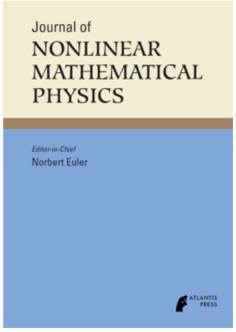


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# THE TRANSFORMATION BETWEEN THE AKNS HIERARCHY AND THE KN HIERARCHY WITH SELF-CONSISTENT SOURCES

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It is shown that the AKNS hierarchy with self-consistent sources can transform to KN hierarchy with self-consistent sources through a transformation operator and gauge transformation. Besides, there exists transformation in their conservation laws and Hamiltonian structures.

*Keywords*: The AKNS hierarchy; the KN hierarchy; self-consistent sources; transformation.

Mathematics Subject Classification 2000: 35Q51, 37K40, 47J35

## 1. Introduction

We consider the transformation between the AKNS hierarchy with self-consistent sources (AKNSSCSH) and the KN hierarchy with self-consistent sources (KNSCSH). The

AKNSSCSH is

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = K_n - \sum_{j=1}^{N} \eta_j \begin{pmatrix} \phi_{1j}^2 \\ \phi_{2j}^2 \end{pmatrix}, \quad (n = 0, 1, 2, \dots), \tag{1.1a}$$

$$\phi_{1j,x} = -\eta_j \phi_{1j} + q\phi_{2j}, \quad \phi_{2j,x} = r\phi_{1j} + \eta_j \phi_{2j}, \quad (j = 1, 2, \dots, N), \tag{1.1b}$$

where

$$K_n = L^n \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (n = 0, 1, 2, \dots), \quad L = \sigma\partial - 2\sigma u \partial^{-1}(r, q), \quad \partial = \frac{\partial}{\partial x}, \tag{1.2}$$

$K_n$ the isospectral AKNS flows, $L$ is the AKNS recursion operator. The KNSCSH is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \partial L'^{n-1} \begin{pmatrix} u \\ v \end{pmatrix} - \frac{1}{2} \sum_{j=1}^{N} \eta_j \begin{pmatrix} -\psi_{1j}^2 \\ \psi_{2j}^2 \end{pmatrix}_x, \quad (n = 0, 1, 2, \dots), \tag{1.3a}$$

$$\psi_{1j,x} = -\eta_j^2 \psi_{1j} + \eta_j u \psi_{2j}, \quad \psi_{2j,x} = \eta_j v \psi_{1j} + \eta_j^2 \psi_{2j}, \quad (j = 1, 2, \dots, N), \tag{1.3b}$$

where

$$L' = \sigma\partial - \begin{pmatrix} u \\ v \end{pmatrix} \partial^{-1}(v, u)\partial, \tag{1.4}$$

is the KN recursion operator. Both hierarchies can be found in [23] and references therein. They have been studied through the constraint flows [11, 23], source-generating approach [3, 20] and introducing a new time variable in the frame of Sato' theory [12, 15]. Several methods are used to solve those equations including the inverse scattering transform (IST) [1, 2, 11, 23], Daboux transform [13, 14, 21, 24], bilinear method [3, 20, 25, 26], etc. Among those methods, the IST has been successfully applied to solve many hierarchies with self-consistent sources, such as the KdVSCS hierarchy, the AKNSSCS hierarchy, the mKdVSCS hierarchy, the NLSSCS hierarchy, the KNSCS hierarchy and the DNLSSCS hierarchy [11, 23], as well as in the nonisospectral case [8, 9]. For example, the sources in the AKNSSCS hierarchy, squared eigenfunctions [4, 5, 22], is the square of the Jost functions $\phi_j(x,t)$ $(j = 1, 2, \dots, N)$, which can be solved via the IST. The IST is based on the approach to the spectral analysis of a linear problem and the soliton equations are the compatibility condition of the linear problem. Besides, there are relation among the IST, the Bäcklund transform and the infinitely many conservation laws [18]. The infinitely many conservation laws and the Hamiltonian structure of the soliton equation with self-consistent sources are investigated [7, 10].

In the letter, motivated by the relation between the sources and the eigenfunctions, we would like to consider the transformation between the AKNSSCSH and the KNSCSH. Making use of a transformation, gauge transformations, we deduce the relation between the AKNSSCSH and the KNSCSH. Besides, the transformation between their infinitely many conservation laws and their Hamiltonian structures are decribed.

## 2. The Transform Between the AKNSSCSH and the KNSCSH

First, we begin with the ZS-AKNS spectral problem with the time evolution [1, 23]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = \boldsymbol{M}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix}-\lambda & q\\ r & \lambda\end{pmatrix}, \tag{2.1a}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t = \boldsymbol{U}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix}A & B\\ C & -A\end{pmatrix}. \tag{2.1b}$$

From the related zero curvature equation

$$\boldsymbol{M}_t - \boldsymbol{U}_x + [\boldsymbol{M}, \boldsymbol{U}] = 0, \tag{2.2}$$

one can derive the AKNSSCSH (1.1) and the relation

$$L\begin{pmatrix}\phi_{1j}^2\\ \phi_{2j}^2\end{pmatrix} = 2\eta_j^2\begin{pmatrix}\phi_{1j}^2\\ \phi_{2j}^2\end{pmatrix}. \tag{2.3}$$

From the spectral problem with the time evolution [23]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = \boldsymbol{M'}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \boldsymbol{M'} = \begin{pmatrix}-\eta^2 & \eta u\\ \eta v & \eta^2\end{pmatrix}, \tag{2.4a}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = \boldsymbol{U'}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \boldsymbol{U'} = \begin{pmatrix}A' & B'\\ C' & -A'\end{pmatrix} \tag{2.4b}$$

and the related zero curvature equation

$$\boldsymbol{M'}_t - \boldsymbol{U'}_x + [\boldsymbol{M'}, \boldsymbol{U'}] = 0, \tag{2.5}$$

one can derived the KNSCSH (1.3) and the relation

$$L'\begin{pmatrix}-\psi_{1j}^2\\ \psi_{2j}^2\end{pmatrix} = 2\eta_j^2\begin{pmatrix}-\psi_{1j}^2\\ \psi_{2j}^2\end{pmatrix}. \tag{2.6}$$

In order to get the relation between the AKNSSCSH and the KNSCSH, one should take the eigenfunction transform

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = T\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad T = \begin{pmatrix}-\dfrac{\gamma}{\eta^2} & \dfrac{u}{2\eta^2\gamma}\\ 0 & \dfrac{1}{\eta\gamma}\end{pmatrix}, \quad \gamma = e^{-\frac{1}{2}\partial^{-1}uv}. \tag{2.7}$$

Such transformation is called the gauge transformation [19]. It follows that the spectral problem (2.1) can transform to the spectral problem (2.4). In the following, we give the transformation between the AKNSSCSH and the KNSCSH.

**Theorem 2.1.** *The AKNSSCSH (1.1) and the KNSCSH (1.3) have the relation*

$$SL'\partial^{-1}\begin{pmatrix}u\\ v\end{pmatrix}_t = L\begin{pmatrix}q\\ r\end{pmatrix}_t, \tag{2.8}$$

*where*

$$S = \begin{pmatrix} \frac{1}{2\gamma^2}\partial & \frac{1}{4\gamma^2}u^2 \\ 0 & \gamma^2 \end{pmatrix} \tag{2.9}$$

*is called the transformation operator.*

The proof can be finished by some direct computations where one may make use of (2.3) and the relation

$$\boldsymbol{U} = T^{-1}\boldsymbol{U'}T - T^{-1}T_t, \quad LS = SL'. \tag{2.10}$$

## 3. The Transformation Between the Infinitely Many Conservation Laws of the AKNSSCSH and the KNSCSH

In the section, we turn to deduce the transformation between the infinitely many conservation laws of the two hierarchies. Let us recall the infinitely conservation laws of the AKNSSCSH briefly. Setting

$$w(x,\eta) = \frac{\phi_2}{\phi_1}, \tag{3.1}$$

into the spectral problem (2.1a), we have

$$qw_x(x,\eta) = -q^2w^2(x,\eta) + 2\eta w(x,\eta) + qr. \tag{3.2}$$

Expanding

$$qw(x,\eta) = \sum_{n=0}^{\infty} \frac{w_n(x,\eta)}{(2\eta^2)^{n+1}} \tag{3.3}$$

and inserting into (3.2) yield

$$w_0(x) = -qr, \quad w_1(x) = -qr_x, \tag{3.4a}$$

$$w_2(x) = q^2r^2 - qr_{xx}, \quad w_3(x) = -qr_{xxx} + qr^2q + 4q^2rr_x, \tag{3.4b}$$

$$w_{n+1}(x) = q\left(\frac{w_n(x)}{q}\right)_x + \sum_{j=0}^{n-1} w_j(x)w_{n-j-1}(x), \quad (n = 1,2,3,\ldots). \tag{3.4c}$$

From (2.4) and compatible condition $\phi_{x,t} = \phi_{t,x}$, we have

$$[qw(x,\eta)]_t = [A + Bw(x,\eta)]_x, \tag{3.5}$$

which is the infinitely conservation laws of the AKNSSCSH.

Now we deduce the conservation laws of KNSCSH from (2.7) and (3.5) directly. Denoting that

$$w'(x,\eta) = \frac{\psi_2}{\psi_1}. \tag{3.6}$$

Using (3.1) and (3.6), the gauge transformation (2.7) changes to

$$qw(x,\eta) = \eta u w'(x,\eta) + \frac{1}{2}uv. \tag{3.7}$$

Substituting (2.10) and (3.7) into (3.5), by some computations, we arrive the infinitely conservation laws of the KNSCSH

$$[\eta\mu w'(x,\eta)]_t = [A' + B'w'(x,\eta)]_x. \tag{3.8}$$

Then expand

$$\eta u w'(x,\eta) = \sum_{n=0}^{\infty} \frac{w'_n(x,\eta)}{(2\eta^2)^{n+1}}. \tag{3.9}$$

Inserting (3.3) and (3.9) into the transformation (3.7), we derive

$$\sum_{n=0}^{\infty} \frac{w_n(x,\eta)}{(2\eta^2)^{n+1}} = \sum_{n=0}^{\infty} \frac{w'_n(x,\eta)}{(2\eta^2)^{n+1}} + \frac{1}{2}uv. \tag{3.10}$$

Comparing the coefficient of $\eta$ yields

$$w'_0(x,\eta) = -\frac{1}{2}uv, \quad w'_k(x,\eta) = w_{k-1}(x,\eta), \quad (k = 1, 2, \ldots), \tag{3.11}$$

which is just the explicit transformation relation of the conservation densities between the two hierarchies.

## 4. The Relation Between the Hamiltonian Structures of the AKNSSCSH and the KNSCSH

In what follows, we shall derive the Hamiltonian structure of the KNSCSH directly from that of the AKNSSCSH by a formulae of constrained functional derivative. The Hamiltonian structure of AKNSSCSH (1.1) is [10, 23]

$$\begin{pmatrix} \dfrac{\delta H_l}{\delta r} \\ \dfrac{\delta H_l}{\delta r} \end{pmatrix} = -L^{*l}\sigma_1 u - \sum_{j=1}^{N} \begin{pmatrix} \phi_{2j}^2 \\ -\phi_{1j}^2 \end{pmatrix}, \tag{4.1a}$$

where $H_l$ is governed by

$$H_l = \int_{-\infty}^{\infty} \omega_l dx. \tag{4.1b}$$

The following theorem and corollary should be used.

**Theorem 4.1 [2].** *Assume that the functional*

$$H(q,r) = \int_{-\infty}^{\infty} f(q, r, q_x, r_x, \ldots, q_x^{(l)}, r_x^{(l)})dx, \tag{4.2}$$

*where*

$$q = q(u, v, u_x, v_x, \omega), \quad w = \partial^{-1} g(u, v), \tag{4.3a}$$

$$r = r(u, v, u_x, v_x, \omega). \tag{4.3b}$$

*If the second order partial derivatives of $q, r$ and $g$ are continuous with respect to each variables*, then

$$\begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial q}{\partial u} - \partial \dfrac{\partial q}{\partial u_x} - \dfrac{\partial g}{\partial u} \partial^{-1} \dfrac{\partial q}{\partial \omega} & \dfrac{\partial r}{\partial u} - \partial \dfrac{\partial r}{\partial u_x} - \dfrac{\partial g}{\partial u} \partial^{-1} \dfrac{\partial r}{\partial \omega} \\ \dfrac{\partial q}{\partial v} - \partial \dfrac{\partial q}{\partial v_x} - \dfrac{\partial g}{\partial v} \partial^{-1} \dfrac{\partial q}{\partial \omega} & \dfrac{\partial r}{\partial v} - \partial \dfrac{\partial r}{\partial v_x} - \dfrac{\partial g}{\partial v} \partial^{-1} \dfrac{\partial r}{\partial \omega} \end{pmatrix} \begin{pmatrix} \dfrac{\delta H}{\delta q} \\ \dfrac{\delta H}{\delta r} \end{pmatrix}. \tag{4.4}$$

**Corollary 1.** *Assume that the functional*

$$H = \int_{-\infty}^{\infty} q\omega dx. \tag{4.5}$$

*The following holds*

$$\begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta v} \end{pmatrix} = \frac{1}{2} \sigma_1 L' S^{-1} \theta \begin{pmatrix} \dfrac{\delta H}{\delta q} \\ \dfrac{\delta H}{\delta r} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.6}$$

The proof can be finished by taking $g = -uv$ in Eq. (4.3) specially and using

$$q = \frac{u}{\gamma^2}, \quad r = \frac{\gamma^2}{2} \left( v_x - \frac{1}{2} u v^2 \right) \tag{4.7}$$

implied by (2.7).

To acquire the relation between the Hamiltonian structure of the two hierarchies, we integrate (3.7), and arrive

$$W = H - \frac{1}{2} \int_{-\infty}^{\infty} uv dx, \tag{4.8}$$

where $W \doteq \int_{-\infty}^{\infty} \eta u \omega' dx$. The functional derivative of (4.8) with respective to $u, v$ is

$$\begin{pmatrix} \dfrac{\delta W}{\delta u} \\ \dfrac{\delta W}{\delta v} \end{pmatrix} = \begin{pmatrix} \dfrac{\delta H}{\delta u} \\ \dfrac{\delta H}{\delta v} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} v \\ u \end{pmatrix}. \tag{4.9}$$

Further, we have

$$\begin{pmatrix} \dfrac{\delta W}{\delta u} \\ \dfrac{\delta W}{\delta v} \end{pmatrix} = \frac{1}{2} \sigma_1 L' S^{-1} \theta \begin{pmatrix} \dfrac{\delta H}{\delta q} \\ \dfrac{\delta H}{\delta r} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} v \\ u \end{pmatrix}, \tag{4.10}$$

where we make use of (4.6). Then expanding $W = \sum_{n=0}^{\infty} \frac{W_n}{(2\eta)^n}$ and comparing the coefficient of the same power of $\eta$ of (4.10), we obtain

$$\begin{pmatrix} \frac{\delta W_0}{\delta u} \\ \frac{\delta W_0}{\delta v} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} v \\ u \end{pmatrix}, \quad \begin{pmatrix} \frac{\delta W_n}{\delta u} \\ \frac{\delta W_n}{\delta v} \end{pmatrix} = \frac{1}{2}\sigma_1 L' S^{-1}\theta \begin{pmatrix} \frac{\delta H_{n-1}}{\delta q} \\ \frac{\delta H_{n-1}}{\delta r} \end{pmatrix}, \quad n = 1, 2, \ldots, \tag{4.11}$$

where $H_{n-1}$ is governed by (4.1). Noting the relation (2.6), (4.11) and (4.1), we have

$$\begin{pmatrix} \frac{\delta W_n}{\delta u} \\ \frac{\delta W_n}{\delta v} \end{pmatrix} = \frac{1}{2}\sigma_1 L'^n S^{-1}\theta \left[ L^{*n-1}\sigma_1 \begin{pmatrix} q \\ r \end{pmatrix} - \sum_{j=1}^{N} \eta_j \begin{pmatrix} \phi_{2,j}^2 \\ -\phi_{1,j}^2 \end{pmatrix} \right]. \tag{4.12}$$

Further we arrive

$$\begin{pmatrix} \frac{\delta W_n}{\delta u} \\ \frac{\delta W_n}{\delta v} \end{pmatrix} = -\frac{1}{2}\sigma_1 L'^n \begin{pmatrix} u \\ v \end{pmatrix} - \sum_{j=1}^{N} \eta_j^5 \begin{pmatrix} \psi_{2,j}^2 \\ -\psi_{1,j}^2 \end{pmatrix}, \tag{4.13}$$

where (2.10) and $\sigma\sigma_1 L^* = L\sigma\sigma_1$ are used. Substituting $\eta_j^4 \binom{-\psi_{1,j}^2}{\psi_{2,j}^2}$ for $\binom{-\psi_{1,j}^2}{\psi_{2,j}^2}$, $-w_n$ for $w_n$, (4.13) becomes

$$\begin{pmatrix} \frac{\delta W_n}{\delta u} \\ \frac{\delta W_n}{\delta v} \end{pmatrix} = \frac{1}{2}\sigma_1 L'^n \begin{pmatrix} u \\ v \end{pmatrix} + \sum_{j=1}^{N} \eta_j \sigma_1 \begin{pmatrix} -\psi_{1,j}^2 \\ \psi_{2,j}^2 \end{pmatrix}, \tag{4.14}$$

which is the Hamiltonian structure of the KNSCSH (1.3).

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