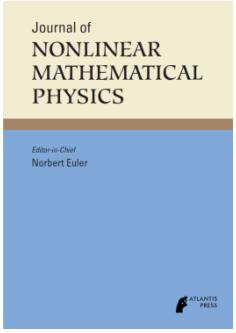

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INVARIANT SOLUTIONS OF NONLINEAR DIFFUSION EQUATIONS WITH MAXIMAL SYMMETRY ALGEBRA

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Nonlinear n-dimensional second-order diffusion equations admitting maximal Lie algebras of point symmetries are considered. Examples of invariant solutions, as well as of solutions on invariant subspaces for some nonlinear operators, are constructed for arbitrary n. A complete description of all possible types of invariant solutions is given in the case $n = 2$ for the equation possessing an infinitely dimensional symmetry algebra. The results obtained are generalized for the hyperbolic and other fourth-order parabolic equations of thin film and nonlinear dispersion type.

Keywords: Partial differential equations; Lie symmetries; invariant solutions.

Mathematics Subject Classification 2000: 35A30, 35C05, 34L30

1. Introduction

In this paper, second-order nonlinear diffusion equations of the form $u_t = \Delta f(u)$, where $u = u(t,x) > 0$, $(t,x) \in R \times R^n$, $f : R \to R$, $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$, admitting the maximal Lie algebra of the point symmetries are considered. From the group classification fulfilled in [13] (for $n = 1$) and [4] (for $n = 2, 3$), it follows that the "most symmetric" equations are

$$u_t = k\Delta u^{\frac{n-2}{n+2}} \quad (n \neq 2), \tag{1.1}$$

where $k = \mathrm{sign}(n-2)$ (this is necessary for the parabolicity of the equation), and

$$u_t = \Delta \ln u \quad (n = 2), \tag{1.2}$$

as well as equations obtained from (1.1), (1.2) by a transformation $u \to au + b$, where a and b are arbitrary constants, $a \neq 0$. It can be shown that the restriction $n \leq 3$ assumed in [4] is not essential and (1.1) is the most symmetric equation for arbitrary dimensions $n \neq 2$.

The following statements hold; see [4,13] and also [1, Ch. 10]. The convention of summation over repeated indices is adopted everywhere below.

Theorem 1.1. *A basis of the algebra of point symmetries for Eq.* (1.1) *consists of the infinitesimal operators*

$$\begin{gathered} X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x_i}, \quad X_{ij} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \\ Y_i = (2x_i x_j - r^2 \delta_{ij}) \frac{\partial}{\partial x_j} - (n+2) x_i u \frac{\partial}{\partial u}, \\ Z_1 = 2t \frac{\partial}{\partial t} + x_i \frac{\partial}{\partial x_i}, \quad Z_2 = 2x_i \frac{\partial}{\partial x_i} - (n+2) u \frac{\partial}{\partial u}, \end{gathered} \tag{1.3}$$

where $i, j = \overline{1, n}$, $r = (\sum_{i=1}^n x_i^2)^{1/2}$, δ_{ij} *is Kronecker's delta.*

Theorem 1.2. *The algebra of point symmetries for Eq.* (1.2) *is infinitely dimensional and has the following basis*:

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}, \\ X_\xi = \xi_1(x_1, x_2) \frac{\partial}{\partial x_1} + \xi_2(x_1, x_2) \frac{\partial}{\partial x_2} - 2(\xi_1(x_1, x_2))_{x_1} u \frac{\partial}{\partial u}, \end{gathered} \tag{1.4}$$

where $\xi_1(x_1, x_2)$ *and* $\xi_2(x_1, x_2)$ *are arbitrary conjugate harmonic functions*, $(\xi_1)_{x_1} = (\xi_2)_{x_2}$, $(\xi_1)_{x_2} = -(\xi_2)_{x_1}$.

Note that both Eqs. (1.1) and (1.2) belong to the type of the so-called *fast diffusion* ones and play an important role in the qualitative theory of parabolic equations (see [16, Ch. 2]). In particular, it is known [5], that the asymptotic properties of solutions of the Cauchy problem for the equation $u_t = k\Delta u^m$ substantially change for m reciprocal to the Sobolev exponent $p = p_s \equiv (n+2)/(n-2)$ appearing for the elliptic operator $\Delta u + |u|^{p-1}u$, when the compact embedding of the corresponding functional spaces for these operators, $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, for any bounded smooth domain Ω fails precisely at $p = p_s$ (see the seminal paper [15] for important consequences of such a failure and further references), i.e., exactly in the case of Eq. (1.1). Equation (1.2) arises in description of the thin films on a smooth surface; see [12], where new exact solutions for this equation are also obtained with the use of its local and nonlocal symmetries.

The present paper is devoted to the construction of invariant solutions for Eqs. (1.1) and (1.2). In the next section, Sec. 2, solutions of Eq. (1.1) invariant with respect to the sum of operators X_1 and Y_1 from (1.3) are considered. For construction of the exact solution of the factor-equation obtained, an approach [7,9,20] based on the notion of linear functional subspace invariant under the operator appearing on the right-hand side of equation is used.

In Sec. 3, a symmetry transformation for Eq. (1.2), reducing the operator X_ξ from (1.4) to the operator of the translations along x_2 is suggested. The utilization of such a transformation allows us to describe all types of invariant solutions of Eq. (1.2).

It is worth mentioning that similar results remain valid for more general equations than (1.1) and (1.2) with linear differential operators $l[u]$ including derivatives in t of arbitrary

orders with coefficients depending on t in the left-hand sides. As typical examples, in Sec. 4, hyperbolic equations

$$u_{tt} = k\Delta u^{\frac{n-2}{n+2}} \quad (n \neq 2) \quad \text{and} \quad u_{tt} = \Delta \ln u \quad (n = 2)$$

are considered, which turn out to be the most symmetric nonlinear equations of the form $u_{tt} = \Delta f(u)$; see group classification (for $n \leq 3$) in [2, 3]. In Appendix A, using some common ideas, we briefly describe other nonlinear PDEs admitting solutions on invariant subspaces.

2. Invariant Solutions of Eq. (1.1)

Note that Eq. (1.1) admits the inversion

$$x \to \frac{x}{r^2}, \quad u \to ur^{n+2}, \tag{2.1}$$

mapping operators X_i and Y_i into each other and not changing other operators in (1.3). An arbitrary solution $u = F(t, x)$ of this equation is then transformed into the solution

$$u = \frac{1}{r^{n+2}} F\left(t, \frac{x}{r^2}\right).$$

The sum of operators X_1 and Y_1,

$$\begin{aligned} X_1 + Y_1 = {} & (1 + x_1^2 - x_2^2 - \cdots - x_n^2)\frac{\partial}{\partial x_1} \\ & + 2x_1\left(x_2\frac{\partial}{\partial x_2} + \cdots + x_n\frac{\partial}{\partial x_n}\right) - (n+2)x_1 u\frac{\partial}{\partial u}, \end{aligned} \tag{2.2}$$

also does not change under the transformation (2.1). Let us next consider solutions, which are invariant with respect to this operator. Using [18], one gets the following:

Lemma 2.1. *A basis of invariants for the operator* (2.2) *consists of functions*

$$t, \quad \frac{x_i}{1 + r^2} \quad (i = 2, \ldots, n), \quad u(1 + r^2)^{\frac{n+2}{2}}.$$

The corresponding invariant solution of Eq. (1.1) *has the form*

$$u = (1 + r^2)^{-\frac{n+2}{2}} F\left(t, \frac{\tilde{x}}{1 + r^2}\right), \quad \tilde{x} = (x_2, \ldots, x_n). \tag{2.3}$$

Note that every solution of the form (2.3) is also invariant with respect to the transformation (2.1). Let us restrict the analysis to the *cylindrically symmetric* solutions,

$$u = (1 + r^2)^{-\frac{n+2}{2}} F(t, \rho), \tag{2.4}$$

where $\rho = \frac{\tilde{r}}{1+r^2}$, $\tilde{r} = (x_2^2 + \cdots + x_n^2)^{1/2}$. Substituting this expression into Eq. (1.1) leads to the equation

$$kF_t = (1 - 4\rho^2)\left(F^{\frac{n-2}{n+2}}\right)_{\rho\rho} + \left(\frac{n-2}{\rho} - 4n\rho\right)\left(F^{\frac{n-2}{n+2}}\right)_{\rho} + (2 - n)nF^{\frac{n-2}{n+2}}, \tag{2.5}$$

whose particular solution can be constructed using the method of invariant subspaces [7, 9].

Namely, by the transformation $F = \varphi^{-\frac{n+2}{4}}$, Eq. (2.5) takes the form

$$\varphi_t = k\frac{n-2}{n+2}\,\Phi[\varphi], \tag{2.6}$$

where the operator notation

$$\Phi[\varphi] \equiv (1-4\rho^2)\left[\varphi\varphi_{\rho\rho} - \frac{n+2}{4}(\varphi_\rho)^2\right] + \left(\frac{n-2}{\rho} - 4n\rho\right)\varphi\varphi_\rho + 4n\varphi^2$$

is introduced. This quadratic operator $\Phi[\varphi]$ admits the 2-dimensional invariant subspace

$$W = \{C_0 + C_1\rho^2 \,|\, C_0, C_1 \in R\}.$$

Indeed, for any C_0, $C_1 \in R$, the following holds:

$$\Phi[C_0 + C_1\rho^2] = 4nC_0^2 + 2(n-1)C_0C_1 + \big[(n-4)C_1^2 - 8C_0C_1\big]\rho^2,$$

i.e., $\Phi[W] \subseteq W$. Therefore, Eq. (2.6) possesses exact solutions of the form

$$\varphi = C_0(t) + C_1(t)\rho^2,$$

where the expansion coefficients $C_0(t)$ and $C_1(t)$ satisfy the following dynamical system:

$$\begin{cases} C_0'(t) = k\dfrac{n-2}{n+2}\big[4nC_0^2 + 2(n-1)C_0C_1\big], \\ C_1'(t) = k\dfrac{n-2}{n+2}\big[(n-4)C_1^2 - 8C_0C_1\big]. \end{cases} \tag{2.7}$$

Returning to the original variables yields the following exact solution for Eq. (1.1):

$$u = (C_0(t)(1+r^2)^2 + C_1(t)\tilde{r}^2)^{-\frac{n+2}{4}}, \tag{2.8}$$

where $\{C_0(t), C_1(t)\}$ is an arbitrary solution of the system (2.7).

In the case $n = 1$ (then $k = -1$), Eq. (1.1) takes the form

$$u_t = -(u^{-1/3})_{xx}, \tag{2.9}$$

while (2.3) becomes the solution in separated variables, $u = (1+x^2)^{-3/2}F(t)$. Substituting in (2.9) and finding F yields $u = \big(C - \frac{4}{3}t\big)^{3/4}(1+x^2)^{-3/2}$.

The existence of solutions of a more complicated structure for Eq. (2.9),

$$u = (C_0(t) + C_1(t)x + \cdots + C_4(t)x^4)^{-3/4}, \tag{2.10}$$

with the expansion coefficients $\{C_0(t), \ldots, C_4(t)\}$ satisfying a dynamical system, was observed in [7] (see Example 1.25). Note that the inversion

$$x \to 1/x, \quad u \to u|x|^3 \tag{2.11}$$

maps the solution (2.10) to the form $u = (C_0x^4 + C_1x^3 + \cdots + C_4)^{-3/4}$. Assuming that $C_4 = C_0$, $C_3 = C_1$, we single out those solutions (2.10), which are invariant with respect

to the transformation (2.11). Then the dynamical system for $\{C_i(t)\}$ simplifies and, after introducing the notation $C = 2C_0 + C_2$, takes the form

$$\begin{cases} C_0' = \dfrac{1}{3}\left(2C_0C - 4C_0^2 - \dfrac{3}{4}C_1^2\right), \\ C_1' = \dfrac{1}{3}(8C_0 - C)C_1, \\ C' = \dfrac{1}{3}(8C_0 - C)C. \end{cases}$$

Consider the case $C \neq 0$. From two last equations, it follows that $C_1 = qC$ ($q = \text{const.}$) and hence the system takes the form

$$\begin{cases} C_0' = \dfrac{1}{3}\left(2C_0C - 4C_0^2 - \dfrac{3}{4}q^2C^2\right), \\ C' = \dfrac{1}{3}(8C_0 - C)C. \end{cases}$$

Excluding C_0 yields the equation for C,

$$3\frac{C''}{C} = 2\left(\frac{1}{4} - q^2\right)C^2 + \frac{3}{2}\frac{C'^2}{C^2}.$$

Restricting to the case $q = \frac{1}{2}$, up to the translations in t, one obtains, $C = 2at^2$, $a = \text{const.}$, and then

$$C_0 = \frac{1}{4}\left(at^2 + \frac{3}{t}\right), \quad C_1 = at^2, \quad C_2 = \frac{3}{2}\left(at^2 - \frac{1}{t}\right).$$

Substituting these expressions in (2.10) and changing t to $t - b$, where $b = \text{const.}$, yields the following exact solution of Eq. (2.9):

$$u(t,x) = \left(\frac{(x+1)^2}{4}\left[a(b-t)^2(x+1)^2 - \frac{3}{b-t}(x-1)^2\right]_+\right)^{-\frac{3}{4}}, \tag{2.12}$$

where, dealing with nonnegative *real* solutions, we have to take the positive part $[\cdot]_+$ in the square brackets.

Setting here, as a key example, $a = 1$ and $b = 1$, we will next describe evolution properties of

$$u(t,x) = \left(\frac{(x+1)^2}{4}\left[(1-t)^2(x+1)^2 - \frac{3}{1-t}(x-1)^2\right]_+\right)^{-\frac{3}{4}}, \tag{2.13}$$

Namely, the evolution of (2.13) is quite curious and unusual. It is shown on Fig. 1 for $t \in [0;1)$, and on Fig. 2 for $t \in (1;+\infty)$.

First of all, there exist *blow-up free boundaries*, at which the solution takes infinite values, $u = +\infty$. Secondly, Fig. 1 shows contraction of blow-up free boundaries as $t \to 1^-$ to the single point $x = 1$. This asymptotic blow-up behaviour is approximately

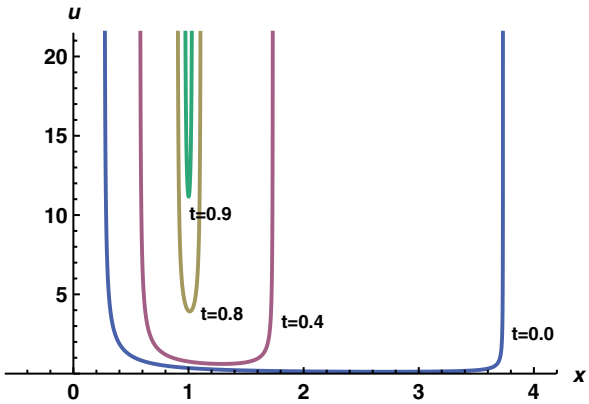

Fig. 1.

self-similar: one can calculate from the simple explicit expression (2.13) that, as $t \to 1^-$,

$$u(t,x) = (1-t)^{-\frac{3}{2}}(g(\xi) + o(1)), \quad \text{where } g(\xi) = 2^{-\frac{3}{2}}\left(1 - \frac{3}{4}\xi^2\right)^{-\frac{3}{4}}, \quad \xi = \frac{x-1}{(1-t)^{3/2}}. \tag{2.14}$$

On one hand, this looks like a standard blow-up limit, but, on the other hand, we should bear in mind that the formula (2.14) describes an unusual phenomenon of *self-focusing of two blow-up interfaces to a single point*. Then the "invariant" variable ξ in (2.14) shows that such a singular process occur on shrinking compact subsets of the order $O((1-t)^{\frac{3}{2}}$ around the point $x = 1$.

Moreover, it is natural to declare that, actually, (2.14) contains convergence to a measure: in the sense of distributions,

$$\frac{(1-t)^{-3}}{u(t,x)} \to c_0\,\delta(x-1) \quad \text{as } t \to 1^-, \tag{2.15}$$

where $\delta(x-1)$ is Dirac's delta concentrated at $x = 1$, and $c_0 > 0$ is an easy computed constant:

$$c_0 = 2^{\frac{3}{2}} \int_{-\infty}^{+\infty} \left(1 - \frac{3}{4}\xi^2\right)_+^{\frac{3}{4}} \mathrm{d}\xi = 2^{\frac{5}{2}} 3^{-\frac{1}{2}} B\left(\frac{7}{4}, \frac{1}{2}\right),$$

$B(\cdot,\cdot)$ being Euler's beta function. It is worth mentioning that singular blow-up limits given by formulae (2.14), (2.15) for the fast diffusion equations such as (2.9) are practically unknown in modern parabolic PDE (porous medium) theory, regardless its huge success and development in recent decades; see [21] as an advanced updated guide. Even for (2.9), proving such convergence for a more general class of blow-up solutions represents a difficult open problem.

Next, on Fig. 2, the solution decays to zero as $t \to +\infty$ almost everywhere, except the single point $x = -1$, where it remains infinite (a permanent standing blow-up point).

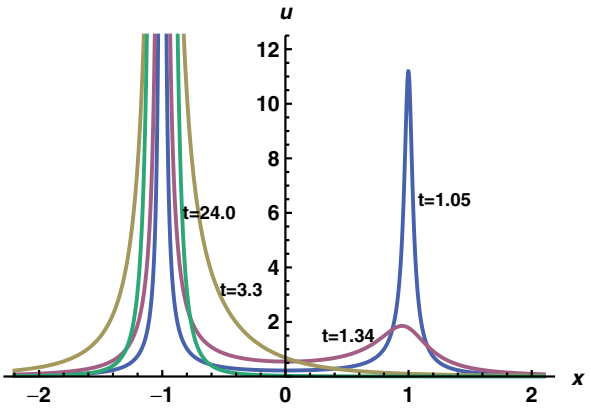

Fig. 2.

Calculating this limit yields the following asymptotics of (2.13) as $t \to +\infty$:

$$u(t,x) = t^3 \hat{g}(\eta)(1+o(1)), \quad \text{where } \hat{g}(\eta) = 3^{-\frac{3}{4}} \eta^{-\frac{3}{2}} \left(1 + \frac{\eta^2}{12}\right)^{-\frac{3}{4}}, \quad \eta = (x+1)t^{3/2}. \tag{2.16}$$

Here the similarity-like variable η reflects the fact that "focusing" as $t \to +\infty$ of such a blow-up structure at $x = -1$ occurs on shrinking compact subsets of the order $O(t^{-3/2})$. Note also that the function $\hat{g}(\eta)$ is not locally measurable close to $\eta = 0$, i.e., for $x \approx -1$, so that this singularity belongs to the *very singular* type.

Let us mention again that the mathematics concerning such strong complicated singularities is rather involved. In particular, proving existence and uniqueness (after such a blow-up) in corresponding PDE theory represent a challenging problem even nowadays. Here, we just get some interesting evolutions/singularity phenomena via explicit solutions on invariant subspaces for such fast diffusion processes.

3. Invariant Solutions of Eq. (1.2)

Setting $x_1 = x$, $x_2 = y$, we rewrite Eq. (1.2) as

$$u_t = \left(\frac{u_x}{u}\right)_x + \left(\frac{u_y}{u}\right)_y, \tag{3.1}$$

while the symmetry operators (1.4) take the form

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ X_\xi = \xi_1(x,y)\frac{\partial}{\partial x} + \xi_2(x,y)\frac{\partial}{\partial y} - 2(\xi_1(x,y))_x u\frac{\partial}{\partial u}, \end{gathered} \tag{3.2}$$

where $(\xi_1)_x = (\xi_2)_y$, $(\xi_1)_y = -(\xi_2)_x$. Following [18], along with the pair of conjugate harmonic functions $\xi_1(x,y)$ and $\xi_2(x,y)$, we introduce new functions $\eta(x,y)$ and $\delta(x,y)$

satisfying conditions

$$\delta_x = \eta_y = \frac{\xi_1}{(\xi_1)^2 + (\xi_2)^2}, \quad \delta_y = -\eta_x = \frac{\xi_2}{(\xi_1)^2 + (\xi_2)^2}, \tag{3.3}$$

and also being conjugate harmonic functions.

From the results obtained in [18], the next statement follows.

Lemma 3.1. *Equation* (3.1) *is invariant with respect to the transformation*

$$t \to t, \quad x \to \eta(x,y), \quad y \to \delta(x,y), \quad u \to \frac{u}{\eta_x^2 + \eta_y^2}. \tag{3.4}$$

The transformation maps any solution $u = F(t,x,y)$ *of Eq.* (3.1) *into the solution*

$$u = (\eta_x^2 + \eta_y^2) F(t, \eta(x,y), \delta(x,y)). \tag{3.5}$$

The operators X_1 *and* X_2 *from* (3.2) *do not change under the transformation* (3.4), *while* X_ξ *reduces to* $\partial/\partial y$.

Lemma 3.2. *The functions*

$$t, \quad \eta(x,y) \quad \text{and} \quad \frac{u}{\eta_x^2 + \eta_y^2}$$

form a basis of invariants for the operator X_ξ.

Formula (3.5) allows one, starting with some particular solutions of Eq. (3.1), to obtain new solutions depending on arbitrary conjugate harmonic functions $\eta(x,y)$ and $\delta(x,y)$.

Example 3.1. Equation (3.1) possesses solution ([6, Example 14]),

$$u = (C_0(t) + C_1(t)\cos 2x + C_2(t)\mathrm{e}^{2y} + C_3(t)\mathrm{e}^{y}\cos x)^{-1},$$

where functions $C_i(t)$ satisfy some dynamical system. Applying (3.4), one obtains the solution

$$u = ((\eta_x)^2 + (\eta_y)^2)(C_0(t) + C_1(t)\cos 2\eta(x,y) + C_2(t)\mathrm{e}^{2\delta(x,y)} + C_3(t)\mathrm{e}^{\delta(x,y)}\cos\eta(x,y))^{-1},$$

including two arbitrary conjugate harmonic functions η and δ.

Example 3.2. Starting with the solution ([7, Example 4.8]),

$$u = (C_0(t) + C_1(t)\cos 2x + C_2(t)\cosh 2y + C_3(t)\cosh y\cos x)^{-1}$$

yields

$$\begin{aligned} u = {} & ((\eta_x)^2 + (\eta_y)^2)(C_0(t) + C_1(t)\cos 2\eta(x,y) + C_2(t)\cosh 2\delta(x,y) \\ & + C_3(t)\cosh\delta(x,y)\cos\eta(x,y))^{-1}. \end{aligned}$$

The transformation (3.4) can be applied to construct all types of invariant solutions (of the rang 1) for Eq. (3.1). Consider an arbitrary linear combination of operators (3.2)

$$X = C_0 X_\xi + C_1 X_1 + C_2 X_2.$$

Then, the change of variables

$$\bar{t} = t, \quad \bar{x} = \eta(x,y), \quad \bar{y} = \delta(x,y), \quad \bar{u} = \frac{u}{\eta_x^2 + \eta_y^2},$$

yields (omitting the bar)

$$X = C_0 \frac{\partial}{\partial y} + C_1 \frac{\partial}{\partial t} + C_2 \left(t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} \right) = C_0 \frac{\partial}{\partial y} + (C_2 t + C_1) \frac{\partial}{\partial t} + C_2 u \frac{\partial}{\partial u}. \tag{3.6}$$

Note that, if $C_2 \neq 0$, then using translations in t it is possible to vanish C_1 and, dividing by C_2, reduce the operator (3.6) to the form $X = s\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$ ($s = \text{const.}$). If $C_2 = 0$, $C_1 \neq 0$, then, dividing by C_1, yields $X = s\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$. Finally, if $C_2 = C_1 = 0$, $C_0 \neq 0$, then (3.6) takes the form $X = \frac{\partial}{\partial y}$. Taking into account dilations in x, y, and u admitted by Eq. (3.1) in the first two cases, it is sufficient to consider only two variants: $s = 0$ and $s = 1$.

Thus, the operator (3.6) reduces to one of the following forms:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \quad X_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + u\frac{\partial}{\partial u}. \tag{3.7}$$

Calculating the invariants for every operator (3.7), one obtains five possible types of invariant solutions (of the rang 1) for Eq. (3.1) and the corresponding factor-equations:

(1) $u = F(x,y)$, $\Delta \ln F = 0$;
(2) $u = F(t,x)$, $F_t = \left(\frac{F_x}{F}\right)_x$;
(3) $u = F(x,h)$, $-F_h = \left(\frac{F_x}{F}\right)_x + \left(\frac{F_h}{F}\right)_h$, $h = y - t$;
(4) $u = tF(x,y)$, $F = \Delta \ln F$ (the *Liouville equation*);
(5) $u = tF(x,h)$, $F - F_h = \left(\frac{F_x}{F}\right)_x + \left(\frac{F_h}{F}\right)_h$, $h = y - \ln|t|$.

Returning to the original variables, one obtains the following result [19]:

Theorem 3.1. *Equation* (3.1) *admits the following five types of invariant solutions and corresponding reductions*:

(1) $u = (\eta_x^2 + \eta_y^2)F(\eta,\delta)$, $\Delta_{\eta,\delta} \ln F = 0$;
(2) $u = (\eta_x^2 + \eta_y^2)F(t,\eta)$, $F_t = \left(\frac{F_\eta}{F}\right)_\eta$;
(3) $u = (\eta_x^2 + \eta_y^2)F(\eta,h)$, $-F_h = \left(\frac{F_\eta}{F}\right)_\eta + \left(\frac{F_h}{F}\right)_h$, $h = \delta - t$;
(4) $u = t(\eta_x^2 + \eta_y^2)F(\eta,\delta)$, $F = \Delta_{\eta,\delta} \ln F$;
(5) $u = t(\eta_x^2 + \eta_y^2)F(\eta,h)$, $F - F_h = \left(\frac{F_\eta}{F}\right)_\eta + \left(\frac{F_h}{F}\right)_h$, $h = \delta - \ln|t|$.

Note that, for the first and the fourth [11] equations for the function F in this list, their general solutions are written down in the explicit form as

$$F = \mathrm{e}^{\Phi(\eta,\delta)} \quad \text{and} \quad F = 2\frac{(\Phi_\eta)^2 + (\Phi_\delta)^2}{\Phi^2},$$

where $\Phi(\eta,\delta)$ is an arbitrary harmonic function, $\Delta_{\eta,\delta}\Phi = 0$. For the other equations given in Theorem 3.1, the construction of particular solutions is possible as well, e.g., by using some symmetries. For example, the second equation is invariant with respect to the dilations $t \to at$, $\eta \to a\eta$, $F \to a^{-1}F$ $(a \neq 0)$ and, then, possesses a self-similar solution, $F = \left(\alpha t + \frac{\eta^2}{2t}\right)^{-1}$ $(\alpha = \text{const.})$.

4. Final Remarks: Extensions to Hyperbolic Wave Equations

The remarkable quality of Eqs. (1.1) and (1.2) studied above is determined by the fact that their Lie point symmetry group includes, as a subgroup, the complete group of conformal transformations in R^n (extended to the variable u).

Indeed, this property holds for any equations obtained from (1.1) and (1.2) by changing their left-hand sides to an arbitrary linear differential operator $l[u]$, which do not depend on x and include derivatives in t of arbitrary order. Then, setting $l[u] \equiv u_{tt}$, one obtains hyperbolic (or wave) equations

$$u_{tt} = k\Delta u^{\frac{n-2}{n+2}} \quad (n \neq 2) \tag{4.1}$$

and

$$u_{tt} = \Delta \ln u \quad (n = 2). \tag{4.2}$$

From the results of the group classification [2,3], it follows (at least for $n \leq 3$) that (4.1) and (4.2) are the most symmetric nonlinear wave equations of the form $u_{tt} = \Delta f(u)$.

For Eqs. (4.1) and (4.2), the results similar to those that obtained above hold. For example, these equations admit transformations (2.1) and (3.4) respectively. Equation (4.1) possesses an invariant solution of the form (2.4) (or (2.3)), and the function F satisfies an equation similar to (2.5) (with F_{tt} on the left-hand side rather than F_t). Finally, for Eq. (4.2) it is possible to list all possible types of invariant solutions and corresponding reductions. In particular, it has the solution

$$u = (\eta_x^2 + \eta_y^2)F(t, \eta(x,y)),$$

where $\eta(x,y)$ is an arbitrary harmonic function, and $F(t,\eta)$ satisfies the equation

$$F_{tt} = \left(\frac{F_\eta}{F}\right)_\eta.$$

The last equation admits dilations defined by the operator

$$X = t\frac{\partial}{\partial t} + m\eta\frac{\partial}{\partial \eta} + 2(1-m)F\frac{\partial}{\partial F}, \quad m = \text{const.},$$

and possesses a self-similar solution of the form $F = t^{2-2m}f(\eta t^{-m})$, where the function f satisfy some ordinary differential equation.

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Appendix A. On Some Exact Solutions on Invariant Subspaces for Higher-Order PDEs

A.1. *Thin film equation*

We begin with the fourth-order *thin film equation* (the TFE–4)

$$u_t = -\nabla \cdot (u^m \nabla \Delta u), \tag{A.1}$$

where $m \neq 0$ is a fixed exponent and solutions $u(t,x)$ are assumed to be nonnegative in a free-boundary setting; see [9, Ch. 4] for a survey concerning physical motivation and mathematics for such PDE models. Similarly, $m < 0$ can be attributed to a "fast diffusion case". We will use a technique that is similar to that applied to (2.9). We refer to the key papers [10, 17] containing basic ideas.

A.2. *One-dimensional TFE*

We begin with a typical explanation how exact solutions on invariant subspaces occur for the operator of the TFE (1.1) in 1D:

$$u_t = -(u^m u_{xxx})_x. \tag{A.2}$$

Setting in (A.2)

$$u = v^\mu, \quad \text{where } \mu = \tfrac{3}{m}, \tag{A.3}$$

splits the thin film operator into five primitive monomials of the algebraic homogenuity four:

$$\begin{aligned} v_t = -\big[&v^3 v_{xxxx} + (4\mu - 1)v^2 v_x v_{xxx} \\ &+ 3(\mu-1)v^2(v_{xx})^2 + 3(\mu-1)(2\mu-1)v(v_x)^2 v_{xx} + \mu(\mu-1)(\mu-2)(v_x)^4\big]. \end{aligned} \tag{A.4}$$

We are interested in solutions of typical bell-shaped forms localized on a bounded interval in x. This requires quadratic polynomial subspaces:

Proposition A.1. *The operator given in* (A.4) *preserves the subspace* $W_3 = \mathcal{L}\{1, x, x^2\}$ *iff* $m = 3$, $m = 6$, *or* $m = -2$.

Proof. Plugging $v = C_1 + C_2 x + C_3 x^2 \in W_3$ into the quadratic operator yields that the terms on $\mathcal{L}\{x^3, x^4\}$ vanish iff $4(\mu-1)[3 + 6(2\mu-1) + 4\mu(\mu-2)] = 0$, which yields either $\mu = 1$, or $4\mu^2 + 4\mu - 3 = 0$, i.e., $\mu = \frac{1}{2}$, or $\mu = -\frac{3}{2}$. □

Consider a particularly interesting case $m = 6$ ($\mu = \frac{1}{2}$). Then, the diffusion-absorption equation takes the form

$$v_t = -\left[v^3 v_{xxxx} + v^2 v_x v_{xxx} - \frac{3}{2} v^2 (v_{xx})^2 + \frac{3}{8}(v_x)^4\right].$$

We have the following exact solutions on $W_2 = \mathcal{L}\{1, x^2\}$:

$$v(t,x) = [C_1(t) + C_3(t)x^2]_+ \,, \tag{A.5}$$

with the dynamical system on the expansion coefficients

$$\begin{cases} C_1' = 6C_1^2C_3^2, \\ C_3' = 12C_1C_3^3. \end{cases}$$

The positive part $[\cdot]_+$ in (A.5) means a certain free-boundary setting for the TFE–4, where special Stefan–Florin type conditions are assumed at the interfaces. We do not treat those here and refer to [9, p. 107] for further details.

The above dynamical system for $\{C_1(t), C_3(t)\}$ is easy to solve explicitly: $C_1(t) = [18A_0^2(1-t)]^{-1/3}$ and $C_3(t) = A_0[18A_0^2(1-t)]^{-2/3}$, where $A_0 \neq 0$ is a constant of integration. Thus, we obtain an explicit blow-up solution of the present TFE–4, which, similar to (2.13), deserves further study and physical interpretations.

A.3. *Solutions of TFE–4 with zero contact angle*

Such solutions play quite a special role in TFE theory, where the requirement of the zero contact angle ($u_x = 0$) at free boundaries expresses a deep physical meaning; [9, § 3.1]. Namely, consider the TFE (A.2) for $m = 1$, i.e., in the quadratic case:

$$u_t = -(uu_{xxx})_x. \tag{A.6}$$

We now impose the zero contact angle condition, so, at each finite interface $x = s(t)$,

$$u = u_x = uu_{xxx} = 0 \quad \text{at} \quad x = s(t). \tag{A.7}$$

Setting next $u = v^2$ yields an equation with the cubic operator,

$$v_t = -\frac{1}{2}\, v(v^2)_{xxxx} - v_x(v^2)_{xxx} \equiv F_3[v], \tag{A.8}$$

exhibiting the following invariant property:

Proposition A.2. *Operator F_3 in (A.8) admits the subspace $W_3 = \mathcal{L}\{1, x, x^2\}$.*

Indeed, taking $v = C_1 + C_2x + C_3x^2$ yields

$$F_3[v] = -12(C_1C_3 + C_2^2)C_3 - 60C_2C_3^2x - 60C_3^3x^2 \in W_3. \tag{A.9}$$

For simplicity, setting $C_2(t) \equiv 0$ (meaning even and symmetric in x patterns) gives the following solutions of the original PDE (A.6) on $W_2 = \mathcal{L}\{1, x^2\}$:

$$u(t,x) = v^2(t,x) \equiv [C_1(t) + C_3(t)x^2]_+^2, \tag{A.10}$$

which, clearly, satisfy all three free-boundary conditions (A.7). In this case, (A.9) gives the dynamical system

$$\begin{cases} C_1' = -12C_1C_3^2, \\ C_3' = -60C_3^3. \end{cases} \tag{A.11}$$

The last ODE is solved explicitly, $C_3(t) = \pm\frac{1}{\sqrt{120}}\frac{1}{\sqrt{t}}$ for $t > 0$ and, from the first one, $C_1(t) = A_0t^{-\frac{1}{10}}$, where $A_0 =$ constant. For a more general analysis of such solutions for

TFEs with extra absorption phenomena, we refer to [9, p. 120], where further results and references can be found.

A.4. *TFE in n dimensions*

The above approach extends to the n-dimensional TFE (A.1). Consider radially symmetric functions from the subspace

$$W_2 = \mathcal{L}\{1, |x|^2\}. \tag{A.12}$$

Using the same transformation (A.3) yields a PDE with a quartic operator which can be analyzed in a manner similar to Proposition A.1. Namely, we then obtain the PDE

$$v_t = -\frac{1}{\mu}\, v^{1-\mu} \nabla \cdot (v^3 \nabla \Delta v^\mu) \equiv F_4[v]. \tag{A.13}$$

Using the radial form of this operator,

$$F_4[v] = -\frac{1}{\mu}\, v^{1-\mu}\, \frac{1}{r^{n-1}} \left[r^{n-1} v^3 \left(\frac{1}{r^{n-1}} [r^{n-1} (v^\mu)'_r]'_r \right)'_r \right]'_r,$$

we arrive at the following conclusion:

Proposition A.3. *Operator F_4 in* (A.13) *admits subspace* (A.12) *iff*

$$(\mu - 1)(4\mu^2 + 4n\mu + n^2 - 4) = 0,$$

i.e., in the following three cases: $m = 3$ $(\mu = 1)$; $m = -\frac{6}{n-2}$ *for* $n \neq 2$ $\big(\mu = -\frac{n-2}{2}\big)$; *and* $m = -\frac{6}{n+2}\big(\mu = -\frac{n+2}{2}\big)$.

Fix now the negative $n = -\frac{6}{n+2}$ (a fast diffusion range) and consider the TFE

$$u_t = -\nabla \cdot \big(u^{-\frac{6}{n+2}} \nabla \Delta u\big). \tag{A.14}$$

Then, the corresponding exact solutions are

$$u(t,x) = v^{\frac{3}{n}}(t,x) = [C_1(t) + C_2(t)|x|^2]_+^{-\frac{n+2}{2}}, \quad \text{where}$$

$$\begin{cases} C_1' = aC_1^2 C_2^2, \\ C_2' = -bC_1 C_2^3, \end{cases} \tag{A.15}$$

with $a = 2n(n+2)(n+4)$, $b = 12(n+2)(n+4)$, so that $a - b = 2(n+2)(n+4)(n-6) > 0$ for all dimensions $n > 6$. The dynamical system can be easily solved that gives: for $n \neq 6$,

$$C_1(t) = [(6-n)aA_0 t/n]^{n/(6-n)}, \quad C_2(t) = A_0[(6-n)aA_0 t/n]^{-6/(6-n)},$$

where A_0 is a constant of integration. For $n = 6$, the functions involved are exponential: $C_1(t) = \mathrm{e}^{aA_0 t}$ and $C_2(t) = \mathrm{e}^{-aA_0 t}$. Concerning other types of exact solutions of the TFEs in 1D and in n dimensions, we refer to [9, Ch. 3] and [9, Ch. 6], respectively, where more detailed mathematical and applied interpretations of those can be found.

A.5. *Towards third-order NDEs*

As a last example, consider an odd-order PDE from the family of the third-order *nonlinear dispersion equations* (NDEs–3):

$$u_t = u^m u_{xxx}\,, \tag{A.16}$$

where, as usual, $m \neq 0$. For $m = 3$, (A.16) gives the *Harry Dym equation* $u_t = u^3 u_{xxx}$, which is integrable and is one of the most exotic soliton equations. It is associated with the classical *string problem* and is linearizable by the inverse spectral transform method; see [9, Ch. 4] for details and a survey on various NDEs.

Let us show that, for the PDE (A.16), the same method applies, so setting

$$u = v^\mu, \quad \text{with the exponent } \mu = \frac{2}{m},$$

yields the following equation with a homogeneous cubic operator:

$$v_t = F[v] \equiv v^2 v_{xxx} + 3(\mu-1) v v_x v_{xx} + (\mu-1)(\mu-2)(v_x)^3. \tag{A.17}$$

Substituting $v = x^2$ yields that the operator F in (A.17) preserves the extended 3D subspace

$$W_3 = \mathcal{L}\{1, x, x^2\}, \quad \text{if } \ 12(\mu-1) + 8(\mu-1)(\mu-2) = 0, \tag{A.18}$$

i.e., for $\mu = 1$ ($m = 2$, the trivial case: $u = v$ and $v_{xxx} = 0$ on W_3) and for $\mu = \frac{1}{2}$ ($m = 4$) that gives some applications and extensions. Namely, consider the NDE

$$u_t = u^4 u_{xxx} \quad (u \geq 0). \tag{A.19}$$

Setting $u = \sqrt{v}$ $\left(\mu = \frac{1}{2}\right)$ yields the PDE

$$v_t = v^2 v_{xxx} - \frac{3}{2} v v_x v_{xx} + \frac{3}{4}(v_x)^3 \tag{A.20}$$

that possesses solutions $v \in W_3$,

$$v(t,x) = u^2(t,x) = C_1(t) + C_2(t)x + C_3(t)x^2, \tag{A.21}$$

where the coefficients satisfy the dynamical system

$$\begin{cases} C_1' = -3C_1C_2C_3 + \dfrac{3}{4}C_2^3, \\ C_2' = -6C_1C_3^2 + \dfrac{3}{2}C_2^2C_3, \\ C_3' = 0. \end{cases}$$

Setting $C_3(t) \equiv 1$ reduces this to a simpler 2D system,

$$\begin{cases} C_1' = -3C_1C_2 + \dfrac{3}{4}\,C_2^2, \\ C_2' = -6C_1 + \dfrac{3}{2}\,C_2^2. \end{cases}$$

Though this cannot be integrated explicitly, the asymptotic study of exact solutions is not that difficult. This last example shows that exact solutions on finite-dimensional invariant

subspaces exist for various nonlinear PDEs: from even-order parabolic (and hyperbolic) ones to odd-order NDEs and others.

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