patterns* (Springer, 2003).
[18] D. Levi and R. I. Yamilov, Conditions for the existence of higher symmetries of evolutionary equations on the lattice, *J. Math. Phys.* **38** (1997) 6648.
[19] X. Li, Y. Zhang and Q. Zhao, Positive and negative integrable hierarchies, associated conservation laws and Darboux transformation, *J. Comput. Appl. Math.* **233** (2009) 1096.
[20] W. Oevel, H. Zhang and B. Fuchssteiner, Master symmetries and multi-Hamiltonian formulations for some integrable lattice systems, *Progr. Theoret. Phys.* **81** (1989) 294.
[21] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1993).
[22] R. Sahadevan and S. Balakrishnan, Similarity reduction, generalized symmetries, recursion operator, and integrability of coupled Volterra system, *J. Math. Phys.* **49** (2008) 113510.
[23] R. Sahadevan and S. Balakrishnan, Complete integrability of two-coupled discrete modified Korteweg–de Vries equations, *J. Phys. A: Math. Theor.* **42** (2009) 415208 (11pp).
[24] J. A. Sanders and J. P. Wang, *Talk Delivered During the Euro Workshop "Discrete Systems and Integrability"*, Isaac Newton Institute for Mathematical Sciences, University of Cambridge, Cambridge, UK.
[25] A. B. Shabat and R. I. Yamilov, Symmetries of nonlinear chains, *Leningrad Math. J* **2** (1991) 377.
[26] Y. Sun, D. Chen and X. Xu, A hierarchy of nonlinear differential-difference equations and a new Bragmann type integrable system, *Phys. Lett. A* **359** (2006) 47.

[27] K. M. Tamizhmani and S. Kanaga Vel, Differential-difference Kadomstev–Petviashvili equation: properties and integrability, *J. Indian Inst. Sci.* **78** (1998) 311.
[28] M. Toda, *Theory of Lattices* (Springer, New York, 1981).
[29] V. E. Zakharov and A. B. Shabat, Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering problem. II, *Funct. Anal. Appl.* **13** (1979) 166.
[30] Q. Zhao, X. Xu and X. Li, The Liouville integrability of integrable couplings of Volterra lattice equation, *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010) 1664.
[31] Q. Zhao and X. Wang, The integrable coupling system of a 3×3 discrete matrix spectral problem, *Appl. Math. Comput.* **216** (2010) 730.
[32] H.-Y. Zhi, Some new Lie symmetry groups of differential-difference equations obtained from a simple direct method, *Commun. Theor. Phys.* (*Beijing*) **52** (2009) 385.
[33] Z.-N. Zhu, X. Wu, W. Xue and Z.-M. Zhu, Infinitely many conservation laws for the Blaszak–Marciniak four-field integrable lattice hierarchy, *Phys. Lett. A* **296** (2002) 280.
[34] Z.-N. Zhu, X. Wu, W. Xue and Q. Ding, Infinitely many conservation laws of two Blaszak–Marciniak three-field lattice hierarchies, *Phys. Lett. A* **297** (2002) 387.