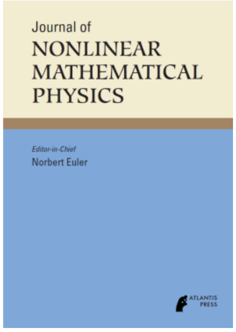

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PROPAGATION OF SPIKES IN NONRESONANT ATOMIC MEDIA: THE REDUCED MAXWELL-DUFFING MODEL

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We obtain spikes or extremely short pulses for reduced Maxwell-Duffing equations under a general boundary condition, connecting the electric field and electron amplitude, using a Möbius transformation. For a nonzero background, this system is shown to admit two families of exact solutions in the form of solitary waves and kink-type of solutions. Parameter domains are delineated where localized solutions signifying both bright and dark spikes, as also singular solutions indicating self-focusing effect, exist in this dynamical system, in nonresonant atomic media.

Keywords: Extremely short pulses; Maxwell-duffing equations; Möbius transformation.

1. Introduction

Ultrashort pulse propagation in nonresonant atomic media is currently attracting considerable attention, because of its relevance to femto-second pulses. Usually an atomic medium is modeled by N-level atoms. The medium is called resonant if the pulse width is well detuned from the resonant frequency, whence the system is effectively described by an ensemble of two level atoms. This is characterized by the Bloch equations. A number of studies have been carried out regarding the propagation of extremely short pulses (ESPs) in resonant atomic medium [1, 3–5, 7, 8, 11, 16, 20]. Two level resonant approximation is seen to be inaccurate in several physical situations [5, 21], like dense atomic media and systems involving three level atoms. Thus the resonant model needs to be generalized and extended to the nonresonant scenario. Recently, the study of propagation of ESPs in nonresonant media has become an area of active research [6, 12–14]. In Ref. 17, the propagation of vector ultra-short pulses has been studied. Also, in Ref. 10, solitary waves in two-wave Maxwell-Duffing type model has been explored. More recently, propagation of ESPs in a doubly resonant medium has been considered [9]. One of the well studied approaches is to consider the response of the medium as weakly nonlinear. Such situation leads to the Duffing oscillator model, where the nonlinear response of the medium is assumed to be cubic. This is the simplest generalization of the Lorentz model, where atoms are assumed to be ideal oscillators. A very recent study of the stabilization of ultra-short pulses in cubic nonlinear media is found in Ref. 6. Maxwell wave equation, for a linearly polarized light, allows pulse propagation in both the directions. However, this can be approximated to unidirectional wave propagation when anharmonic contribution to the polarization is very small. As a result, the wave equation reduces from second order to a first order equation. Here, we consider the reduced Maxwell-Duffing model (RMD), which is the combination of unidirectional wave propagation approximation [7] and the Duffing oscillator model for the propagation of ESP in a nonresonant atomic medium. Soliton and multi-soliton solutions are found as exact solutions in Refs. 1, 8, 15 and 19. A detailed study of the ultra short pulse propagation in an atomic media can be found in Ref. 2. A class of exact solutions of this RMD system have been obtained recently [13], where the coupled dynamical system involving electron and field variables satisfy a variety of initial conditions and boundary conditions. The paper deals with a general boundary condition, where field amplitude can possess a constant background. We employ a fractional linear transform or Möbius transformation and systematically obtain both bright and dark ESPs, as also singular solutions indicative of self-focusing effect. We emphasize that these steady-state ESPs are nonperturbative in nature and cannot be obtained by integrating the system of RMD equations using quadratures. Furthermore, we observe that these steady-state ESPs obtained here also satisfy modified KdV equation — a well studied equation in literature. In the following section we present a brief derivation of the reduced Maxwell-Duffing (RMD) model and the general boundary conditions involving the electron and field variables. We then outline the method of solving the system for localized pulses in Sec. 3. The proper parameter ranges are also specified as compared with the previous one. We end with the conclusions in Sec. 4.

2. The Constitutive Model

The propagation of electromagnetic waves in a media is described by the Maxwell wave equation, which for unidirectional wave propagation can be reduced to a first order equation,

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi}{c}\frac{\partial P}{\partial t}. \tag{2.1}$$

Here E is the electric field and P being the polarization of the medium. In a homogeneous broadening medium electrons (effective mass m and charge q_0) are assumed to oscillate with frequency ω_0 and displacement Φ from their equilibrium position due to the influence of electromagnetic field:

$$\frac{\partial^2\Phi}{\partial t^2} + \omega_0^2\Phi + \beta\Phi^3 = \frac{q_0}{m}E. \tag{2.2}$$

β is the nonlinear coefficient. The medium polarization is defined as $P = nq_0\Phi$, where n is the number density of the oscillators in the medium. Considering the scalings: $\tau = z/l$, $x = \omega_0(t-z/c)$, $q = E/a_0$, $g = \Phi/\Phi_0$ and the parameters: $a_0 = m\omega_0^2\Phi_0/q_0 = m\omega_0^3 q_0^{-1}(2\mu/|\beta|)^{1/2}$,

$\Phi_0 = (2\mu\omega_0^2/|\beta|)^{1/2}$, $l = mc\omega_0/(2\pi n q_0^2)$, and $2\mu = \beta\Phi_0^2/\omega_0^2$, Eqs. (2.1) and (2.2) can be written in terms of the rescaled variables as

$$\frac{\partial q}{\partial \tau} = -\frac{\partial g}{\partial x}, \quad \frac{\partial^2 g}{\partial x^2} + g + 2\mu g^3 = q. \tag{2.3}$$

The system of equations in (2.3) together are called reduced Maxwell-Duffing equations. For finding propagating solutions, one defines $\eta = x - \tau/\alpha$, where α is related to the velocity of the pulse. The first equation in Eq. (2.3) can be integrated with respect to the single variable η to yield

$$q(\tau, x) = \alpha g(\tau, x) + C, \tag{2.4}$$

where C is a constant, which signifies the background electric field, when electron amplitude g goes to zero. The boundary condition corresponding to the earlier case was for $C = 0$. Here, the electric field contains an independent part in addition to the g-dependent term. Then Eq. (2.3), takes the form:

$$\frac{d^2 g}{d\eta^2} + (1-\alpha)g + 2\mu g^3 = C. \tag{2.5}$$

For a pulse propagating in the right direction, $\alpha > 1$ and $\mu > 0$. Let us consider the following initial and boundary conditions:

$$q(\tau = 0, x) = q_0(x),$$

and at $x \to \pm\infty$, we consider a constant electron amplitude $g = g_0$, for which

$$\partial g(\tau, x)/\partial x = 0. \tag{2.6}$$

The above condition can be used to obtain the background electric field C from Eq. (2.5) as

$$C = g_0 \left[(1-\alpha) + 2\mu g_0^2\right]. \tag{2.7}$$

At $x \to \pm\infty$, initial electric field is determined as $q_0(x) = g_0 + 2\mu g_0^3$. Thus, C is fixed by uniform electron amplitude g_0, which also fixes the initial constant electric field $q_0(x)$.

3. Localized Solutions for the ESP

For propagating solitons, Eq. (2.5) yields nonlinear Schrödinger equation with a source [18], making it imperative to investigate localized solutions of this nonlinear dynamical system. It is very interesting to note that the traveling wave solutions of RMD equations also include the solutions of modified KdV equation, $V_{xxx} + a_1 V^2 V_x + a_2 V_t = 0$, a well studied equation in literature. This can be easily seen by simply going to traveling coordinate $\zeta = x - vt$ and integrating the resultant equation to obtain $V_{\zeta\zeta} + a_1/3\ V^3 - v a_2 V = s$, where s is the integration constant. By writing $a_1 = 6\mu$, $a_2 = -(1-\alpha)/v$ and $s = C$, we arrive at Eq. (2.5). Hence, the fractional transformation solutions obtained for the RMD system in this paper also satisfy modified KdV equation. For finding out explicit solutions we consider Eq. (2.5) in the form:

$$g'' + \lambda g^3 + \epsilon g = C, \tag{3.1}$$

provided $\lambda = 2\mu$ and $\epsilon = (1-\alpha)$. Prime indicates differentiation with respect to η. It is known earlier [18], that this equation can be connected to the elliptic equation $g'' + ag + bg^3 = 0$ through the following fractional transformation (FT):

$$g(\eta) = \frac{A + Bf(\eta, m)^\delta}{1 + Df(\eta, m)^\delta} \tag{3.2}$$

where A, B and D are real constants, δ is an integer, and $f(\eta, m)$ is a Jacobian elliptic function, with the modulus parameter m $(0 \le m \le 1)$. Here we describe two families of localized solutions of Eq. (3.1) for $\delta = 1$, $f(\eta, m) = cn(\eta, m)$, and $f(\eta, m) = sn(\eta, m)$ in the FT.

3.1. *Steady state solitary waves*

In the following, we illustrate the steady state solutions for $f(\eta, m) = cn(\eta, m)$. The consistency conditions for $m = 1$ are given by

$$A^3\lambda + A\epsilon - C = 0, \tag{3.3}$$

$$3A^2B\lambda - AD(1-2\epsilon) + B(1+\epsilon) - 3CD = 0, \tag{3.4}$$

$$3AB^2\lambda + AD^2(1+\epsilon) - BD(1-2\epsilon) - 3CD^2 = 0, \tag{3.5}$$

$$B^3\lambda + 2AD - 2B + BD^2\epsilon - CD^3 = 0. \tag{3.6}$$

It should be pointed out that $cn(\eta, 1) = \text{sech}(\eta)$. Note that $A = 0$ does not allow any solution for $C \ne 0$. Although a wide class of solutions is allowed, here we only outline a few of the interesting solutions.

Case 1: General solution. One can see from the above FT (Eq. (3.2)) that $AD = B$ implies only a constant or trivial solution and is not considered here. An $(AD - B)$ factor can be taken out of all the conditions by tactically using the first consistency Eq. (3.3). Therefore Eq. (3.4) is made to solve for A:

$$A = \pm\sqrt{-\frac{(1+\epsilon)}{3\lambda}}. \tag{3.7}$$

We simplify Eqs. (3.5) and (3.6) in similar manner and solve for D and B, respectively to arrive at

$$D = \pm\sqrt{-\frac{2(1+\epsilon)}{(1-2\epsilon)}} \quad \text{and} \quad B = \pm\sqrt{\frac{2(\epsilon-2)^2}{3\lambda(1-2\epsilon)}}. \tag{3.8}$$

As we are have three unknown variables A, B and D, which are already solved in terms of the equation parameters. Thus, Eq. (3.3) would imply a constraint condition between the equation parameters — ϵ, λ and C:

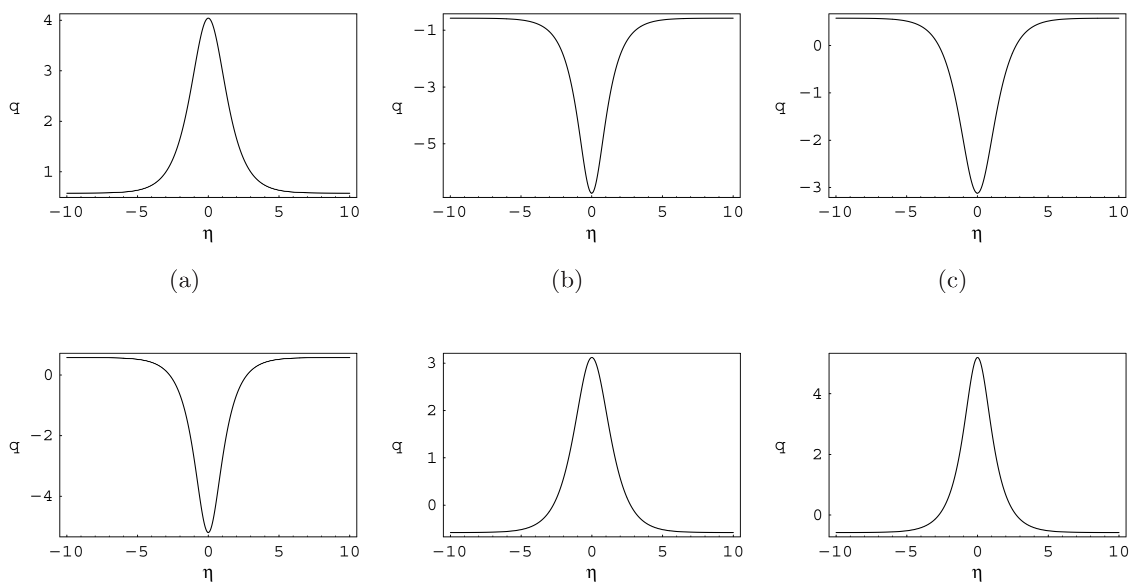
$$C^2 = -\frac{(1+\epsilon)(1-2\epsilon)^2}{27\lambda}. \tag{3.9}$$

Thus, the electric field of the electromagnetic wave is given as

$$q(\tau, x) = g_0 + 2\mu g_0^3 + \frac{A + B\text{sech}(x - \tau/\alpha)}{1 + D\text{sech}(x - \tau/\alpha)}. \tag{3.10}$$

As is already mentioned that pulse propagation in the right direction means $\mu > 0$ and $\alpha > 1$, which imply $\lambda > 0$ and $\epsilon < 0$, respectively. Again from Eq. (3.7) reality of A demands $\epsilon < -1$. This last condition simultaneously satisfies the reality of D, B and C. It is worth pointing out that all of the solutions are nonsingular in nature. This is because the magnitude of D remains always less than unity. Localized solutions are depicted in Fig. 1 for all possible signs of A, B and D. These contain both bright as well as dark localized pulses. In this particular case, the source becomes $C = 0.616$ for $\mu = 0.05$ and $\alpha = 2.1$. The variations of the pulse with μ and α are shown in Fig. 2, where A, B and D are all positive. The amplitude and the background of the bright solitonic electron amplitude decreases with increase in μ for a fixed value of α (Fig. 2(a)). Variation of the solution with α is more interesting as shown in Fig. 2(b). The background electron amplitude rises slowly with α, whereas the peak value of the electron amplitude initially decreases then slowly increases. In both the cases the localized pulses gradually broaden with μ and α, respectively. We would like to emphasize that all these steady state solutions of the RMD equations are new and they are nonperturbative in nature.

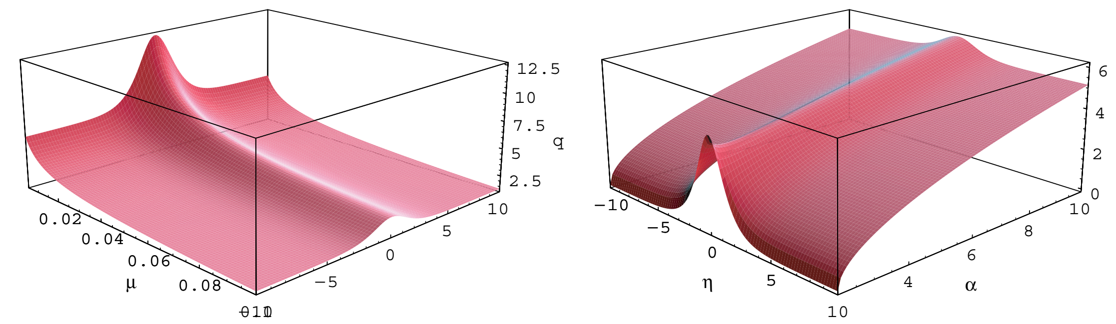
Case 2: Solution for $B = 0$. In this case, there are four consistency conditions and two unknowns. Thus the solution is more restrictive compared to the earlier case ($B \neq 0$). Comparison of the second and third consistency conditions fixes the value of $\epsilon = 2$, which



(a) (b) (c)

(d) (e) (f)

Fig. 1. Localized pulses for $\mu = 0.05$ and $\alpha = 2.1$, (a) $A > 0$, $B > 0$ and $D > 0$; (b) $A < 0$, $B < 0$ and $D < 0$; (c) $A > 0$, $B < 0$ and $D > 0$; (d) $A > 0$, $B < 0$ and $D < 0$; (e) $A < 0$, $B > 0$ and $D > 0$; (f) $A < 0$, $B > 0$ and $D < 0$.



(a) (b)

Fig. 2. Variation of bright localized pulse of Fig. 1(a) with (a) $\alpha = 3$ and (b) $\mu = 0.05$.

immediately provides the value of A from the third condition: $A = C$. Last condition yields $D = \pm\sqrt{2}$. So far we have not used the first consistency condition, which would obviously imply a constraint:

$$C^2 = -\frac{1}{\lambda}. \tag{3.11}$$

In this case, the allowed domain of localized solutions is $g < 0$, where $\epsilon = 2$. For this solution we have the following relation

$$q(x, \tau) = g_0 + 2\mu g_0^3 + \frac{A}{1 + D\mathrm{sech}(x - \tau/\alpha)}. \tag{3.12}$$

This would imply a unidirectional propagation in the left direction. Figure 3 shows the dark and bright pulses with $C = 10$ and $C = -10$, respectively, for $D = \sqrt{2}$ and $D = -\sqrt{2}$ implying singular pulse, indicative of self-focusing effect.

3.2. *Kink-type of steady state solutions*

In the following we elucidate the steady state solutions of the RMD system signifying kink-type of solitary waves. For this purpose, we put $f(\eta, m) = sn(\eta, m)$ in the FT and take the

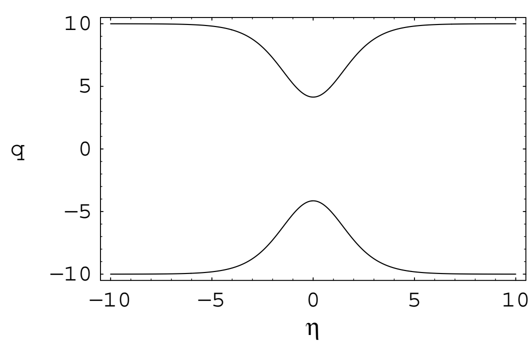


Fig. 3. Localized dark pulse ($C = 10$) and bright pulse ($C = -10$) for $B = 0$.

limit $m = 1$. We obtain the following consistency conditions

$$-2D(B - AD) + A^3\lambda + A\epsilon - C = 0, \tag{3.13}$$

$$2(B - AD) + B^3\lambda + BD^2\epsilon - CD^3 = 0, \tag{3.14}$$

$$2D(B - AD) + 3AB^2\lambda + (AD^2 + 2BD)\epsilon - 3CD^2 = 0, \tag{3.15}$$

$$-2(B - AD) + 3A^2B\lambda + (2AD + B)\epsilon - 3CD = 0. \tag{3.16}$$

The above consistency conditions clearly indicate that for solitonic limit, various kink-type of solutions are possible for the RMD system. We illustrate below various interesting cases.

Case 1: General solution. To obtain the general kink-type solution, we keep all the parameters in the FT. From Eqs. (3.15) and (3.16), we obtain $D = -\frac{3\lambda}{(\epsilon+4)}AB$. Substituting this value in Eq. (3.16), we get an equation, quadratic in A, which is solved to give

$$A = \frac{-b \pm \sqrt{b^2 + 4d}}{2}, \tag{3.17}$$

where $b = \frac{3C}{2-\epsilon}$ and $d = (\epsilon + 4)/3\lambda$. Further, Eq. (3.13) is solved to give

$$B = \pm\sqrt{\frac{(C - \lambda A^3 - \epsilon A)(\epsilon + 4)^2}{6\lambda A(\epsilon + 4) + 18\lambda^2 A^3}}. \tag{3.18}$$

And Eq. (3.14) would imply a constraint condition between the equation parameters — ϵ, λ, and C:

$$C = -\frac{(2 + \lambda B^2)(\epsilon + 4)^3 + 6\lambda A^2(\epsilon + 4)^2 + 9\lambda^2 A^2 B^2 \epsilon(\epsilon + 4)}{27\lambda^3 A^3 B^2}.$$

Thus the general kink-type of solitary wave solution signifying ultrashort pulse is

$$g(\eta) = \frac{A + B\tanh(\eta)}{1 + D\tanh(\eta)}. \tag{3.19}$$

Localized kink-solutions are demonstrated in Fig. 4. This includes all the possible solutions in Case 1. The condition for the localized solution is $D < 1$ or $|AB| < |\frac{\epsilon+4}{3\lambda}|$.

Case 2: Solution for $A = 0$. In this case, we obtain a special kink-type of solitary solution, which surprisingly is a solution in the absence of Duffing term in RMD system. The steady state solution is given by

$$g(\eta) = \frac{B\tanh(\eta)}{1 + D\tanh(\eta)}, \tag{3.20}$$

where $B = C/2$, $D = \pm 1$ and $\epsilon = -4$.

Case 3: Solution for $B = 0$. We obtain yet another interesting kink-type of solution given by

$$g(\eta) = \frac{A}{1 + D\tanh(\eta)}, \tag{3.21}$$

where $A = (3C)/(\epsilon - 2)$, $D^2 = 6$, $\epsilon = 1$ and $\lambda = -40/27C^2$.

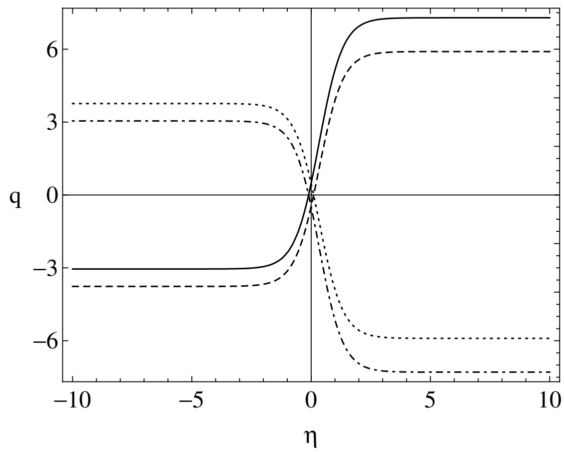


Fig. 4. Localized kink-type solution in the general case (Eq. (3.19)) for $\alpha = 3$, $\mu = 0.05$ and $C = 5$. Solid and dashed solutions correspond to two roots of A, dotted and dot-dashed solutions correspond to the same with opposite sign of B.

4. Conclusion

In conclusion, two families of novel localized solutions have been obtained for the reduced Maxwell-Duffing equations, which are unidirectional, using a Möbius transformation. These are the new steady state pulses in nonzero background electric field, signifying electromagnetic spikes propagating in a nonlinear medium. Our procedure yields both bright and dark ESPs, as also singular solutions indicating self-focusing effect. We emphasize that all these novel solutions obtained in this paper are new and they are nonperturbative in nature and cannot be obtained by mere integration of RMD system of equations using quadratures. These ones correspond to the situation where electric field has a constant part. Hence the coupled dynamical system lead to dark and bright solutions having the above constant value in the asymptotic condition.

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