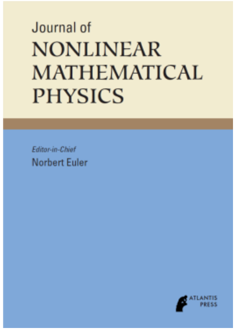

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LIE ALGEBRA OF THE SYMMETRIES OF THE MULTI-POINT EQUATIONS IN STATISTICAL TURBULENCE THEORY

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We briefly derive the infinite set of multi-point correlation equations based on the Navier–Stokes equations for an incompressible fluid. From this we reconsider the previously derived set of Lie symmetries, i.e. those directly induced by the ones from classical mechanics and also new symmetries. The latter are denoted statistical symmetries and have no direct counterpart in classical mechanics. Finally, we considerably extend the set of symmetries by Lie algebra methods and give the corresponding commutator tables. Due to the infinite dimensionality of the multi-point correlation equations completeness of its symmetries is not proven yet and is still an open question.

Keywords: Turbulence; multi-point statistics, Lie symmetry; Lie algebra.

2000 Mathematics Subject Classification: 76F02, 76M60

1. Introduction

Although there has been considerable progress in the theory as well as in experiments, turbulence remains still one of the most important and in fact challenging unresolved problems in classical mechanics.

In the last one and a half decades it has been observed that statistical turbulence quantities such as the mean velocity have the strong tendency to establish invariant solutions. Using the Lie symmetries which originate from the Euler or Navier–Stokes equations leads to classical solutions such as the well known logarithmic law of the wall but also a broad set of new solutions has been derived which can clearly be observed in experimental or numerical data (see e.g. [8–10]).

There were clear hints that statistical turbulence quantities can be described with even more general invariant solutions which are apparently based on an extended set of symmetries not directly stemming from the Euler or Navier–Stokes equations. This is in contrast to the common expectation that scaling laws arise from classical symmetries. This rather remarkable results may have been first explicit demonstrated in [7] where a new scaling group was derived and used to understand two-point correlations in a turbulent boundary layer flow. Based on this only recently [11] it became clear that an even considerably extended set of infinite new symmetries exists, in the following called statistical symmetries, which allow for a very broad number of invariant solutions and, in fact, may give a very deep understanding of statistical turbulence behavior.

The present paper is an extension of the work in [11] in the sense that the classical symmetries and the new statistical symmetries will be unified using commutator theory and constructing the Lie algebra. For this we first consider the Navier–Stokes equations as the essential basis which in turn will be employed to derive the infinite set of multi-point equations describing the statistics of fluid turbulence. Part of the symmetries of the latter equations, those of classical mechanics, may be found by transferring the symmetries of the Navier–Stokes equations (Subsec. 3.2.1) to the multi-point formulation. A direct computation of all symmetries using Lie's algorithm is rather difficult because of the infinite dimensionality of the system. Instead the presently known statistical set of symmetries in [11] has been employed as a basis.

In calculating the Lie algebra of these symmetries, it turns out that the Lie algebra has to be considerably extended (Subsec. 4.1), in order to obtain an algebra (Subsec. 4.2).

It should be remarked that only recently Khabirov and Ünal [6] computed all symmetries of the two-point equation for homogeneous isotropic case, the von Kármán–Howarth equation. Those presently derived symmetries of the full multi-point correlation equation relevant for isotropic turbulence transfer to some of the symmetries derived therein.

2. The Infinite Set of the Multi-point Equations

The following introduction of the governing equations and the outline of some general ideas of the mathematics of statistical multi-point turbulence theory is a brief repetition of our previous work (see [11]).

2.1. *Navier–Stokes equations*

The starting point of the entire analysis is based on the three dimensional Navier–Stokes equations describing an incompressible fluid under the assumption of a Newtonian material with constant density and viscosity. In Cartesian tensor notation the continuity equation and the momentum equations read as follows

$$\frac{\partial U_k}{\partial x_k} = 0 \tag{2.1}$$

$$\mathcal{M}_i(\boldsymbol{x}) = \frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} + \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} = 0, \quad i = 1, 2, 3. \tag{2.2}$$

Here $t \in \mathbb{R}^+$, $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x}, t)$ and $P = P(\boldsymbol{x}, t)$ respectively represent time, position vector, instantaneous velocity vector and pressure. Furthermore, pressure has been normalized by the constant density.

2.2. *Statistical averaging*

In the following we define the classical Reynolds averaging. Let Z represent an arbitrary statistical variable, i.e. $\boldsymbol{U}$ and P, which in the following we also denote as *instantaneous value*. According to the classical definition by Reynolds all instantaneous quantities are decomposed into their mean and their fluctuation value

$$Z = \bar{Z} + z. \tag{2.3}$$

The overbar always marks a statistically averaged quantity whereas the lower-case z denotes the fluctuation value of Z. A rather general definition of a statistically averaged quantity, sometimes referred to as mean value or expectation value, is given by an ensemble average operator $\mathcal{K}$

$$\bar{Z}(\boldsymbol{x}, t) = \mathcal{K}\left[Z(\boldsymbol{x}, t)\right] = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} Z_n(\boldsymbol{x}, t)\right), \tag{2.4}$$

where Z_n refers to one realization.

Employing the above we may define the fluctuation value of Z

$$z = Z - \bar{Z}. \tag{2.5}$$

2.3. *Reynolds averaged transport equations*

Implementing the decomposition of $\boldsymbol{U}$ and P according to (2.3) and applying the operator $\mathcal{K}$ to the continuity equation (2.1) and the momentum equations (2.2) we respectively obtain its averaged versions, i.e. the continuity equation

$$\frac{\partial \bar{U}_k}{\partial x_k} = 0 \tag{2.6}$$

and the momentum equations

$$\frac{\partial \bar{U}_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} = -\frac{\partial \bar{P}}{\partial x_i} + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} - \frac{\partial \overline{u_i u_k}}{\partial x_k}, \quad i = 1, 2, 3. \tag{2.7}$$

At this point we observe the well-known closure problem of turbulence since, compared to the original set of equations, the unknown Reynolds stress tensor $\overline{u_i u_k}$ has appeared. However, rather different from the classical approach we will not proceed with deriving a transport equation for the Reynolds stress tensor which contains additional four unclosed tensors. Instead the multi-point correlation approach is put forward the reason being twofold.

First, if the infinite set of correlation equations is considered the closure problem is somewhat bypassed. Second, the multi-point correlation delivers additional information on the turbulence statistics such as length scale information which may not be gained from the Reynolds stress tensor, which is a single-point approach.

2.4. *Multi-point correlation equations*

The idea of two- and multi-point correlation equations in turbulence was presumably first established by Friedmann and Keller [5]. In the beginning it was assumed that all correlation equations of orders higher than two may be neglected but theoretical considerations led to the result that all higher correlations have to be taken into account. Consequently, all multi-point correlation (MPC) equations have to be considered in the symmetry analysis.

2.4.1. *MPC equations*: *fluctuation approach*

The classical approach in the theory of turbulence deals with correlation functions which are based on the fluctuating quantities $\boldsymbol{u}$ and p as introduced by Reynolds. The multi-point correlation for the fluctuation velocity,

$$R_{i_{\{n+1\}}} = R_{i_{(0)}i_{(1)}\ldots i_{(n)}} = \overline{u_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot u_{i_{(n)}}(\boldsymbol{x}_{(n)})}\,, \tag{2.8}$$

is a tensor of order $n+1$, where the first index of the $\mathbf{R}$ tensor defines the tensor character of the term and the second index in braces denotes the order of the tensor. The curly brackets point out that not an index of a tensor but an enumeration is meant. It is important to mention that the indices start with 0 which is advantageous when introducing a new coordinate system based on the Euclidean distance of two space points.

2.4.2. *MPC equations*: *instantaneous approach*

In order to write the MPC equations in a very compact form, we introduce the following notation. Similar to (2.8) we have the multi-point correlation for the instantaneous velocity

$$H_{i_{\{n+1\}}} = H_{i_{(0)}i_{(1)}\ldots i_{(n)}} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}. \tag{2.9}$$

Apparently we have the connection to the mean velocity according to $H_{i_{\{1\}}} = H_{i_{(0)}} = \bar{U}_i$.

From (2.3) it is apparent that there is a unique relation between the instantaneous and the fluctuation approach though the actual crossover is somewhat cumbersome in particular with increasing tensor order. The first relations are given by

$$\begin{aligned}
H_{i_{(0)}} &= \bar{U}_{i_{(0)}} && (2.10)\\
H_{i_{(0)}i_{(1)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}} + R_{i_{(0)}i_{(1)}} && (2.11)\\
H_{i_{(0)}i_{(1)}i_{(2)}} &= \bar{U}_{i_{(0)}}\bar{U}_{i_{(1)}}\bar{U}_{i_{(2)}} \\
&\quad + R_{i_{(0)}i_{(1)}}\bar{U}_{i_{(2)}} + R_{i_{(0)}i_{(2)}}\bar{U}_{i_{(1)}} + R_{i_{(1)}i_{(2)}}\bar{U}_{i_{(0)}} + R_{i_{(0)}i_{(1)}i_{(2)}} && (2.12)\\
&\vdots \qquad \vdots
\end{aligned}$$

where the indices also refer to the spatial points as indicated.

To obtain new symmetries and to calculate its algebra it is much more elaborate to deal with the instantaneous approach of the multi-point equations as the representation of the new symmetries, later called statistical symmetries, becomes simpler in the way that we only have to deal with linear functions. Hence, we only deal with the instantaneous approach which is equivalent to the fluctuation approach through (2.10)–(2.12).

In some cases the list of indices is interrupted by one or more other indices which is pointed out by attaching the replaced value in brackets to the index, i.e.

$$H_{i_{\{n+1\}}[i_{(l)}\mapsto k_{(l)}]} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)})U_{k_{(l)}}(\boldsymbol{x}_{(l)})U_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}. \tag{2.13}$$

This is further extended by

$$H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}] = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)}) U_{k_{(l)}}(\boldsymbol{x}_{(l)})}\,, \tag{2.14}$$

where not only the index $i_{(n+1)}$ is replaced by $k_{(l)}$, but also the independent variable $\boldsymbol{x}_{(n+1)}$ is replaced by $\boldsymbol{x}_{(l)}$. If indices are missing e.g. between $i_{(l-1)}$ and $i_{(l+1)}$ we define

$$H_{i_{\{n\}}[l]_\emptyset} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)}) U_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}\,. \tag{2.15}$$

Finally, if the pressure is involved we write

$$I_{i_{\{n\}}[l]_p} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(l-1)}}(\boldsymbol{x}_{(l-1)}) P(\boldsymbol{x}_{(l)}) U_{i_{(l+1)}}(\boldsymbol{x}_{(l+1)}) \cdot \ldots \cdot U_{i_{(n)}}(\boldsymbol{x}_{(n)})}\,, \tag{2.16}$$

which is, considering all the above definitions, sufficient to derive the MPC equations from Eqs. (2.1) and (2.2) of instantaneous velocity and pressure.

Applying the Reynolds averaging operator (2.4) according to the sum below

$$\mathcal{S}_{i_{\{n+1\}}}(\boldsymbol{x}_{(0)}, \ldots, \boldsymbol{x}_{(n)}) = \overline{\sum_{a=0}^{n} \mathcal{M}_{i_{(a)}}(\boldsymbol{x}_{(a)}) \prod_{b=0, b\neq a}^{n} U_{i_{(b)}}(\boldsymbol{x}_{(b)})}\,, \tag{2.17}$$

we obtain the $\mathcal{S}$-equation which writes

$$\begin{aligned} \mathcal{S}_{i_{\{n+1\}}} = \frac{\partial H_{i_{\{n+1\}}}}{\partial t} + \sum_{l=0}^{n} \Bigg[& \frac{\partial H_{i_{\{n+2\}}[i_{(n+1)} \mapsto k_{(l)}]}[\boldsymbol{x}_{(n+1)} \mapsto \boldsymbol{x}_{(l)}]}{\partial x_{k_{(l)}}} \\ & + \frac{\partial I_{i_{\{n\}}[l]_p}}{\partial x_{i_{(l)}}} - \nu \frac{\partial^2 H_{i_{\{n+1\}}}}{\partial x_{k_{(l)}} \partial x_{k_{(l)}}} \Bigg] = 0 \quad \text{for } n = 1, \ldots, \infty. \end{aligned} \tag{2.18}$$

Loosely speaking Eq. (2.18) implies the statistical information on the Navier–Stokes equations at the expense to deal with an infinite dimensional chain of differential equations starting with order 2, i.e. $n = 1$. Similar to the $\mathcal{S}$-equation we can formulate a corresponding equation for $\mathbf{R}$ which can be obtained by (2.8). The rather remarkable difference between these equations is that (2.18) based on $\mathbf{H}$ is a linear equation which considerably simplifies the analysis of Lie symmetries to be pointed out below.

From Eq. (2.1) continuity equations for $H_{i_{\{n+1\}}}$ and $I_{i_{\{n\}}[l]}$ can be derived. This leads to

$$\frac{\partial H_{i_{\{n+1\}}[i_{(l)} \mapsto k_{(l)}]}}{\partial x_{k_{(l)}}} = 0 \quad \text{for } l = 0, \ldots, n \tag{2.19}$$

and

$$\frac{\partial I_{i_{\{n\}}[k][i_{(l)} \mapsto m_{(l)}]}}{\partial x_{m_{(l)}}} = 0 \quad \text{for } k, l = 0, \ldots, n \quad \text{and} \quad k \neq l. \tag{2.20}$$

3. Symmetries of the Governing Equations

In this section we first recapitulate the Lie symmetries of the Euler and Navier–Stokes equations. In turn they will all be transferred to its corresponding ones for the MPC equations. In the second part we show that the MPC equations admit even more Lie symmetries which are not reflected in the original Euler and Navier–Stokes equations.

All Lie groups will only be presented in infinitesimal forms (see e.g. [3]).

3.1. *Symmetries of the Euler and Navier–Stokes equations*

The Euler equations, i.e. Eqs. (2.1) and (2.2) with $\nu = 0$, admit a ten-parameter symmetry group,

$$
\begin{aligned}
\mathrm{X}_t &= \frac{\partial}{\partial t},\\
\mathrm{X}_{s1} &= x_i\frac{\partial}{\partial x_i} + U_j\frac{\partial}{\partial U_j} + 2P\frac{\partial}{\partial P},\\
\mathrm{X}_{s2} &= t\frac{\partial}{\partial t} - U_i\frac{\partial}{\partial U_i} - 2P\frac{\partial}{\partial P},\\
\mathrm{X}_{rx_3} &= -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - U_2\frac{\partial}{\partial U_1} + U_1\frac{\partial}{\partial U_2}\\
\mathrm{X}_{rx_1} &= -x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} - U_3\frac{\partial}{\partial U_2} + U_2\frac{\partial}{\partial U_3}\\
\mathrm{X}_{rx_2} &= -x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3} - U_3\frac{\partial}{\partial U_1} + U_1\frac{\partial}{\partial U_3}\\
\mathrm{X}_{g1} &= f_1(t)\frac{\partial}{\partial x_1} + \frac{\mathrm{d}f_1(t)}{\mathrm{d}t}\frac{\partial}{\partial U_1} - x_1\frac{\mathrm{d}^2 f_1(t)}{\mathrm{d}t^2}\frac{\partial}{\partial P},\\
\mathrm{X}_{g2} &= f_2(t)\frac{\partial}{\partial x_2} + \frac{\mathrm{d}f_2(t)}{\mathrm{d}t}\frac{\partial}{\partial U_2} - x_2\frac{\mathrm{d}^2 f_2(t)}{\mathrm{d}t^2}\frac{\partial}{\partial P},\\
\mathrm{X}_{g3} &= f_3(t)\frac{\partial}{\partial x_3} + \frac{\mathrm{d}f_3(t)}{\mathrm{d}t}\frac{\partial}{\partial U_3} - x_3\frac{\mathrm{d}^2 f_3(t)}{\mathrm{d}t^2}\frac{\partial}{\partial P},\\
\mathrm{X}_p &= f_4(t)\frac{\partial}{\partial P},
\end{aligned}
\tag{3.1}
$$

with $f_1(t) - f_4(t)$ being arbitrary but smooth functions.

The symmetries $\mathrm{X}_{g1}-\mathrm{X}_{g3}$ comprise translational invariance in space for constant f_1-f_3 as well as the classical Galilei group if f_1-f_3 are linear in time. In its rather general form $\mathrm{X}_{g1}-\mathrm{X}_{g3}$ and X_{10} are direct consequences of an incompressible flow and do not have a counterpart in the case of compressible flows. The complete record of all point-symmetries (3.1) was first published by Pukhnachev [12].

Making the formal transfer from the Euler to the Navier–Stokes equations symmetry properties change and a recombination of the two scaling symmetries X_2 and X_3 is observed

$$
\mathrm{X}_{ScaleNS} = 2t\frac{\partial}{\partial t} + x_i\frac{\partial}{\partial x_i} - U_j\frac{\partial}{\partial U_j} - 2P\frac{\partial}{\partial P},
\tag{3.2}
$$

while the remaining groups stay unaltered.

It should be noted that further symmetries exist for dimensional restricted cases such as plane or axisymmetric flows (see e.g. [1, 4]).

3.2. *Symmetries of the MPC equations*

We consider two different sets of symmetries depending on their origin. First, the symmetries of the Euler– and Navier–Stokes equations, i.e. (3.1) and (3.2), may be transferred to the **H** formulation. The latter set of symmetries, which has no proof of completeness yet, has been taken from [11]. They have no corresponding symmetries in the Navier–Stokes equations, and will be called *statistical symmetries*. In the following we will only deal with infinite Reynolds' number flows, and hence only consider the symmetries of the Euler equations, though knowing very well that this is a singular limit.

3.2.1. *Transferred symmetries*

Before the symmetries (3.1) are transferred into the **H** notation we need to define some special sum convention for a full understanding of the generators with the indices of space and velocity as summation indices ($i = 1, 2, 3$). In particular for the higher order tensors we also have summations of the type

$$n \cdot H_{i_{\{n\}}} \frac{\partial}{\partial H_{i_{\{n\}}}} = \sum_{n=2}^{\infty} \sum_{i_{\{n\}}} n \cdot H_{i_{\{n\}}} \frac{\partial}{\partial H_{i_{\{n\}}}},$$

where the second sum expresses all possible sequences of indices $i_{\{j\}} = 1, 2, 3$ of the length n. This means, there are 3^n possibilities. In the case there is a summation over $I_{i_{\{n-1\}}[q]}$, the summation is taken over all sequences of the length $n-1$ and over all possible positions q of the pressure as defined in (2.16).

The Galilei invariance has a further index $\alpha \in \{1, 2, 3\}$, so that all three generators in (3.1) can be written in one equation.

With these definitions at hand we obtain the classical symmetries in **H** form

$$\begin{aligned}
\mathrm{Z}_t &= \frac{\partial}{\partial t}, \\
\mathrm{Z}_{s1} &= x_i \frac{\partial}{\partial x_i} + \bar{U}_i \frac{\partial}{\partial \bar{U}_i} + 2\bar{P}\frac{\partial}{\partial \bar{P}} + n \cdot H_{i_{\{n\}}} \frac{\partial}{\partial H_{i_{\{n\}}}} + (n+1) \cdot I_{i_{\{n-1\}}[q]_p} \frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \\
\mathrm{Z}_{s2} &= t\frac{\partial}{\partial t} - \bar{U}_i \frac{\partial}{\partial \bar{U}_i} - 2\bar{P}\frac{\partial}{\partial \bar{P}} - n \cdot H_{i_{\{n\}}} \frac{\partial}{\partial H_{i_{\{n\}}}} - (n+1) \cdot I_{i_{\{n-1\}}[q]_p} \frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \\
\mathrm{Z}_{rx_3} &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \bar{U}_1 \frac{\partial}{\partial \bar{U}_2} - \bar{U}_2 \frac{\partial}{\partial \bar{U}_1} \\
&\quad + \sum_{b=0}^{n-1} (\delta_{i_{(b)},2} H_{i_{\{n\}}[i_{(b)} \mapsto 1]} - \delta_{i_{(b)},1} H_{i_{\{n\}}[i_{(b)} \mapsto 2]}) \frac{\partial}{\partial H_{i_{\{n\}}}} \\
&\quad + \sum_{b=0, b\neq q}^{n-1} (\delta_{i_{(b)},2} I_{i_{\{n-1\}}[q]_p[i_b \mapsto 1]} - \delta_{i_{(b)},1} I_{i_{\{n-1\}}[q]_p[i_b \mapsto 2]}) \frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \\
\mathrm{Z}_{rx_1} &= x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + \bar{U}_2 \frac{\partial}{\partial \bar{U}_3} - \bar{U}_3 \frac{\partial}{\partial \bar{U}_2} \\
&\quad + \sum_{b=0}^{n-1} (\delta_{i_{(b)},3} H_{i_{\{n\}}[i_{(b)} \mapsto 2]} - \delta_{i_{(b)},2} H_{i_{\{n\}}[i_{(b)} \mapsto 3]}) \frac{\partial}{\partial H_{i_{\{n\}}}} \\
&\quad + \sum_{b=0, b\neq q}^{n-1} (\delta_{i_{(b)},3} I_{i_{\{n-1\}}[q]_p[i_{(b)} \mapsto 2]} - \delta_{i_{(b)},2} I_{i_{\{n-1\}}[q]_p[i_{(b)} \mapsto 3]}) \frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Z}_{rx_2} &= x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3} + \bar{U}_3\frac{\partial}{\partial \bar{U}_1} - \bar{U}_1\frac{\partial}{\partial \bar{U}_3} \\
&\quad + \sum_{b=0}^{n-1}(\delta_{i_{(b)},1}H_{i_{\{n\}}[i_{(b)}\mapsto 3]} - \delta_{i_{(b)},3}H_{i_{\{n\}}[i_{(b)}\mapsto 1]})\frac{\partial}{\partial H_{i_{\{n\}}}} \\
&\quad + \sum_{b=0,b\neq q}^{n-1}(\delta_{i_{(b)},1}I_{i_{\{n-1\}}[q]_p[i_{(b)}\mapsto 3]} - \delta_{i_{(b)},3}I_{i_{\{n-1\}}[q]_p[i_{(b)}\mapsto 1]})\frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \\
\mathrm{Z}_{g\alpha} &= f_\alpha(t)\frac{\partial}{\partial x_\alpha} + f'_\alpha(t)\frac{\partial}{\partial \bar{U}_\alpha} - x_\alpha f''_\alpha(t)\frac{\partial}{\partial \bar{P}} + f'_\alpha(t)\sum_{b=0}^{n-1}\delta_{i_{(b)},1}H_{i_{\{n-1\}}[b]_\emptyset}\frac{\partial}{\partial H_{i_{\{n\}}}} \\
&\quad + \left(-x_\alpha f''_\alpha(t)H_{i_{\{n-1\}}[q]_\emptyset} + \sum_{c=0,c\neq q}^{n-1}\delta_{i_{(c)}1}f'_\alpha(t)I_{i_{\{n-2\}}[q]_p[c]_\emptyset}\right)\frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \\
\mathrm{Z}_p &= f_4(t)\frac{\partial}{\partial \bar{P}} + f_4(t)H_{i_{\{n-1\}}[q]_\emptyset}\frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}.
\end{aligned} \tag{3.3}$$

3.2.2. *Statistical symmetries*

Additionally, to (3.3) we have to consider the statistical symmetries

$$\mathrm{Z}_{sH} = \bar{U}_i\frac{\partial}{\partial \bar{U}_i} + \bar{P}\frac{\partial}{\partial \bar{P}} + H_{i_{\{n\}}}\frac{\partial}{\partial H_{i_{\{n\}}}} + I_{i_{\{n-1\}}[q]}\frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \tag{3.4}$$

$$\mathrm{Z}_{I,i_{\{n-1\}},q} = f_{I,i_{\{n-1\}}[q]}(t)\frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \tag{3.5}$$

which are directly taken from [11]. The summation over all possible indices in formula (3.4) is equivalent to the transferred symmetries (3.3). Every arbitrary choice of the indices $i_{\{n\}}$, $n \geq 1$, and any q $(0 \leq q \leq n)$ in the symmetry (3.5), yields an independent symmetry.

Further the set of symmetries in [11] has to be extended employing commutator theory and Lie algebra, so that two more sets of symmetries

$$Z_{H,i_{\{n\}},q} = g_{i_{\{n\}},q}(t)\frac{\partial}{\partial H_{i_{\{n\}}}} - x_{i_{(q)}}g'_{i_{\{n\}},q}(t)\frac{\partial}{\partial I_{i_{\{n-1\}}[q]_p}}, \tag{3.6}$$

$$Z_{I2,i_{\{m\}},a,r,\beta} = -x_\beta h_{i_{\{m\}},a,r,\beta}(t)\frac{\partial}{\partial I_{i_{\{m\}}[\downarrow a]_p}} + \begin{cases} x_{k_{(r)}}h_{i_{\{m\}},a,r,\beta}(t)\frac{\partial}{\partial I_{\hat{i}^\beta_{\{m\}}[r]_p}} & r < a \\ x_{k_{(r)}}h_{i_{\{m\}},a,r,\beta}(t)\frac{\partial}{\partial I_{\hat{i}^\beta_{\{m\}}[r+1]_p}} & r \geq a \end{cases}, \tag{3.7}$$

are invoked, where $0 \leq q \leq n-1$, $0 \leq a \leq m$, $0 \leq r \leq m$, $b \in \{1, 2, 3\}$, $\hat{i}^\beta_{\{m+1\}} = i_{\{m\}}[\downarrow a : \beta]$ and two arbitrary but smooth functions $g(t)$ and $h(t)$. Here $\hat{i}^\beta_{\{m+1\}} = i_{\{m\}}[\downarrow a : \beta]$ means that between the position $a-1$ and a of the sequence $i_{\{m\}}$ one element (β) is inserted. Furthermore the definition

$$I_{i_{\{m\}}[\downarrow a]_p} = \overline{U_{i_{(0)}}(\boldsymbol{x}_{(0)}) \cdot \ldots \cdot U_{i_{(a-1)}}(\boldsymbol{x}_{(a-1)})P(\boldsymbol{x}_{(b)})U_{i_{(a)}}(\boldsymbol{x}_{(a)}) \cdot \ldots \cdot U_{i_{(m-1)}}(\boldsymbol{x}_{(m-1)})}$$

holds. As before any choice of the sequences $i_{\{n\}}$ and $i_{\{m\}}$ and the constants leads to a different symmetry.

For the sake of completeness we will finally give the infinite dimensional symmetry admitted by all linear differential equations, i.e.

$$\mathrm{Z}_\infty = \bar{V}_i \frac{\partial}{\partial \bar{U}_i} + \bar{Q} \frac{\partial}{\partial \bar{P}} + F_{i_{\{n\}}} \frac{\partial}{\partial H_{i_{\{n\}}}} + G_{i_{\{n\}}} \frac{\partial}{\partial I_{i_{\{n\}}}}. \tag{3.8}$$

Here the infinitesimals $\bar{V}_i$, $\bar{Q}$, $F_{i_{\{n\}}}$ and $G_{i_{\{n\}}}$ refer to any solution of the original system (2.18)–(2.20) with the variables respectively associated to $\bar{U}_i$, $\bar{P}$, $H_{i_{\{n\}}}$ and $I_{i_{\{n\}}}$. This symmetry in fact implies the superposition principle of linear differential equations and, as usual, it will not be invoked for the Lie algebra to follow in the next chapter. This completes the presently known set of symmetries of the MPC equations.

4. Lie Algebra

The key motivation for obtaining all Lie symmetries of the MPC equations is the desire to obtain a deeper understanding of statistical turbulence quantities based on turbulent scaling laws. For this it would be rather worthwhile to have a complete set of symmetries which, however, is difficult to obtain directly using Lie's algorithm since the MPC equation constitutes an infinite dimensional set of equations.

A first step in this direction is the calculation of the Lie algebra of the presently known symmetries in [11]. The ratio behind this is that if any commutator

$$[Z_i, Z_j] = Z_i Z_j - Z_j Z_i \tag{4.1}$$

of any two symmetries in operator form, which again is a first order operator, is not a linear combination of the set of all symmetries, there must exist an additional symmetry (see [2]). By this way new symmetries are generated and a Lie algebra may be computed.

Still, completeness of all symmetries of the MPC equation is not necessarily proven with this even if we show the symmetries form an algebra. The reason is we cannot ensure that we generated the largest algebra possible and not only a sub-algebra.

4.1. *The derivation of new statistical symmetries*

In [11] the symmetries

$$\mathrm{Z}_{old,i_{\{n\}}} = \frac{\partial}{\partial H_{i_{\{n\}}}} \tag{4.2}$$

have been observed which do not lead to a Lie algebra with the other symmetries therein. In detail we have to consider the commutator between Z_{g1} and Z_{sH}

$$\begin{aligned}\left[Z_{old,i_{\{n\}}}, X_{G1}\right] = {} & f_1'(t) \sum_{b=0}^{n} \sum_{j_{\{n+1\}}} \delta_{j_{(b)},1} \delta_{j_{\{n\}}[b],i_{\{n\}}} \frac{\partial}{\partial H_{j_{\{n+1\}}}} \\ & - x_1 f_1''(t) \sum_{q=0}^{n} \sum_{j_{\{n\}}[q]_p} \delta_{j_{\{n\}}[q],i_{\{n\}}} \frac{\partial}{\partial I_{j_{\{n\}}[q]_p}},\end{aligned} \tag{4.3}$$

which cannot be expressed as a linear combination of the symmetries in [11]. A Kronecker delta between two sequences $i_{\{n\}}$ and $j_{\{n\}}$ means that elements at the same position $i_{(k)}$ and $j_{(k)}$, $0 \leq k \leq n-1$, are equal.

Hence, the symmetry (3.6) can be inferred by (4.3) and it is straightforward to show that (4.3) is indeed a linear combination of (2.18). Further if $g_{i_{\{n\}},q}(t)$ in (3.6) is a constant only $Z_{old,i_{\{n\}}}$ remains. This means that $Z_{H,i_{\{n\}},q}$ is an extension of $Z_{old,i_{\{n\}}}$ and for the subsequent calculations $Z_{old,i_{\{n\}}}$ may not be considered any further.

Even this extended set of symmetries does not form a Lie algebra. Considering the commutator

$$\begin{aligned}
[Z_{g1}, Z_{H,k_{\{m\}},r}] = &- \delta_{k_{(r)},1}\tilde{Z}_{I,k_{\{m-1\}}[r]} - \sum_{a=0}^{m}\sum_{i_{\{m+1\}}[a]_p} \delta_{i_{(b)},1}\delta_{i_{\{m\}}[a]_\emptyset,k_{\{m\}}}\tilde{Z}_{H,i_{\{m+1\}},a} \\
&- x_1 f_1'(t)g'(t)\sum_{q=0}^{m}\frac{\partial}{\partial I_{k_{\{m\}}[\downarrow q]_p}} \\
&+ x_{k_{(r)}} f_1'(t)g'(t)\sum_{i_{\{m\}}[r+1]_p}\sum_{c<r+1}\delta_{i_{(c)},1}\delta_{i_{\{m-1\}}[r+1,c],k_{\{m-1\}}[r]}\frac{\partial}{\partial I_{i_{\{m\}}[r+1]_p}} \\
&+ x_{k_{(r)}} f_1'(t)g'(t)\sum_{i_{\{m\}}[r+1]_p}\sum_{c>r}\delta_{i_{(c)},1}\delta_{i_{\{m-1\}}[r,c],k_{\{m-1\}}[r]}\frac{\partial}{\partial I_{i_{\{m\}}[r]_p}}, \qquad (4.4)
\end{aligned}$$

we observe that the last three lines cannot be expressed in terms of the previous symmetries, so that we again get a new symmetry. The last part of (4.4) implies the symmetry (3.7) and it may be directly proven that this is indeed a linear combination of (2.18).

With this our calculation of new symmetries is complete and in the following section we will prove this employing Lie algebra methods.

4.2. *Complete algebra of all symmetries*

4.2.1. *Sub-algebra of the classical symmetries*

Since the symmetries (3.3) originally stem from the Euler equations, we may naturally expect that the classical symmetries generate a sub-algebra, which can be verified in Table 1. Note that a tilde over any of the symmetries simply denotes that the arbitrary function of this symmetry has changed its form but does not refer to a new symmetry.

4.2.2. *Sub-algebra of the statistical symmetries*

Further, also the statistical symmetries (3.4)–(3.5) form a sub-algebra as may be taken from Table 2. Note, that here an abbreviatory notation has been introduced since $\mathrm{Z}_{H,i_{\{n\}}[q]}$ refers to an infinite number of symmetries since the tensor order n in principle varies from 1 to ∞. This has been taken into account and pairs of two different translation symmetries from H and I have been commutated against each other, leaving only zeros for the related structure constants (see [2]).

Table 1. Commutator table of the classical symmetries.

	Z_t	Z_{s1}	Z_{s2}	Z_{rx_3}	Z_{rx_1}	Z_{rx_2}	Z_{g1}	Z_{g2}	Z_{g3}	Z_p
Z_t	0	0	Z_t	0	0	0	$\tilde{Z}_{g1}$ [a]	$\tilde{Z}_{g2}$ [b]	$\tilde{Z}_{g3}$ [c]	$\tilde{Z}_p$ [d]
Z_{s1}	0	0	0	0	0	0	$-Z_{g1}$	$-Z_{g2}$	$-Z_{g3}$	$-2Z_p$
Z_{s2}	$-Z_t$	0	0	0	0	0	$\tilde{Z}_{g1}$ [e]	$\tilde{Z}_{g2}$ [f]	$\tilde{Z}_{g3}$ [g]	$-\tilde{Z}_p$ [h]
Z_{rx_3}	0	0	0	0	$-Z_{rx_2}$	Z_{rx_1}	$-\tilde{Z}_{g2}$ [i]	$\tilde{Z}_{g1}$ [j]	0	0
Z_{rx_1}	0	0	0	Z_{rx_2}	0	$-Z_{rx_3}$	0	$-\tilde{Z}_{g3}$ [k]	$\tilde{Z}_{g2}$ [l]	0
Z_{rx_2}	0	0	0	$-Z_{rx_1}$	Z_{rx_3}	0	$\tilde{Z}_{g3}$ [m]	0	$-\tilde{Z}_{g1}$ [n]	0
Z_{g1}	$-\tilde{Z}_{g1}$ [a]	Z_{g1}	$-\tilde{Z}_{g1}$ [e]	$\tilde{Z}_{g2}$ [i]	0	$-\tilde{Z}_{g3}$ [m]	0	0	0	0
Z_{g2}	$-\tilde{Z}_{g2}$ [b]	Z_{g2}	$-\tilde{Z}_{g2}$ [f]	$-\tilde{Z}_{g1}$ [j]	$\tilde{Z}_{g3}$ [k]	0	0	0	0	0
Z_{g3}	$-\tilde{Z}_{g3}$ [c]	Z_{g3}	$-\tilde{Z}_{g3}$ [g]	0	$-\tilde{Z}_{g2}$ [l]	$\tilde{Z}_{g1}$ [n]	0	0	0	0
Z_p	$-\tilde{Z}_p$ [d]	$2Z_p$	$\tilde{Z}_p$ [h]	0	0	0	0	0	0	0

[a]with $\tilde{f}_1(t) = f_1'(t)$
[b]with $\tilde{f}_2(t) = f_2'(t)$
[c]with $\tilde{f}_3(t) = f_3'(t)$
[d]with $\tilde{f}_4(t) = f_4'(t)$
[e]with $\tilde{f}_1(t) = tf_1'(t)$
[f]with $\tilde{f}_2(t) = tf_2'(t)$
[g]with $\tilde{f}_3(t) = tf_3'(t)$
[h]with $\tilde{f}_4(t) = tf_4'(t) + 2f_4(t)$
[i]with $\tilde{f}_2(t) = f_1(t)$
[j]with $\tilde{f}_1(t) = f_2(t)$
[k]with $\tilde{f}_3(t) = f_2(t)$
[l]with $\tilde{f}_2(t) = f_3(t)$
[m]with $\tilde{f}_3(t) = f_1(t)$
[n]with $\tilde{f}_1(t) = f_3(t)$

Table 2. Commutator table of the statistical symmetries.

	Z_{sH}	$Z_{H,i_{\{n\}},q}$	$Z_{H,j_{\{m\}},r}$	$Z_{I,i_{\{n-1\}},q}$	$Z_{I,j_{\{m-1\}},r}$	$Z_{I2,i_{\{m\}},a,r,\beta}$
Z_{sH}	0	$-Z_{H,i_{\{n\}},q}$	$-Z_{H,j_{\{m\}},r}$	$-Z_{I,i_{\{n-1\}},q}$	$-Z_{I,j_{\{m-1\}},r}$	$-Z_{I2,i_{\{m\}},a,r,\beta}$
$Z_{H,i_{\{n\}},q}$	$Z_{H,i_{\{n\}},q}$	0	0	0	0	0
$Z_{H,j_{\{m\}},r}$	$Z_{H,j_{\{m\}},r}$	0	0	0	0	0
$Z_{I,i_{\{n-1\}},q}$	$Z_{I,i_{\{n-1\}},q}$	0	0	0	0	0
$Z_{I,j_{\{m-1\}},r}$	$Z_{I,j_{\{m-1\}},r}$	0	0	0	0	0
$Z_{I2,i_{\{m\}},a,r,\beta}$	$Z_{I2,i_{\{m\}},a,r,\beta}$	0	0	0	0	0

4.2.3. *Commutator table for combinations of the classical and the statistical symmetries*

The most interesting part is the commutation of both sets of symmetries. In Table 3 we omitted one half of it for brevity because it is always skew-symmetric.

Since some of the commutators are rather complicated, we introduce the symbol $L(X)$ or $L(Y, Z)$ referring to the fact that they may be expressed as a linear combination of the generators of X or Y and Z. The complete linear expressions in Table 3 are given by

$$\begin{aligned} L_{1,\alpha,\beta}(Z_H) = \sum_{b=0,b\neq q}^{n-1} \sum_{j_{\{n-1\}}} &(\delta_{j_{(b)},\beta}\delta_{j_{\{n\}}[i_{(b)}\mapsto\alpha],i_{\{n\}}} - \delta_{j_{(b)},\alpha}\delta_{j_{\{n\}}[i_{(b)}\mapsto\beta],i_{\{n\}}})Z_{H,j_{\{n\}},q} \\ &+ \delta_{i_{(q)},\alpha}Z_{H,\tilde{\tilde{i}}_{\{n\}},q} - \delta_{i_{(q)},\beta}Z_{H,\tilde{i}_{\{n\}},q} \end{aligned}$$

with

$$\tilde{\tilde{i}}_{\{n\}} = i_{\{n\}}[i_{(q)} \mapsto \beta], \quad \tilde{i}_{\{n\}} = i_{\{n\}}[i_{(q)} \mapsto \alpha]$$

Table 3. Commutator table of the combinations between both sets.

	Z_{sH}	$Z_{H,i_{\{n\}},q}$	$Z_{I,i_{\{n-1\}},q}$	$Z_{I2,i_{\{m\}},a,r,\beta}$
Z_t	0	$\tilde{Z}_{H,i_{\{n\}}}$ [a]	$\tilde{Z}_{I,i_{\{n-1\}},q}$ [b]	$\tilde{Z}_{I2,i_{\{m\}},a,r,\beta}$ [h]
Z_{s1}	0	$-nZ_{H,i_{\{n\}}}$	$-(n+1)Z_{I,i_{\{n-1\}},q}$	$-(m+1)Z_{I2,i_{\{m\}},a,r,\beta}$
Z_{s2}	0	$\tilde{Z}_{H,i_{\{n\}}}$ [c]	$(n+1)Z_{I,i_{\{n-1\}},q}$ [d]	$\tilde{Z}_{I2,i_{\{m\}},a,r,\beta}$ [i]
Z_{rx_3}	0	$L_{1,2,1}(Z_H)$	$L_{2,1,2}(Z_I)$	$L_{6,1,2}(Z_I,Z_{I2})$
Z_{rx_1}	0	$L_{1,3,2}(Z_H)$	$L_{2,2,3}(Z_I)$	$L_{6,2,3}(Z_I,Z_{I2})$
Z_{rx_2}	0	$L_{1,1,3}(Z_H)$	$L_{2,3,1}(Z_I)$	$L_{6,3,1}(Z_I,Z_{I2})$
Z_{g1}	$\tilde{Z}_{H,i_{\{1\}}[i_{(0)}\to 1],0}$ [e]	$L_{3,1}(Z_I,Z_H,Z_{I2})$	$L_{4,1}(Z_I)$	$L_{7,1}(Z_I,Z_{I2})$
Z_{g2}	$\tilde{Z}_{H,i_{\{1\}}[i_{(0)}\to 2],0}$ [f]	$L_{3,2}(Z_I,Z_H,Z_{I2})$	$L_{4,2}(Z_I)$	$L_{7,2}(Z_I,Z_{I2})$
Z_{g3}	$\tilde{Z}_{H,i_{\{1\}}[i_{(0)}\to 3],0}$ [g]	$L_{3,3}(Z_I,Z_H,Z_{I2})$	$L_{4,3}(Z_I)$	$L_{7,3}(Z_I,Z_{I2})$
Z_p	$\tilde{Z}_{I,\{i_0\},0}$	$L_5(Z_I)$	0	0

[a]with $\tilde{g}_{i_{\{n\}},q}(t)=g'_{i_{\{n\}},q}(t)$

[b]with $\tilde{f}_{I,i_{\{n-1\}},q}(t)=f'_{I,i_{\{n-1\}},q}(t)$

[c]with $\tilde{g}_{i_{\{n\}},q}(t)=tg'_{i_{\{n\}},q}(t)+ng_{i_{\{n\}},q}(t)$

[d]with $\tilde{f}_{I,i_{\{n-1\}},q}(t)=tf'_{I,i_{\{n-1\}},q}(t)+nf_{I,i_{\{n-1\}},q}(t)$

[e]with $g_{i_{\{n\}},q}(t)=f'_1(t)$

[f]with $g_{i_{\{n\}},q}(t)=f'_2(t)$

[g]with $g_{i_{\{n\}},q}(t)=f'_3(t)$

[h]with $\tilde{h}_{i_{\{m\}},a,r,\beta}(t)=h'_{i_{\{m\}},a,r,\beta}(t)$

[i]with $\tilde{h}_{i_{\{m\}},a,r,\beta}(t)=(m+2)h_{i_{\{m\}},a,r,\beta}(t)+th'_{i_{\{m\}},a,r,\beta}(t)$

$$L_{2,\alpha,\beta}(Z_I)=-\sum_{j_{\{n-1\}}[q]}\sum_{b=0,b\neq q}^{n-1}(\delta_{j_{(b)},\beta}\delta_{j_{\{n-1\}}[q][j_{(b)}\mapsto\alpha],i_{\{n-1\}}[q]}$$
$$-\delta_{j_{(b)},\alpha}\delta_{j_{\{n-1\}}[q][j_{(b)}\mapsto\beta],i_{\{n-1\}}[q]})Z_{I,j_{\{n-1\}},q}$$

$$L_{3,\alpha}(Z_I,Z_H,Z_{I2})=-\delta_{i_{(q)},\alpha}\tilde{Z}_{I,i_{\{n-1\}}[q]}-\sum_{a=0}^{n}\sum_{j_{\{n+1\}}}\delta_{j_{(a)},\alpha}\delta_{j_{\{n\}}[a],i_{\{n\}}}\tilde{Z}_{H,j_{\{n+1\}},a}$$
$$+\sum_{a=0}^{n}Z_{I2,i_{\{n\}},a,q,\beta}$$

with

$$\tilde{f}_I(t)=f_\alpha(t)g'_{i_{\{m\}},a,r,\beta}(t),$$
$$\tilde{g}(t)=f'_\alpha(t)g_{i_{\{m\}},a,r,\beta}(t),\tilde{h}(t)=f'_\alpha(t)g'_{i_{\{m\}},a,r,\beta}(t)$$

$$L_{4,\alpha}(Z_I)=\sum_{c<q}\sum_{j_{\{n\}}[q+1]}\delta_{j_{(c)},\alpha}\delta_{j_{\{n-1\}}[q+1,c],i_{\{n-1\}}[q]}\tilde{Z}_{I,j_{\{n\}}[q+1]}$$
$$+\sum_{c>q}\sum_{j_{\{n\}}[q]}\delta_{j_{(c)},\alpha}\delta_{j_{\{n-1\}}[q,c],i_{\{n-1\}}[q]}\tilde{Z}_{I,j_{\{n\}}[q]}$$

with

$$\tilde{f}_{I,j_{\{n\}}[q]}(t)=\tilde{f}_{I,j_{\{n\}}[q+1]}(t)=-f_{I,i_{\{n-1\}}[q]}(t)f'_1(t)$$

$$L_5(Z_I)=\sum_{r=0}^{n}\tilde{Z}_{I,i_{\{n\}}[\downarrow r]}\quad\text{with }\tilde{f}_I(t)=-f_4(t)g_{i_{\{n\}}[q]}(t)$$

$$
\begin{aligned}
L_{6,\alpha,\gamma}(Z_{I2}) = {} & \delta_{k_{(r)},\gamma} Z_{I2,\tilde{\tilde{k}}_{\{m\}},a,r,\beta} - \delta_{k_{(r)},\alpha} Z_{I2,\tilde{k}_{\{m\}},a,r,\beta} + \delta_{\beta,\gamma} Z_{I2,k_{\{m\}},a,r,1} \\
& - \delta_{\beta,\alpha} Z_{I2,k_{\{m\}},a,r,2} + \sum_{b<a,\left\{ \substack{a>r:b\neq r \\ a\leq r:b\neq r+1} \right.} (\delta_{k_{(b)},\gamma} Z_{I2,\hat{k}^1_{\{m\}},a,r,\beta} - \delta_{k_{(b)},\alpha} Z_{I2,\hat{k}^2_{\{m\}},a,r,\beta}) \\
& + \sum_{b>a,\left\{ \substack{a>r:b\neq r \\ a\leq r:b\neq r+1} \right.} (\delta_{k_{(b-1)},\gamma} Z_{I2,\hat{\hat{k}}^1_{\{m\}},a,r,\beta} - \delta_{k_{(b-1)},\alpha} Z_{I2,\hat{\hat{k}}^2_{\{m\}},a,r,\beta})
\end{aligned}
$$

with

$$
\begin{aligned}
& \tilde{k}_{\{m\}} = k_{\{m\}}[k_{(r)} \mapsto \gamma], \tilde{\tilde{k}}_{\{m\}} = k_{\{m\}}[k_{(r)} \mapsto \alpha], \hat{k}^2_{\{m\}} = k_{\{m\}}[k_{(b)} \mapsto \gamma], \\
& \qquad \hat{k}^1_{\{m\}} = k_{\{m\}}[k_{(b)} \mapsto \alpha], \hat{\hat{k}}^2_{\{m\}} = k_{\{m\}}[k_{(b-1)} \mapsto \gamma], \hat{\hat{k}}^1_{\{m\}} = k_{\{m\}}[k_{(b-1)} \mapsto \alpha].
\end{aligned}
$$

The expression L_7 depends on the relation between a and r, so that we consider two cases. If $a \leq r$ we gain

$$
\begin{aligned}
L^{a\leq r}_{7,\beta}(Z_I, Z_{I2}) = {} & -\delta_{\gamma,\beta} \tilde{Z}_{I,k_{\{m\}}[\downarrow a]} + \delta_{\gamma,k_{(a)}} \sum_{j_{\{m\}}[r+1]} \delta_{j_{(a)}\beta} \delta_{j_{\{m-1\}}[r+1,a],k_{\{m-1\}}[r]} \tilde{Z}_{I,j_{\{m\}}[r+1]} \\
& + \sum_{c\leq a} \tilde{Z}_{I2,\hat{k}^c_{\{m+1\}},a+1,r+1,\beta} + \sum_{a<c<r+2} \tilde{Z}_{I2,\tilde{k}^c_{\{m+1\}},a,r+1,\beta} + \sum_{c>r+1} \tilde{Z}_{I2,\tilde{k}^c_{\{m+1\}},a,r,\beta}
\end{aligned}
$$

with

$$
\begin{aligned}
& \hat{k}^c_{\{m+1\}} = k_m[\downarrow c : \gamma], \tilde{k}^c_{\{m+1\}} = k_m[\downarrow c-1 : \gamma], \\
& \quad \tilde{h}(t) = f'_\gamma(t) h_{i_{\{m\}},a,r,\beta}(t), \tilde{f}_{I,k_{\{m\}}[a]}(t) = \tilde{f}_{I,j_{\{m\}}[r]}(t) = f_\gamma(t) h_{i_{\{m\}},a,r,\beta}(t),
\end{aligned}
$$

where the summation index c only appears in the formulation of the sequences $\tilde{k}$ and $\hat{k}$. The latter case is $a > r$

$$
\begin{aligned}
L^{a>r}_{7,\gamma}(Z_I, Z_{I2}) = {} & -\delta_{\gamma,\beta} \tilde{Z}_{I,k_{\{m\}}[\downarrow a]} + \delta_{\gamma,k_{(a)}} \sum_{j_{\{m\}}[r]} \delta_{j_{(a)}\beta} \delta_{j_{\{m-1\}}[r,a],k_{\{m-1\}}[r]} \tilde{Z}_{I,j_{\{m\}}[r]} \\
& + \sum_{c>a} \tilde{Z}_{I2,\tilde{k}^c_{\{m+1\}},a,r,\beta} + \sum_{r<c<a+1} \tilde{Z}_{I2,\hat{k}^c_{\{m+1\}},a+1,r,\beta} + \sum_{c<r+1} \tilde{Z}_{I2,\hat{k}^c_{\{m+1\}},a+1,r+1,\beta}
\end{aligned}
$$

with

$$
\begin{aligned}
& \hat{k}^c_{\{m+1\}} = k_{\{m\}}[\downarrow c : \gamma], \tilde{k}^c_{\{m+1\}} = k_{\{m\}}[\downarrow c-1 : \gamma], \\
& \quad \tilde{h}(t) = f'_\gamma(t) h_{i_{\{m\}},a,r,\beta}(t), \tilde{f}_{I,k_{\{m\}}[a]}(t) = \tilde{f}_{I,j_{\{m\}}[r]}(t) = f_\gamma(t) h_{i_{\{m\}},a,r,\beta}(t).
\end{aligned}
$$

5. Conclusion

In extending the results in [11] we have presently shown that the symmetries of the MPC equations form a algebra with sub-algebras for symmetries stemming from classical mechanics and the statistical symmetries. With this approach there is no potential to extend the symmetries any further. Hence, in order to show completeness of all symmetries of the MPC equations other techniques have to be employed such as Lie's algorithm though the computations may be cumbersome for this infinite dimensional system.

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