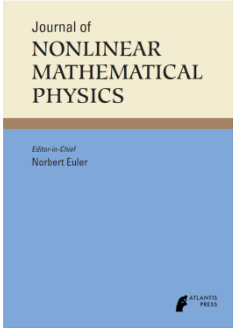

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GENERALIZATION OF AN INVERTIBLE TRANSFORMATION AND EXAMPLES OF ITS APPLICATIONS

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Recently we highlighted the remarkable nature of an *explicitly invertible* transformation, we reported some generalizations of it and examples of its expediency in several mathematical contexts: algebraic and Diophantine equations, dynamical systems (with continuous and discrete time), nonlinear PDEs, analytical geometry, functional equations. In this paper we report a significant generalization of this approach and we again illustrate via some analogous examples its expediency to identify problems which appear *far from trivial* but are in fact *explicitly solvable*.

Keywords: Explicitly invertible transformations; solvable algebraic equations; Diophantine equations; isochronous discrete-time dynamical systems; solvable systems of nonlinear PDEs; solvable nonautonomous PDEs; functional equations.

1. Introduction

This paper follows a previous article [1], in which we highlighted the remarkable nature of a simple, *explicitly invertible* transformation, we reported some generalizations of it and examples of its expediency in several mathematical contexts: algebraic and Diophantine equations, dynamical systems (with continuous and discrete time), nonlinear PDEs, analytical geometry, functional equations. In this paper we report a significant generalization of this approach and we again illustrate via some analogous examples its expediency to identify problems which appear *far from trivial* but are in fact *explicitly solvable*.

In the following Sec. 2 we tersely review the basic *invertible transformation* that is at the core of our treatment and we describe the generalization of it which constitutes the main contribution of this paper; some related developments are confined to the appendix. In Sec. 3 we report some representative examples of applications of this approach: the topics treated are identified by the titles of the subsections (note that, at the cost of some minor repetitions, the presentation allows the reader only interested in one of those topics to jump directly to the relevant subsection).

2. The *Explicitly Invertible* Transformation and Its Generalization

In this section we review tersely the original *explicitly invertible* transformation — without repeating the considerations about this approach proffered in [1] — and we then report its generalization.

The simplest avatar of the *explicitly invertible* transformation that provided the point of departure of our treatment [1] consists of a change of variables, involving two *arbitrary* functions $F_1(w), F_2(w)$, from two quantities u_1, u_2 to two quantities x_1, x_2 and *vice versa*. It reads as follows:

$$x_1 = u_1 + F_1(u_2), \quad x_2 = u_2 + F_2(x_1) = u_2 + F_2(u_1 + F_1(u_2)), \tag{2.1a}$$

$$u_2 = x_2 - F_2(x_1), \quad u_1 = x_1 - F_1(u_2) = x_1 - F_1(x_2 - F_2(x_1)). \tag{2.1b}$$

The most remarkable aspect of this transformation is its *explicitly invertible* character: note that both the *direct* respectively the *inverse* changes of variables, (2.1a) respectively (2.1b), involve *only* (albeit also in a nested manner) the two *arbitrary* functions $F_1(w), F_2(w)$, and *not* their inverses. This, for instance, implies that if the two functions $F_1(w), F_2(w)$ are both *polynomials*, then (the right-hand sides of) both the direct and the inverse transformations, (2.1a) and (2.1b), are *polynomials*.

The extensions of these transformations to more than two variables are rather obvious [1]; see also below.

Remark 2.1. The transformation (2.1a) can also be characterized as resulting from the sequential application of the following two ("lower triangular" respectively "upper triangular"), obviously invertible, "seed" transformations:

$$y_1 = u_1 + F_1(u_2), \quad y_2 = u_2, \tag{2.2a}$$

$$x_1 = y_1, \quad x_2 = y_2 + F_2(y_1). \tag{2.2b}$$

Since 1942 it is known that *all* invertible transformations of two variables to two variables that are *polynomial* in both directions are in fact obtainable by iterating such seed transformations [3] (terminology: they are *all* "tame"). Three decades later it was conjectured that this is *not* the case for three variables [4] ("Nagata conjecture": for three variables there exist automorphisms — *polynomial* in both direction — that are not tame, namely that are "wild"). And it took three more decades to validate this famous conjecture, by showing [6–8] that the *polynomial* Nagata transformation

$$x_1 = u_1 + (u_1^2 - u_2 u_3) u_3, \tag{2.3a}$$

$$x_2 = u_2 + (u_1^2 - u_2 u_3)[(u_1^2 - u_2 u_3) u_3 + 2u_1], \tag{2.3b}$$

$$x_3 = u_3, \tag{2.3c}$$

whose inversion,

$$u_1 = x_1 - (x_1^2 - x_2 x_3) x_3, \tag{2.4a}$$

$$u_2 = x_2 + (x_1^2 - x_2 x_3)[(x_1^2 - x_2 x_3) x_3 - 2x_1], \tag{2.4b}$$

$$u_3 = x_3, \tag{2.4c}$$

is clearly also *polynomial*, cannot be obtained by iterating seed transformations of the type

$$y_1 = u_1 + F_1(u_1, u_2), \quad y_2 = u_2, \quad y_3 = u_3, \tag{2.5a}$$

$$z_1 = y_1, \quad z_2 = y_2 + F_2(y_1, y_3), \quad z_3 = y_3, \tag{2.5b}$$

$$x_1 = z_1, \quad x_2 = z_2, \quad x_3 = z_3 + F_3(z_1, z_2). \tag{2.5c}$$

These seed transformations entail

$$x_1 = u_1 + F_1(u_2, u_3), \quad x_2 = u_2 + F_2(x_1, u_3), \quad x_3 = u_3 + F_3(x_1, x_2) \tag{2.6}$$

(of course in the right-hand side of the second of these three equations x_1 should be replaced by its expression given by the preceding formula, and likewise for x_1 and x_2 in the third equation). Note the coincidence of these equations with the Equations (3.5a) of [1].

To arrive at our generalization we take as point of departure two (or more) assumedly known *invertible transformations*, which we write in operatorial form as follows:

$$z = T_n \cdot y, \quad y = T_n^{-1} \cdot z, \quad n = 1, 2, \ldots. \tag{2.7a}$$

And let us assume that the direct and inverse versions, T_n and T_n^{-1}, of each of these transformations depend on an arbitrary number of parameters f_k, which themselves may be functions of another variable u (or possibly several variables u_j):

$$T_n \equiv T_n(f_{nk}), \quad f_{nk} \equiv f_{nk}(u_j). \tag{2.7b}$$

Hereafter we assume for simplicity that these functions $f_{nk}(u_j)$ are one-valued functions of their arguments.

For instance a simple example of *invertible* transformation ("Möbius"), from y to z and *vice versa*, reads as follows:

$$z = \frac{y f_1(u) + f_2(u)}{y f_3(u) + f_4(u)}, \quad y = -\frac{z f_4(u) - f_2(u)}{z f_3(u) - f_1(u)}, \tag{2.8a}$$

where the four *a priori arbitrary* functions $f_k(u)$ of the single variable u are only restricted by the condition that the combination $D(u) = f_1(u) f_4(u) - f_2(u) f_3(u)$ *not vanish identically*,

$$D(u) = f_1(u) f_4(u) - f_2(u) f_3(u) \neq 0 \tag{2.8b}$$

(indeed if $D(u)$ were to vanish identically, $z = f_2(u)/f_4(u) = f_1(u)/f_3(u)$ and $y = -f_2(u)/f_1(u) = -f_4(u)/f_3(u)$ would both become functions of u only, related to each other only via u, in a generally complicated manner).

Let us now consider the transformation — from two quantities u_1, u_2 to two quantities x_1, x_2 — reading as follows:

$$x_1 = T_1(f_{1k}(u_2)) \cdot u_1, \tag{2.9a}$$

$$x_2 = T_2(f_{2k}(x_1)) \cdot u_2 = T_2(f_{2k}(T_1(f_{1k}(u_2)) \cdot u_1)) \cdot u_2, \tag{2.9b}$$

which — under the above assumptions — can clearly be *explicitly inverted*, to read

$$u_1 = T_1^{-1}(f_{1k}(u_2)) \cdot x_1 = T_1^{-1}(f_{1k}(R_2^{-1}(f_{2k}(x_1)) \cdot x_2)) \cdot x_1, \tag{2.10a}$$

$$u_2 = T_2^{-1}(f_{2k}(x_1)) \cdot x_2. \tag{2.10b}$$

Remark 2.2. Note that both the direct transformation (2.9) and the inverse transformation (2.10) involve the functions $f_{nk}(w)$ but *not* their inverses.

For instance if the two transformations T_n are both of Möbius type, see (2.8), then the direct transformation (2.9) reads

$$x_1 = \frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}, \tag{2.11a}$$

$$\begin{aligned} x_2 &= \frac{u_2 f_{21}(x_1) + f_{22}(x_1)}{u_2 f_{23}(x_1) + f_{24}(x_1)}, \\ &= \left[u_2 f_{21}\left(\frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}\right) + f_{22}\left(\frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}\right)\right] \\ &\quad \cdot \left[u_2 f_{23}\left(\frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}\right) + f_{24}\left(\frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}\right)\right]^{-1}, \end{aligned} \tag{2.11b}$$

and the inverse transformation (2.10) reads

$$\begin{aligned} u_1 &= -\frac{x_1 f_{14}(u_2) - f_{12}(u_2)}{x_1 f_{13}(u_2) - f_{11}(u_2)}, \\ &= \left[-x_1 f_{14}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right) + f_{12}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right)\right] \\ &\quad \cdot \left[x_1 f_{13}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right) - f_{11}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right)\right]^{-1}, \end{aligned} \tag{2.12a}$$

$$u_2 = \frac{-x_2 f_{24}(x_1) + f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}. \tag{2.12b}$$

Note that these explicit transformations involve eight *a priori* arbitrary functions $f_{nk}(w), n = 1, 2, k = 1, 2, 3, 4$. Hereafter we assume for simplicity that *all* these functions are well defined, yielding a *unique* outcome for all input values of their argument w.

In the Appendix, we list other examples of elementary invertible transformations that can serve (in alternative to the Möbius transformation (2.8)) as starting point of this approach, and a few representative examples of invertible transformations obtained by combining such transformations. The alert reader will enjoy inventing many more.

Remark 2.3. In the treatment above we assumed the functions $f_{nk}(w)$ to depend on a single argument, and the generalized transformation to relate two quantities u_1, u_2 to two quantities x_1, x_2 and vice versa, see (2.9) and (2.10). The extension of this treatment to *explicitly invertible* transformations from N quantities u_n to N quantities x_n, $n = 1, 2, \ldots, N$, and vice versa, with N an arbitrary integer *larger* than two is rather obvious; they may involve arbitrary functions of $N - 1$ arguments. If in doubt about this development, see the analogous treatment in [1, Sec. 3] and see also some examples in the appendix, in particular the last one.

3. Applications

As in our previous paper [1], in this section we outline various applications of the *invertible transformation* described above. Generally we illustrate typical possibilities via quite elementary examples. The reader will have no difficulty to imagine and explore a multitude of additional examples. And let us reiterate that the reader interested in only one of the topics treated below can jump directly to the relevant subsection.

3.1. *Algebraic and Diophantine equations*

In this subsection we indicate — via a simple example — how highly nonlinear yet explicitly solvable algebraic equations can be manufactured using the *invertible transformation* introduced in the preceding section, see (2.9) and its inverse (2.10). We moreover show how in this manner *Diophantine* equations can also be identified, namely nonlinear algebraic equations featuring *integer* solutions; the assessment whether these *Diophantine* findings are interesting or trivial is left to the cognoscenti.

Let us take as point of departure the single *linear* equation in the two unknowns u_1, u_2 reading

$$a_1 u_1 + a_2 u_2 = b, \tag{3.1}$$

where the three numbers a_1, a_2, b are *a priori* arbitrarily assigned. The *general* solution of this equation clearly reads

$$u_1 = u, \quad u_2 = \frac{b - a_1 u}{a_2}, \tag{3.2a}$$

or equivalently

$$u_1 = \frac{b - a_2 u}{a_1}, \quad u_2 = u, \tag{3.2b}$$

where u is an *arbitrary* number. But if one is interested in the *Diophantine* version of this simple Eq. (3.1) — namely in the consideration of this equation and its solutions in the context of *integer* numbers, requiring *all* the quantities appearing in them to be *integers* — then it is *necessary and sufficient* for the existence of a solution that the largest common divisor of the two *integers* a_1, a_2 be also a divisor of b. It is then easy to show (see, for instance, [5]) that there certainly exist a *Diophantine* solution — $u_1 = v_1$, $u_2 = v_2$, with v_1 and v_2 *integers* — of (3.1),

$$a_1 v_1 + a_2 v_2 = b \tag{3.3a}$$

and the *general Diophantine solution* of (3.1) then clearly reads

$$u_1 = v_1 + a_2 z, \quad u_2 = v_2 - a_1 z, \tag{3.3b}$$

with z an *arbitrary integer*.

Let us now apply to the simple Eq. (3.1) our transformations.

The first case we consider obtains by employing the *linear-linear transformation* (see the appendix) with all the functions $f_{nk}(w)$ also *linear*,

$$f_{nk}(w) = \alpha_{nk} w + \beta_{nk}, \quad n = 1,2, \;\; k = 1,2. \tag{3.4}$$

Via (the inverse version (A.7) of) this transformation Eq. (3.1) becomes (after a bit of standard algebra, including the elimination of a common denominator)

$$c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1^2 + c_4 x_1 x_2 + c_5 x_2^2 + c_6 x_1^3 = 0, \tag{3.5}$$

where

$$c_0 = (b\beta_{21} + a_2\beta_{22})(\alpha_{11}\beta_{22} - \beta_{11}\beta_{21}) + a_1\beta_{21}(\alpha_{12}\beta_{22} - \beta_{12}\beta_{21}), \tag{3.6a}$$

$$c_1 = (b + a_1)(\alpha_{11}\alpha_{22}\beta_{21} - 2\alpha_{21}\beta_{11}\beta_{21} + \alpha_{11}\alpha_{21}\beta_{22}) + a_1\beta_{21}^2 - a_2(\alpha_{22}\beta_{11}\beta_{21} - 2\alpha_{11}\alpha_{22}\beta_{22} + \alpha_{21}\beta_{11}\beta_{22}), \tag{3.6b}$$

$$c_2 = -b\alpha_{11}\beta_{21} - a_1\alpha_{12}\beta_{21} + a_2(\beta_{11}\beta_{21} - 2\alpha_{11}\beta_{22}), \tag{3.6c}$$

$$c_3 = (b\alpha_{21} + a_2\alpha_{22})(\alpha_{11}\alpha_{22} - \alpha_{21}\beta_{11}) + a_1\alpha_{21}(\alpha_{12}\alpha_{22} - \alpha_{21}\beta_{12} + 2\beta_{21}), \tag{3.6d}$$

$$c_4 = -b\alpha_{11}\alpha_{21} - a_1\alpha_{12}\alpha_{21} + a_2(\alpha_{21}\beta_{11} - 2\alpha_{11}\alpha_{22}), \tag{3.6e}$$

$$c_5 = a_2\alpha_{11}, \quad c_6 = a_1\alpha_{21}^2. \tag{3.6f}$$

And, via (A.6) with (3.3b) (and of course (3.3a)), one can assert that the cubic equation (3.5) has the explicit solution

$$x_1 = (v_2 - a_1 z)\left[\alpha_{11}(v_1 + a_2 z) + \alpha_{12}\right] + (v_1 + a_2 z)\beta_{11} + \beta_{12}, \tag{3.7a}$$

$$x_2 = [(v_2 - a_1 z)\alpha_{21} + \alpha_{22}]\{(v_2 - a_1 z)[\alpha_{11}(v_1 + a_2 z) + \alpha_{12}] + (v_1 + a_2 z)\beta_{11} + \beta_{12}\} + (v_2 - a_1 z)\beta_{21} + \beta_{22}, \tag{3.7b}$$

namely x_1 (respectively, x_2) is a polynomial function of z of second (respectively, third) degree. To write this solution we employed, rather than (3.2) (solution of (3.1)), the equivalent version (3.3b) (solution of (3.1) with (3.3a)), because it is more appropriate to discuss the more interesting *Diophantine* case, see below. Note however that, in the general (*non-Diophantine*) context of *real* (or, for that matter, of *rational*) numbers, the 12 numbers z, b and $a_n, \alpha_{nm}, \beta_{nm}$ with $n = 1,2, m = 1,2$ appearing in (3.5) and (3.7) can *all* be *arbitrarily* assigned, while v_1 and v_2 in (3.7) satisfy the condition (3.3a), thus only one of these two parameters can be *arbitrarily* assigned, say v_1; moreover the solution, in this general context of *real* (or, for that matter, *rational*) numbers, depends in fact on only one *arbitrary* parameter, say on $u = v_1 + a_2 z$, not separately on v_1 and z (see (3.2a) and (3.3b)). If instead the parameters z, b, a_n, α_{nm} and β_{nm} are *all* restricted to be *integers*, and we impose v_1 and v_2 to be *integers*, then (3.3a) becomes a trivial linear *Diophantine* equation for v_1 and v_2, whose solution requires that the greatest common divisor of a_1 and a_2 divides b, see above. If such a condition is satisfied, then clearly the cubic equation (3.5) is *Diophantine* as well, since the solutions (3.7) are then clearly *integer* numbers.

Equation (3.5) can be easily recast in the so-called Weierstrass form for an elliptic curve over a general field [2]

$$\tilde{x}_2^2 + \tilde{c}_4 \tilde{x}_1 \tilde{x}_2 + \tilde{c}_2 \tilde{x}_2 = \tilde{x}_1^3 + \tilde{c}_3 \tilde{x}_1^2 + \tilde{c}_1 \tilde{x}_1 + \tilde{c}_0, \tag{3.8}$$

via the rescaling

$$x_1 = -c_5 c_6 \tilde{x}_1, \quad x_2 = c_5 c_6^2 \tilde{x}_2, \tag{3.9a}$$

$$\tilde{c}_0 = -\frac{c_0}{c_5^3 c_6^4}, \quad \tilde{c}_1 = \frac{c_1}{c_5^2 c_6^3}, \quad \tilde{c}_2 = \frac{c_2}{c_5^2 c_6^2}, \quad \tilde{c}_3 = -\frac{c_3}{c_5 c_6^2}, \quad \tilde{c}_4 = -\frac{c_4}{c_5 c_6}. \tag{3.9b}$$

It is then immediate to observe that, for any choice of the 12 parameters z, b, a_n, α_{nm} and β_{nm} in a generic field (for instance $\mathbb{Q}$), the elliptic discriminant Δ of (3.8), associated to (3.5) via (3.9), vanishes,

$$\begin{aligned}\Delta = &-27(4\tilde{c}_0 + \tilde{c}_2^2)^2 - 8(2\tilde{c}_1 + \tilde{c}_2\tilde{c}_4)^3 + 9(4\tilde{c}_0 + \tilde{c}_2^2)(2\tilde{c}_1 + \tilde{c}_2\tilde{c}_4)(4\tilde{c}_3 + \tilde{c}_4^2) \\ &+ (4\tilde{c}_3 + \tilde{c}_4^2)^2[(4\tilde{c}_0 + \tilde{c}_2^2)\tilde{c}_3 + \tilde{c}_0\tilde{c}_4^2 - \tilde{c}_1(\tilde{c}_1 + \tilde{c}_2\tilde{c}_4)] = 0.\end{aligned} \tag{3.10}$$

Hence the variety associated to the cubic curve (3.5) is *singular* and the cubic (3.5) is *not* an elliptic curve.

Remark 3.1. This is actually a straightforward consequence of the fact that the cubic (3.8) is obtained from a *plane*, as defined by (3.1). Since nonsingular elliptic curves are *tori*, the impossibility to obtain such an elliptic curve by our approach is clear on topological grounds.

Indeed the expressions of the seven parameters $c_j, j = 0, 1, \ldots, 6$, in terms of the 11 parameters b, a_n, α_{nm}, β_{nm} is such, see (3.6), that the freedom to assign arbitrarily the latter does not entail the freedom to assign arbitrarily the former; as it happens, the seven parameters c_j are constrained by one (and only one) condition. This can be seen if one tries and expresses the 11 parameters b, a_n, α_{nm}, β_{nm} in terms of the seven parameters c_j, by inverting the expressions (3.6). A convenient way to do so (out of many possible ones) yields the following formulas:

$$a_1 = \frac{c_6}{\alpha_{21}^2}, \quad a_2 = \frac{c_5}{\alpha_{11}}, \tag{3.11a}$$

$$\beta_{11} = \frac{\alpha_{11}[(\alpha_{11} v_1 + \alpha_{12})c_6 + \alpha_{21}(\alpha_{21} v_2 + 2\alpha_{22})c_5 + \alpha_{21} c_4]}{\alpha_{21}^2 c_5}, \tag{3.11b}$$

$$\beta_{12} = -\frac{\gamma_{12}}{\alpha_{21}^2 c_5 c_6}, \quad \beta_{2,1} = \frac{\alpha_{21}(4c_3c_5 - c_4^2 + R)}{12c_5c_6}, \tag{3.11c}$$

$$\beta_{22} = \frac{\gamma_{22}}{24c_5^2c_6}, \tag{3.11d}$$

where

$$\begin{aligned}\gamma_{12} = {} & (\alpha_{11}v_1c_6 + \alpha_{21}\alpha_{22}c_5)^2 + \alpha_{21}(\alpha_{11}v_1c_6 + \alpha_{21}\alpha_{22}c_5)c_4 \\ & + 2\alpha_{21}^2(\alpha_{11}v_1c_6 + \alpha_{21}\alpha_{22}c_5)v_2c_5 + \alpha_{11}\alpha_{12}v_1c_6^2 \\ & + \alpha_{21}^4v_2^2c_5^2 + \alpha_{21}^3v_2c_4c_5 + \alpha_{21}^2\alpha_{12}v_2c_5c_6 + (1/6)(2c_3c_5 + c_4^2 - R)\alpha_{21}^2, \end{aligned} \tag{3.11e}$$

$$\gamma_{22} = 2\alpha_{22}(4c_3c_5 - c_4^2)c_5 + (2\alpha_{22}c_5 + c_4)R + 4c_3c_4c_5 - c_4^3 - 12c_2c_5c_6 \tag{3.11f}$$

and

$$R^2 = (c_4^2 - 4c_3c_5)^2 - 24c_5c_6(2c_1c_5 - c_2c_4). \tag{3.11g}$$

Note that all the parameters appearing in the right-hand sides of these expressions remain free: this includes the two parameters v_1, v_2 and the four parameters α_{nm} (except for the conditions $\alpha_{11} \neq 0, \alpha_{21} \neq 0$), as well of course as the six parameters $c_1, c_2, c_3, c_4, c_5, c_6$. On the other hand the parameter c_0 is given in terms of the other six parameters $c_1, c_2, c_3, c_4, c_5, c_6$ by the following formula:

$$c_0 = \frac{-R^3 + (c_4^2 - 4c_3c_5)[R^2 + 12c_5c_6(c_2c_4 - 2c_1c_5)] + 216c_2^2c_5^2c_6^2}{864c_5^3c_6^2}, \tag{3.12}$$

and by inserting these values in the solution formula (3.7) this solution takes the neat form

$$x_1 = -c_5c_6\zeta^2 + c_4\zeta - \frac{c_3}{3c_6} + \frac{R - c_4^2}{6c_5c_6}, \tag{3.13a}$$

$$x_2 = c_5c_6^2\zeta^3 - c_4c_6\zeta^2 - \frac{R - c_4^2}{4c_5}\zeta - \frac{c_2}{2c_5} + c_4\left(\frac{c_3}{6c_6} + \frac{R - c_4^2}{24c_5^2c_6}\right), \tag{3.13b}$$

where

$$\zeta = \frac{z}{\alpha_{21}} - \frac{(\alpha_{22} + \alpha_{21}v_2)}{c_6}, \tag{3.13c}$$

is an arbitrary parameter, and R is defined by (3.11g).

There are two interesting cases when the algebraic equation (3.5) with (3.6) is *Diophantine*, namely all its coefficients are *integers* and it features a nontrivial class of *integer* solutions.

The first case is characterized by the following assignments:

$$c_1 = \lambda c_4, \quad c_2 = 2\lambda c_5, \quad c_3 = 0, \tag{3.14a}$$

entailing

$$R = c_4^2, \quad c_0 = c_5\lambda^2. \tag{3.14b}$$

Above and below λ is an *arbitrary integer*. Then, after conveniently setting

$$x_1 = x, \quad x_2 = y - \lambda, \tag{3.14c}$$

the algebraic equation (3.5) reads

$$c_4xy + c_5y^2 + c_6x^3 = 0, \tag{3.14d}$$

which is *Diophantine* for any choice of c_4, c_5, c_6 as *arbitrary integers* and which features the solution

$$x = -c_5c_6\zeta^2 + c_4\zeta, \tag{3.14e}$$

$$y = c_5c_6^2\zeta^3 - c_4c_6\zeta^2 = -c_6\zeta x, \tag{3.14f}$$

with ζ an *arbitrary integer*.

The second case is characterized by the following assignments:

$$c_1 = \lambda c_4, \quad c_2 = 2\lambda c_5, \quad c_3 = 3\mu c_5c_6^2, \quad c_4 = 6\rho c_5c_6. \tag{3.15a}$$

entailing

$$R = \pm 12c_5^2c_6^2(\mu - 3\rho^2). \tag{3.15b}$$

Above and below μ and ρ (like λ) are two *arbitrary integers* (of course $\mu \neq 0$). We are then left with two cases depending on the determination of the sign of R in (3.15b).

If the sign in front of the expression of R in (3.15b) is taken to be positive, then

$$c_0 = c_5[\lambda^2 - 4c_5^2c_6^4(\mu - 3\rho^2)^3], \tag{3.15c}$$

and, after conveniently making again the simple change of variables (3.14c), the algebraic equation (3.5) reads

$$6c_5c_6\rho xy + c_5y^2 + c_6x^3 + 3c_5c_6^2\mu x^2 - 4c_5^3c_6^4(\mu - 3\rho^2)^3 = 0. \tag{3.15d}$$

Therefore, if ρ, c_5, c_6, μ are *arbitrary integers*, this equation becomes *Diophantine* and its solutions x, y are clearly *integer numbers*:

$$x = c_5c_6(-\zeta^2 + 6\rho\zeta + \mu - 12\rho^2), \tag{3.15e}$$

$$y = c_5c_6^2[\zeta^3 - 6\rho\zeta^2 - 3(\mu - 6\rho^2)\zeta + 6\rho(\mu - 3\rho^2)], \tag{3.15f}$$

with ζ an *arbitrary integer*.

On the other hand, if the sign in front of the expression of R in (3.15b) is taken to be negative, then

$$c_0 = c_5\lambda^2, \tag{3.15g}$$

and, after conveniently making again the simple change of variables (3.14c), the algebraic equation (3.5) reads

$$6c_5c_6\rho xy + c_5y^2 + c_6x^3 + 3c_5c_6^2\mu x^2 = 0. \tag{3.15h}$$

As above, if ρ, c_5, c_6, μ are *arbitrary integers*, this equation is *Diophantine* and its solutions x, y are clearly *integer numbers*:

$$x = c_5c_6(-\zeta^2 + 6\rho\zeta - 3\mu), \tag{3.15i}$$

$$y = c_5c_6^2(\zeta^3 - 6\rho\zeta^2 + 3\mu\zeta), \tag{3.15j}$$

with ζ an *arbitrary integer*.

This ends our treatment of the algebraic equation obtained from (3.1) or (3.3a) via the *linear–linear* transformation (A.7) with (3.4).

A second possibility we tersely consider employs the transformation (2.12) (instead of (A.7)). Thereby Eq. (3.1) becomes

$$\begin{aligned} a_1 \Bigg\{ \Bigg[-x_1 f_{14}\left(-\frac{x_2f_{24}(x_1)-f_{22}(x_1)}{x_2f_{23}(x_1)-f_{21}(x_1)}\right) + f_{12}\left(-\frac{x_2f_{24}(x_1)-f_{22}(x_1)}{x_2f_{23}(x_1)-f_{21}(x_1)}\right)\Bigg] \\ \cdot \Bigg[x_1 f_{13}\left(-\frac{x_2f_{24}(x_1)-f_{22}(x_1)}{x_2f_{23}(x_1)-f_{21}(x_1)}\right) - f_{11}\left(-\frac{x_2f_{24}(x_1)-f_{22}(x_1)}{x_2f_{23}(x_1)-f_{21}(x_1)}\right)\Bigg]^{-1} \Bigg\} \\ + a_2\left[\frac{-x_2f_{24}(x_1)+f_{22}(x_1)}{x_2f_{23}(x_1)-f_{21}(x_1)}\right] = b, \end{aligned} \tag{3.16}$$

and via (2.11) with (3.3b) with (3.3a) one can assert that the (explicit!) solution of this equation reads

$$x_1 = \frac{(v_1+a_2z)f_{11}(v_2-a_1z)+f_{12}(v_2-a_1z)}{(v_1+a_2z)f_{13}(v_2-a_1z)+f_{14}(v_2-a_1z)}, \tag{3.17a}$$

$$\begin{aligned} x_2 = \Bigg[(v_2-a_1z)f_{21}\left(\frac{(v_1+a_2z)f_{11}(v_2-a_1z)+f_{12}(v_2-a_1z)}{(v_1+a_2z)f_{13}(v_2-a_1z)+f_{14}(v_2-a_1z)}\right) \\ + f_{22}\left(\frac{(v_1+a_2z)f_{11}(v_2-a_1z)+f_{12}(v_2-a_1z)}{(v_1+a_2z)f_{13}(v_2-a_1z)+f_{14}(v_2-a_1z)}\right)\Bigg] \\ \cdot \Bigg[(v_2-a_1z)f_{23}\left(\frac{(v_1+a_2z)f_{11}(v_2-a_1z)+f_{12}(v_2-a_1z)}{(v_1+a_2z)f_{13}(v_2-a_1z)+f_{14}(v_2-a_1z)}\right) \\ + f_{24}\left(\frac{(v_1+a_2z)f_{11}(v_2-a_1z)+f_{12}(v_2-a_1z)}{(v_1+a_2z)f_{13}(v_2-a_1z)+f_{14}(v_2-a_1z)}\right)\Bigg]^{-1}, \end{aligned} \tag{3.17b}$$

where z is again an arbitrary number. This conclusion is of course true for any arbitrary assignment of the eight functions $f_{nk}(y)$.

It is moreover plain that, if the four numbers a_1, a_2, v_1, v_2 are *rational* (implying that b is as well rational, see (3.3a)) and the eight functions $f_{nk}(y)$ are all *rational functions with rational coefficients*, then Eq. (3.16) has an infinity of *rational* solutions, as given by (3.17) with z an arbitrary *rational* number.

3.2. *A class of* isochronous *discrete-time dynamical systems*

We display a class of discrete-time dynamical systems describing the nonlinear evolution of two dependent variables $x_1(\ell), x_2(\ell)$, functions of the *integer* independent variable $\ell = 0, 1, 2, \ldots$ (the "discrete-time"), which are explicitly *solvable* and have moreover the remarkable property that — if and only if the arbitrary *real* constant λ (featured by this class of systems, see below) is a *rational* number,

$$\lambda = \frac{L}{M} \tag{3.18}$$

with L and M two arbitrary co-prime *integers* (hereafter $M \neq 0$ and $L > 0$) — they are *isochronous*: *all* their solutions are then *periodic* with the *fixed* period L,

$$x_n(\ell + L) = x_n(\ell), \quad n = 1, 2. \tag{3.19}$$

As it shall be clear, see below, this class of systems is merely one out of a quite large family of such systems that can be manufactured by extensions of the technique described above: for instance, via sequential applications of the invertible transformations described above. Such systems can involve just two variables, as that exhibited below, or a *larger* number.

The system we now exhibit obtains by applying the transformation (2.12) and its inverse (2.11) to the trivial discrete-time linear dynamical system

$$\tilde{u}_1 = cu_1 - su_2, \quad \tilde{u}_2 = su_1 + cu_2. \tag{3.20a}$$

Here and below $\tilde{u}_n(\ell) \equiv u_n(\ell + 1)$ (and likewise $\tilde{x}_n(\ell) \equiv x_n(\ell + 1)$, see below) and

$$c = \cos\left(\frac{2\pi}{\lambda}\right), \quad s = \sin\left(\frac{2\pi}{\lambda}\right), \tag{3.20b}$$

entailing that the solution of the initial-value problem of (3.20a) reads

$$u_1(\ell) = \cos\left(\frac{2\pi\ell}{\lambda}\right) u_1(0) - \sin\left(\frac{2\pi\ell}{\lambda}\right) u_2(0), \tag{3.21a}$$

$$u_2(\ell) = \sin\left(\frac{2\pi\ell}{\lambda}\right) u_1(0) + \cos\left(\frac{2\pi\ell}{\lambda}\right) u_2(0). \tag{3.21b}$$

Hence clearly this system is *isochronous*,

$$u_n(\ell + L) = u_n(\ell), \quad n = 1, 2, \tag{3.22}$$

if and only if the parameter λ is rational, see (3.18).

It is then a matter of trivial algebra to obtain the corresponding system for the variables $x_1(\ell), x_2(\ell)$ related to $u_1(\ell), u_2(\ell)$ via (2.12) and its inverse (2.11). It reads

$$\begin{aligned} \tilde{x}_1 = {} & [(cu_1 - su_2) f_{11}(su_1 + cu_2) + f_{12}(su_1 + cu_2)] \\ & \cdot [(cu_1 - su_2) f_{13}(su_1 + cu_2) + f_{14}(su_1 + cu_2)]^{-1}, \end{aligned} \tag{3.23a}$$

$$\tilde{x}_2 = [(cu_1 - su_2) f_{11}(\tilde{x}_1) + f_{12}(\tilde{x}_1)] \cdot [(cu_1 - su_2) f_{13}(\tilde{x}_1) + f_{14}(\tilde{x}_1)]^{-1}. \tag{3.23b}$$

In these two equations $u_1 \equiv u_1(\ell), u_2 \equiv u_2(\ell)$ should be expressed in terms of $x_1 \equiv x_1(\ell), x_2(\ell) \equiv x_2(\ell)$ by (2.12), and then, in the second of these two equations, $\tilde{x}_1$ should be replaced by its expression provided by the first. Thereby one obtains the equations of (discrete) motion of the new dynamical system: two (quite explicit if complicated) equations expressing $\tilde{x}_1$ and $\tilde{x}_2$ in terms of x_1 and x_2, via formulas involving (in a nested manner) the eight arbitrary functions $f_{nk}(w), n = 1, 2, k = 1, 2, 3, 4$, appearing in (2.12) and its inverse (2.11). And the, also quite explicit, solution of the initial-value problem for this discrete-time dynamical system is provided by the formulas (2.11) with $u_1 \equiv u_1(\ell), u_2 \equiv u_2(\ell)$ given by (3.21) where $u_1(0), u_2(0)$ are given in terms of the *arbitrary* initial data $x_1(0), x_2(0)$ by (2.12). It is of course plain that this explicit solution entails the *isochrony* property (3.19) whenever the *real* constant λ is rational, see (3.18).

It is also plain, see (3.21), that the original (discrete-time) dynamical system (3.20a) yields, as solutions of its initial-value problem, points $u_1 \equiv u_1(\ell), u_2 \equiv u_2(\ell)$ lying, in the u_1u_2-plane, on the circle identified by the equation

$$u_1^2(\ell) + u_2^2(\ell) = u_1^2(0) + u_2^2(0). \tag{3.24}$$

Hence, see (2.12), the points in the x_1x_2-plane corresponding to the solutions of the initial-value problem of the new discrete-time dynamical system (3.23) lie on the curve characterized by the equation

$$K(\ell) = K(0) \tag{3.25a}$$

with

$$\begin{aligned} K(\ell) = {} & \left[x_1 f_{14}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right) - f_{12}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right)\right]^2 \\ & \cdot \left[x_1 f_{13}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right) - f_{11}\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right)\right]^{-2} \\ & + \left[\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right]^2, \end{aligned} \tag{3.25b}$$

where of course $x_1 \equiv x_1(\ell)$, $x_2 \equiv x_2(\ell)$.

Let us finally emphasize that this class of discrete-time dynamical systems (3.23) is large since it involves eight *arbitrary* functions; the explicit display of examples corresponding to specific assignments of these eight functions are left to the whim of the alert reader. A number of such examples for the special case with $f_{13}(w) = f_{23}(w) = 0$ and $f_{11}(w) = f_{14}(w) = f_{21}(w) = f_{24}(w) = 1$ are provided in [1, Sec. 4.4].

3.3. Solvable *systems of autonomous* nonlinear *partial differential equations*

This subsection is analogous to [1, Sec. 4.5]. The presentation is therefore quite terse, although it reports an example somewhat more general than that presented there.

Indeed, as in [1, Sec. 4.5], we take as point of departure the trivial system of two linear PDEs

$$\varphi_{1,t} = \varphi_{2,x}, \quad \varphi_{2,t} = \varphi_{1,x}, \tag{3.26}$$

where the two functions $\varphi_n \equiv \varphi_n(x,t)$ depend on the two variables x and t. Here and below subscripted variables indicate partial differentiation with respect to them, $\varphi_{n,t}(x,t) \equiv \partial\varphi_n(x,t)/\partial t, \varphi_{n,x}(x,t) \equiv \partial\varphi_n(x,t)/\partial x$. Clearly this system of two linear PDEs has the following *general* solution:

$$\varphi_1(x,t) = \Phi_+(x,t) \equiv \Phi_1(x+t) + \Phi_2(x-t), \tag{3.27a}$$

$$\varphi_2(x,t) = \Phi_-(x,t) \equiv \Phi_1(x+t) - \Phi_2(x-t), \tag{3.27b}$$

where $\Phi_1(z)$ and $\Phi_2(z)$ are two *arbitrary* functions of the single variable z.

If we now apply the transformation (2.12) and its inverse (2.11) — with the two quantities u_1, u_2 replaced by the two functions $\varphi_1(x,t), \varphi_2(x,t)$ and, likewise, the two quantities x_1, x_2 replaced by two functions $\psi_1(x,t), \psi_2(x,t)$ — we find that the system of two linear PDEs (3.26) becomes a, generally quite nonlinear, system of two first-order PDEs for the two functions $\psi_1(x,t), \psi_2(x,t)$. And it is plain that this system features, as *general* solution, the explicit expressions (2.11) with the two quantities x_1, x_2 in the left-hand side replaced by the two functions $\psi_1(x,t), \psi_2(x,t)$ and the two quantities u_1, u_2 in the right-hand side replaced by the two functions $\varphi_1(x,t), \varphi_2(x,t)$ given, in terms of the two *arbitrary* functions $\Phi_1(z)$ and $\Phi_2(z)$, by the simple formulas (3.27).

We leave the explicit display of this system as a task for the diligent reader: a trivial exercise, but yielding quite cumbersome formulas featuring the eight *arbitrary* functions $f_{nk}(w), n = 1,2, k = 1,2,3,4$. We instead limit our presentation here to displaying the specific example corresponding to the following assignment:

$$f_{11}(w) = f_{14}(w) = \frac{c_4 + c_1c_3 + w(c_3 + c_1c_4)}{c_3^2 - c_4^2}, \tag{3.28a}$$

$$f_{12}(w) = f_{13}(w) = -\frac{c_2(c_3 + wc_4)}{c_3^2 - c_4^2}, \tag{3.28b}$$

$$f_{21}(w) = f_{24}(w) = c_3 w, \quad f_{22}(w) = f_{23}(w) = c_4 w, \tag{3.28c}$$

with c_1, c_2, c_3 and c_4 arbitrary constants (of course with $c_3^2 \neq c_4^2$). Then the two functions $\psi_1(x,t)$ and $\psi_2(x,t)$ satisfy the following system of two coupled nonlinear PDEs:

$$\psi_{1,t} = \frac{1}{\beta}[\alpha\psi_{1,x} + (\beta^2 - \alpha^2)\psi_{2,x}], \tag{3.29a}$$

$$\psi_{2,t} = \frac{1}{\beta}(\psi_{1,x} - \alpha\psi_{2,x}), \tag{3.29b}$$

where

$$\alpha = \frac{c_2(1 - \psi_1^2)}{(c_1 + c_2 + \psi_2)(c_1 - c_2 + \psi_2)}, \tag{3.29c}$$

$$\beta = \frac{(c_3^2 - c_4^2)(c_1 + c_2\psi_1 + \psi_2)}{(c_1 + c_2 + \psi_2)(c_1 - c_2 + \psi_2)(c_3 - c_4\psi_2)^2}. \tag{3.29d}$$

And we assert that the *general* solution $\psi_1 \equiv \psi_1(x,t), \psi_2 \equiv \psi_2(x,t)$ of this system reads as follows:

$$\psi_1 = \frac{c_2c_3 - (c_1c_3 + c_4)\Phi_+ + c_2c_4\Phi_- - (c_1c_4 + c_3)\Phi_+\Phi_-}{-c_1c_3 - c_4 + c_2c_3\Phi_+ - (c_1c_4 + c_3)\Phi_- + c_2c_4\Phi_+\Phi_-}, \tag{3.30a}$$

$$\psi_2 = \frac{c_4 + c_3\Phi_-}{c_3 + c_4\Phi_-}, \tag{3.30b}$$

where $\Phi_\pm \equiv \Phi_\pm(x,t)$ are of course defined by (3.27) in terms of the two *arbitrary* functions $\Phi_1(z)$ and $\Phi_2(z)$.

3.4. *Solvable* nonautonomous *partial differential equations*

Here we show via two representative examples how to manufacture *solvable nonautonomous* nonlinear partial differential equations (PDEs).

We start from the trivial (linear) *autonomous* PDE

$$\varphi_u(u,w) = \varphi_w(u,w), \tag{3.31a}$$

the *general* solution of which reads of course

$$\varphi(u,w) = F(u+w), \tag{3.31b}$$

where F is an arbitrary function. In (3.31a) and below subscripted variables denote of course partial differentiations, for instance $\varphi_u(u \cdot w) \equiv \partial\varphi(u \cdot w)/\partial u$.

We then set

$$\varphi(u,w) = \psi(x,y), \tag{3.32}$$

with x, y related to u, w by the relations (2.11) and (2.12), with x_1, x_2 replaced (for notational simplicity) by x, y and likewise u_1, u_2 replaced by u, w.

It is then a matter of trivial if tedious algebra to ascertain that $\psi(x,y)$ satisfies the following (linear) *nonautonomous* PDE:

$$g(x,y)\psi_x(x,y) = h(x,y)\psi_y(x,y), \tag{3.33a}$$

with $g(x,y)$ and $h(x,y)$ expressed as follows in terms of the eight arbitrary functions $f_{nk}(z)$, $n = 1,2,\ k = 1,2,3,4$:

$$g(x,y) = [f_{23}(x)w + f_{24}(x)]^2\chi(u,w), \tag{3.33b}$$

$$\begin{aligned} h(x,y) = \{&[f'_{23}(x)w + f'_{24}(x)][f_{21}(x)w + f_{22}(x)] \\ &- [f'_{21}(x)w + f'_{22}(x)][f_{23}(x)w + f_{24}(x)]\}\chi(u,w) \\ &+ [f_{13}(w)u + f_{14}(w)]^2[f_{21}(x)f_{24}(x) - f_{22}(x)f_{23}(x)], \end{aligned} \tag{3.33c}$$

where

$$\begin{aligned} \chi(u,w) = f_{11}(w)f_{14}(w) - f_{12}(w)f_{13}(w) + [f'_{13}(w)u + f'_{14}(w)][f_{11}(w)u + f_{12}(w)] \\ - [f'_{11}(w)u + f'_{12}(w)][f_{13}(w)u + f_{14}(w)], \end{aligned} \tag{3.33d}$$

and u and w must be expressed in terms of x and y as follows:

$$u = \left[-x f_{14}\left(-\frac{y f_{24}(x) - f_{22}(x)}{y f_{23}(x) - f_{21}(x)}\right) + f_{12}\left(-\frac{y f_{24}(x) - f_{22}(x)}{y f_{23}(x) - f_{21}(x)}\right)\right] \cdot \left[x f_{13}\left(-\frac{y f_{24}(x) - f_{22}(x)}{y f_{23}(x) - f_{21}(x)}\right) - f_{11}\left(-\frac{y f_{24}(x) - f_{22}(x)}{y f_{23}(x) - f_{21}(x)}\right)\right]^{-1}, \tag{3.33e}$$

$$w = \frac{-y f_{24}(x) + f_{22}(x)}{y f_{23}(x) - f_{21}(x)}. \tag{3.33f}$$

The *general* solution of this *nonautonomous* PDE then reads as follows:

$$\psi(x, y) = F(u + w), \tag{3.34}$$

where $F(z)$ is an arbitrary function and u, w are expressed in terms of x, y by the two preceding formulas (3.33e) and (3.33f).

As an example we set

$$f_{11}(z) = z, \quad f_{12}(z) = f_{13}(z) = 0, \quad f_{14}(z) = 1,$$
$$f_{21}(z) = f_{24}(z) = \cos(z), \quad f_{23}(z) = -f_{22}(z) = \sin(z). \tag{3.35a}$$

Then (3.33a) holds with

$$g(x, y) = (x - 1)(y^2 + 1) - (x + 1)[(y^2 - 1)\cos(2x) + 2y\sin(2x)], \tag{3.35b}$$

$$h(x, y) = y(y^2 - 1)\cos(4x) - (1 - 6y^2 + y^4)\frac{\sin(4x)}{4} - [-1 + y + y^3 + y^4 + x(y^4 - 1)]\cos(2x) - (1 + y^2)[1 + 4(1 + x) - y^2]\frac{\sin(2x)}{2} + (x - 1)(1 + y^2)^2. \tag{3.35c}$$

And its general solution reads as follows:

$$\psi(x, y) = F\left(\frac{x\,[\cos(x) - y\sin(x)]}{y\cos(x) + \sin(x)} + \frac{y\cos(x) + \sin(x)}{\cos(x) - y\sin(x)}\right), \tag{3.35d}$$

with $F(z)$ an arbitrary function.

On the face of it, the fact that the PDE (3.33a) with this assignment of $g(x, y)$ and $h(x, y)$ is *explicitly solvable* might well appear quite nontrivial to anybody who does not know how this finding had been arrived at; although verifying it is a relatively trivial task.

To manufacture our second example, we start from the same trivial PDE (3.31a), but for notational convenience we now write it as follows:

$$\varphi_t(u, t) = \varphi_u(u, t), \tag{3.36a}$$

the *general* solution of which reads of course

$$\varphi(u, t) = f(u + t), \tag{3.36b}$$

where we now denote as $f(z)$ an *arbitrary* function.

We then use the most elementary invertible transformation (2.1), in the following guise:

$$\psi(x,t) = \varphi(u,t) + F_1(u), \quad x = u + F_2(\psi(x,t)), \tag{3.37a}$$

$$\varphi(u,t) = \psi(x,t) - F_1(u), \quad u = x - F_2(\psi(x,t)), \tag{3.37b}$$

where of course $F_1(w)$ and $F_2(w)$ are two arbitrary functions.

It is then a matter of standard (if a bit tricky) algebra to obtain the PDE satisfied by the new dependent variable $\psi(x,t)$. It reads as follows:

$$\psi_t(x,t) = \psi_x(x,t) + [\psi_x(x,t)F_2'(\psi(x,t)) - 1]F_1'(x - F_2(\psi(x,t))), \tag{3.38}$$

where of course the primes denote differentiation with respect to the argument of the function they are appended to. And clearly the *general* solution of this (quasilinear, first-order) PDE is the solution of the following nondifferential equation, which is a rather immediate consequence of the transformation (3.37) and of the solution formula (3.36b):

$$\psi(x,t) = f(x + t - F_2(\psi(x,t))) + F_1(x - F_2(\psi(x,t))), \tag{3.39}$$

where of course $f(z)$ is an *arbitrary* function. Note however that in this case, in contrast to the previous one, the solution is provided only in *implicit* form, i.e. as the solution of this nondifferential equation (in contrast to the previous case, when the solution is given by the *explicit* formula (3.34) with (3.33e) and (3.33f)).

3.5. *Functional equations*

In this section we report an, apparently nontrivial, functional equation involving two functions, as an example of the kind of findings obtainable via the approach advertised in this paper. It reads as follows:

$$x_1(z_1 + z_2) = \frac{u_1(z_1)u_1(z_2)f_{11}(u_2(z_1) + u_2(z_2)) + f_{12}(u_2(z_1) + u_2(z_2))}{u_1(z_1)u_1(z_2)f_{13}(u_2(z_1) + u_2(z_2)) + f_{14}(u_2(z_1) + u_2(z_2))}, \tag{3.40a}$$

$$x_2(z_1 + z_2) = \frac{[u_2(z_1) + u_2(z_2)]f_{21}(x_1(z_1 + z_2)) + f_{22}(x_1(z_1 + z_2))}{[u_2(z_1) + u_2(z_2)]f_{23}(x_1(z_1 + z_2)) + f_{24}(x_1(z_1 + z_2))}, \tag{3.40b}$$

where, in the two preceding formulas, firstly $u_1(z)$ should be replaced by the following expression in terms of $x_1(z)$ and $u_2(z)$,

$$u_1(z) = -\frac{x_1(z)f_{14}(u_2(z)) - f_{12}(u_2(z))}{x_1(z)f_{13}(u_2(z)) - f_{11}(u_2(z))}, \tag{3.40c}$$

and subsequently $u_2(z)$ should be replaced by the following expression in terms of $x_1(z)$ and $x_2(z)$,

$$u_2(z) = -\frac{x_2(z)f_{24}(x_1(z)) - f_{22}(x_1(z))}{x_2(z)f_{23}(x_1(z)) - f_{21}(x_1(z))}, \tag{3.40d}$$

so that the resulting formulas relate (*explicitly*, if in a convoluted manner) the values that the two functions $x_1(z)$ and $x_2(z)$ take at the value $z = z_1 + z_2$ of their argument, to the values they take at z_1 and at z_2 (where z_1 and z_2 are of course *arbitrary*).

The (explicit) solution of this functional equation reads as follows:

$$x_1(z) = \frac{\exp(bz) f_{11}(az) + f_{12}(az)}{\exp(bz) f_{13}(az) + f_{14}(az)}, \tag{3.41a}$$

$$x_2(z) = \left[az f_{21} \left(\frac{\exp(bz) f_{11}(az) + f_{12}(az)}{\exp(bz) f_{13}(az) + f_{14}(az)} \right) + f_{22} \left(\frac{\exp(bz) f_{11}(az) + f_{12}(az)}{\exp(bz) f_{13}(az) + f_{14}(az)} \right) \right] \cdot \left[az f_{23} \left(\frac{\exp(bz) f_{11}(az) + f_{12}(az)}{\exp(bz) f_{13}(az) + f_{14}(az)} \right) + f_{24} \left(\frac{\exp(bz) f_{11}(az) + f_{12}(az)}{\exp(bz) f_{13}(az) + f_{14}(az)} \right) \right]^{-1}, \tag{3.41b}$$

where a and b are two *arbitrary* parameters.

This finding clearly obtains via the transformations (2.11) and (2.12) from the two trivial functional equations

$$u_1(z_1 + z_2) = u_1(z_1) u_1(z_2), \quad u_2(z_1 + z_2) = u_2(z_1) + u_2(z_2), \tag{3.42a}$$

whose solutions of course read

$$u_1(z) = \exp(bz), \quad u_2(z) = az. \tag{3.42b}$$

Let us again emphasize that the eight functions $f_{nk}(w)$ appearing in the functional equation (3.40) and in its solution (3.41) are *arbitrary*. And again the reader might get some amusement by inserting in these formulas specific assignments of these eight functions.

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Appendix

In this appendix we list a few, quite elementary, invertible transformations that may be taken as points of departure for our approach (say, in alternative to the Möbius transformation (2.8)), and a few examples of invertible transformations involving arbitrary functions, obtained by combining such transformations.

A.1. *Elementary invertible transformations*

We list them — first the direct transformation and then its inverse — with minimal comments, but assigning to them a name. The quantities f_k can depend on an arbitrary number of variables u_{nj} (see the following subsection), but for notational simplicity this is not explicitly shown in this subsection.

Linear transformation:

$$z = L \cdot y = yf_1 + f_2, \quad y = L^{-1} \cdot z = \frac{z - f_2}{f_1}. \tag{A.1}$$

Exponential transformation:

$$z = E \cdot y = \exp(yf_1 + f_2), \quad y = E^{-1} \cdot z = \frac{\ln(z) - f_2}{f_1}. \tag{A.2}$$

Rational (Möbius) transformation:

$$z = R \cdot y = \frac{yf_1 + f_2}{yf_3 + f_4}, \quad y = R^{-1} \cdot z = -\frac{zf_4 - f_2}{zf_3 - f_1}, \tag{A.3}$$

with the condition that $D = f_1 f_4 - f_2 f_3$ not vanish identically (see (2.8)).

Note that this transformation reduces to the *linear transformation* (A.1) for $f_3 = 0$, $f_4 = 1$.

Matrix transformation:

$$\vec{z} = M \cdot \vec{y}, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} f_5 \\ f_6 \end{pmatrix}, \tag{A.4a}$$

$$z_1 = y_1 f_1 + y_2 f_2 + f_5, \quad z_2 = y_1 f_3 + y_2 f_4 + f_6; \tag{A.4b}$$

$$\vec{y} = M^{-1} \cdot \vec{z}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} f_4 & -f_2 \\ -f_3 & f_1 \end{pmatrix} \begin{pmatrix} z_1 - f_5 \\ z_2 - f_6 \end{pmatrix}, \tag{A.5a}$$

$$y_1 = \frac{(z_1 - f_5) f_4 - (z_2 - f_6) f_2}{D}, \quad y_2 = \frac{-(z_1 - f_5) f_3 + (z_2 - f_6) f_1}{D} \tag{A.5b}$$

$$D = f_1 f_4 - f_2 f_3 \neq 0. \tag{A.5c}$$

Note that — in contrast to the *rational* (*Möbius*) case, see above — now, to avoid the occurrence of a singularity in the inverse transformation (A.5), it is not sufficient that the quantity D not vanish *identically*; the condition (A.5c) should hold for *all* (relevant) values of the arguments of the four functions f_n. Also note that this transformation is linear, and that (for simplicity) we have restricted attention to a matrix of order two.

A.2. *Combined transformations*

In this subsection we list, again with minimal commentary, some invertible transformations obtained by combining the elementary transformations listed in the preceding subsection.

Linear–linear transformation:

$$x_1 = u_1 f_{11}(u_2) + f_{12}(u_2), \tag{A.6a}$$

$$\begin{aligned} x_2 &= u_2 f_{21}(x_1) + f_{22}(x_1) \\ &= u_2 f_{21}(u_1 f_{11}(u_2) + f_{12}(u_2)) + f_{22}(u_1 f_{11}(u_2) + f_{12}(u_2)); \end{aligned} \tag{A.6b}$$

$$u_1 = \left[x_1 - f_{12}\left(\frac{x_2 - f_{22}(x_1)}{f_{21}(x_1)} \right) \right] \left[f_{11}\left(\frac{x_2 - f_{22}(x_1)}{f_{21}(x_1)} \right) \right]^{-1}, \tag{A.7a}$$

$$u_2 = \frac{x_2 - f_{22}(x_1)}{f_{21}(x_1)}. \tag{A.7b}$$

Linear–exponential transformation:

$$x_1 = u_1 f_1(u_2) + f_2(u_2), \tag{A.8a}$$

$$\begin{aligned} x_2 &= \exp(f_{21}(x_1) + u_2 f_{22}(x_1)) \\ &= \exp(f_{21}(u_1 f_{11}(u_2) + f_{12}(u_2)) + u_2 f_{22}(u_1 f_{11}(u_2) + f_{12}(u_2))); \end{aligned} \tag{A.8b}$$

$$u_1 = \left[x_1 - f_{12}\left(\frac{\ln(x_2) - f_{21}(x_1)}{f_{22}(x_1)}\right)\right]\left[f_{11}\left(\frac{\ln(x_2) - f_{21}(x_1)}{f_{22}(x_1)}\right)\right]^{-1}, \tag{A.9a}$$

$$u_2 = \frac{\ln(x_2) - f_{21}(x_1)}{f_{22}(x_1)}. \tag{A.9b}$$

Exponential–linear transformation:

$$x_1 = \exp(u_1 f_{11}(u_2) + f_{12}(u_2)), \tag{A.10a}$$

$$\begin{aligned} x_2 &= u_2 f_{21}(x_1) + f_{22}(x_1) \\ &= u_2 f_{21}(\exp(u_1 f_1(u_2) + f_2(u_2))) + f_{22}(u_1 f_1(u_2) + f_2(u_2)); \end{aligned} \tag{A.10b}$$

$$u_1 = \left[\ln(x_1) - f_{11}\left(\frac{x_2 - f_{22}(x_1)}{f_{21}(x_1)}\right)\right]\left[f_{11}\left(\frac{x_2 - f_{22}(x_1)}{f_{21}(x_1)}\right)\right]^{-1}, \tag{A.11a}$$

$$u_2 = \frac{x_2 - f_{22}(x_1)}{f_{21}(x_1)}. \tag{A.11b}$$

Note that the *exponential–linear transformation* is quite different from the *linear–exponential transformation*.

Rational–rational transformation: see the formulas (2.11) and (2.12). We do not rewrite them, but display some subcases.

Linear–rational transformation:

$$x_1 = u_1 f_{11}(u_2) + f_{12}(u_2), \tag{A.12a}$$

$$\begin{aligned} x_2 &= \frac{u_2 f_{21}(x_1) + f_{22}(x_1)}{u_2 f_{23}(x_1) + f_{24}(x_1)} \\ &= \frac{u_2 f_{21}(u_1 f_{11}(u_2) + f_{12}(u_2)) + f_{22}(u_1 f_{11}(u_2) + f_{12}(u_2))}{u_2 f_{23}(u_1 f_{11}(u_2) + f_{12}(u_2)) + f_{24}(u_1 f_{11}(u_2) + f_{12}(u_2))}; \end{aligned} \tag{A.12b}$$

$$u_1 = \left[x_1 - f_2\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right)\right]\left[f_1\left(-\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}\right)\right]^{-1}, \tag{A.13a}$$

$$u_2 = -\frac{x_2 f_{24}(x_1) - f_{22}(x_1)}{x_2 f_{23}(x_1) - f_{21}(x_1)}. \tag{A.13b}$$

Rational–linear transformation:

$$x_1 = \frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)} \tag{A.14a}$$

$$\begin{aligned} x_2 &= u_2 f_1(x_1) + f_2(x_1) \\ &= u_2 f_1\left(\frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}\right) + f_2\left(\frac{u_1 f_{11}(u_2) + f_{12}(u_2)}{u_1 f_{13}(u_2) + f_{14}(u_2)}\right); \end{aligned} \tag{A.14b}$$

$$u_1 = -\left[x_1 f_{14}\left(\frac{x_2 - f_2(x_1)}{f_1(x_1)}\right) - f_{12}\left(\frac{x_2 - f_2(x_1)}{f_1(x_1)}\right)\right] \cdot \left[x_1 f_{13}\left(\frac{x_2 - f_2(x_1)}{f_1(x_1)}\right) - f_{11}\left(\frac{x_2 - f_2(x_1)}{f_1(x_1)}\right)\right]^{-1}, \tag{A.15a}$$

$$u_2 = \frac{x_2 - f_2(x_1)}{f_1(x_1)}. \tag{A.15b}$$

Note the difference among the *linear–rational* and the *rational–linear* transformations. *Rational–matrix transformation*:

$$x_1 = \frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}, \tag{A.16a}$$

$$\begin{aligned} x_2 &= u_2 f_{21}(x_1) + u_3 f_{22}(x_1) + f_{25}(x_1) \\ &= u_2 f_{21}\left(\frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}\right) + u_3 f_{22}\left(\frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}\right) \\ &\quad + f_{25}\left(\frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}\right), \end{aligned} \tag{A.16b}$$

$$\begin{aligned} x_3 &= u_2 f_{23}(x_1) + u_3 f_{24}(x_1) + f_{26}(x_1) \\ &= u_2 f_{23}\left(\frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}\right) + u_3 f_{24}\left(\frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}\right) \\ &\quad + f_{26}\left(\frac{u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3)}{u_1 f_{13}(u_2, u_3) + f_{14}(u_2, u_3)}\right); \end{aligned} \tag{A.16c}$$

$$u_1 = -\frac{x_1 f_{14}(u_2, u_3) - f_{12}(u_2, u_3)}{x_1 f_{13}(u_2, u_3) - f_{11}(u_2, u_3)}, \tag{A.17a}$$

$$u_2 = \frac{f_{24}(x_1)[x_2 - f_{25}(x_1)] - f_{22}(x_1)[x_3 - f_{26}(x_1)]}{f_{21}(x_1) f_{24}(x_1) - f_{22}(x_1) f_{23}(x_1)}, \tag{A.17b}$$

$$u_3 = \frac{-f_{24}(x_1)[x_2 - f_{25}(x_1)] + f_{21}(x_1)[x_3 - f_{26}(x_1)]}{f_{21}(x_1) f_{24}(x_1) - f_{22}(x_1) f_{23}(x_1)}. \tag{A.17c}$$

In the right-hand side of the first of the three Eqs. (A.17) the quantities u_2, u_3 should be replaced by their expressions in terms of x_1, x_2, x_3 given by the last two of these three formulas.

Note the ("*linear-matrix*") subcases that obtains by setting $f_{13} = 0$ and $f_{14} = 1$.

Matrix–rational transformation:

$$x_1 = u_1 f_{11}(u_3) + u_2 f_{12}(u_3) + f_{15}(u_3), \tag{A.18a}$$

$$x_2 = u_1 f_{13}(u_3) + u_2 f_{14}(u_3) + f_{16}(u_3), \tag{A.18b}$$

$$x_3 = \frac{u_3 f_{21}(x_1, x_2) + f_{22}(x_1, x_2)}{u_3 f_{23}(x_1, x_2) + f_{24}(x_1, x_2)}; \tag{A.18c}$$

$$u_1 = \frac{[x_1 - f_{15}(u_3)]f_{14}(u_3) - [x_2 - f_{16}(u_3)]f_{12}(u_3)}{f_{11}(u_3)f_{14}(u_3) - f_{12}(u_3)f_{13}(u_3)}, \tag{A.19a}$$

$$u_2 = \frac{-[x_1 - f_{15}(u_3)]f_{13}(u_3) + [x_2 - f_{16}(u_3)]f_{11}(u_3)}{f_{11}(u_3)f_{14}(u_3) - f_{12}(u_3)f_{13}(u_3)}, \tag{A.19b}$$

$$u_3 = -\frac{x_3 f_{24}(x_1, x_2) - f_{22}(x_1, x_2)}{x_3 f_{23}(x_1, x_2) - f_{21}(x_1, x_2)}. \tag{A.19c}$$

In the right-hand side of the last one of the three formulas (A.18) x_1 and x_2 should of course be replaced by their expressions in terms of u_1, u_2, u_3 given by the first two of these three formulas; likewise in the right-hand sides of the first two formulas (A.19) u_3 should be replaced by its expression in terms of x_1, x_2, x_3 given by the last one of these three formulas.

Note the ("*matrix-linear*") subcase that obtains by setting $f_{23} = 0$ and $f_{24} = 1$.

And let us emphasize the difference of this *matrix–rational transformation* from the preceding *rational–matrix transformation*.

Matrix–matrix transformation:

$$x_1 = u_1 f_{11}(u_3, u_4) + u_2 f_{12}(u_3, u_4) + f_{15}(u_3, u_4), \tag{A.20a}$$

$$x_2 = u_1 f_{13}(u_3, u_4) + u_2 f_{14}(u_3, u_4) + f_{16}(u_3, u_4), \tag{A.20b}$$

$$x_3 = u_3 f_{21}(x_1, x_2) + u_4 f_{22}(x_1, x_2) + f_{25}(x_1, x_2), \tag{A.20c}$$

$$x_4 = u_3 f_{23}(x_1, x_2) + u_4 f_{24}(x_1, x_2) + f_{26}(x_1, x_2); \tag{A.20d}$$

$$u_1 = \frac{[x_1 - f_{15}(u_3, u_4)]f_{14}(u_3, u_4) - [x_2 - f_{16}(u_3, u_4)]f_{12}(u_3, u_4)}{f_{11}(u_3, u_4)f_{14}(u_3, u_4) - f_{12}(u_3, u_4)f_{13}(u_3, u_4)}, \tag{A.21a}$$

$$u_2 = \frac{-[x_1 - f_{15}(u_3, u_4)]f_{13}(u_3, u_4) + [x_2 - f_{16}(u_3, u_4)]f_{11}(u_3, u_4)}{f_{11}(u_3, u_4)f_{14}(u_3, u_4) - f_{12}(u_3, u_4)f_{13}(u_3, u_4)}, \tag{A.21b}$$

$$u_3 = \frac{[x_3 - f_{25}(x_1, x_2)]f_{24}(x_1, x_2) - [x_4 - f_{26}(x_1, x_2)]f_{22}(x_1, x_2)}{f_{21}(x_1, x_2)f_{24}(x_1, x_2) - f_{22}(x_1, x_2)f_{23}(x_1, x_2)}, \tag{A.21c}$$

$$u_4 = \frac{-[x_3 - f_{25}(x_1, x_2)]f_{23}(x_1, x_2) + [x_4 - f_{26}(x_1, x_2)]f_{21}(x_1, x_2)}{f_{21}(x_1, x_2)f_{24}(x_1, x_2) - f_{22}(x_1, x_2)f_{23}(x_1, x_2)}. \tag{A.21d}$$

In the right-hand sides of the last two of the four formulas (A.20) the quantities x_1, x_2 should of course be replaced by their expressions in terms of u_1, u_2, u_3, u_4 provided by the first two of these four formulas; likewise in the right-hand sides of the first two of the four formulas (A.21) the quantities u_3, u_4 should be replaced by their expressions in terms of x_1, x_2, x_3, x_4 provided by the last two of these four formulas.

Linear–linear–linear transformation:

$$x_1 = u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3), \tag{A.22a}$$

$$x_2 = u_2 f_{21}(u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3), u_3) + f_{22}(u_1 f_{11}(u_2, u_3) + f_{12}(u_2, u_3), u_3), \tag{A.22b}$$

$$x_3 = u_3 f_{31}(x_1, x_2) + f_{32}(x_1, x_2); \tag{A.22c}$$

$$u_1 = \frac{x_1 - f_{12}(u_2, u_3)}{f_{11}(u_2, u_3)}, \tag{A.23a}$$

$$u_2 = \left[x_2 - f_{22}\left(x_1, \frac{x_3 - f_{32}(x_1, x_2)}{f_{31}(x_1, x_2)}\right)\right]\left[f_{21}\left(x_1, \frac{x_3 - f_{32}(x_1, x_2)}{f_{31}(x_1, x_2)}\right)\right]^{-1}, \tag{A.23b}$$

$$u_3 = \frac{x_3 - f_{32}(x_1, x_2)}{f_{31}(x_1, x_2)}. \tag{A.23c}$$

In the right-hand side of the last of the three formulas (A.22) x_1 and x_2 should of course be replaced by their explicit expressions in terms of u_1, u_2, u_3 provided by the first two of these three formulas; and likewise in the right-hand side of the first of the three formulas (A.23) u_2 and u_3 should be replaced by their explicit expressions in terms of x_1, x_2, x_3 provided by the last two of these three formulas.

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