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## CLASSICAL LIE SYMMETRIES AND REDUCTIONS OF A NONISOSPECTRAL LAX PAIR

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The classical Lie method is applied to a nonisospectral problem associated with a system of partial differential equations in  $2 + 1$  dimensions (Maccari A, *J. Math. Phys.* **12** (1998) 6547–6551.). Identification of the classical Lie symmetries provides a set of reductions that give rise to different nontrivial spectral problems in  $1 + 1$  dimensions. The form in which the spectral parameter of the  $1 + 1$  Lax pair is introduced is carefully described.

*Keywords:* Lie symmetries; similarity reductions; problems.

Mathematics Subject Classification 2000: 35C06, 35P30, 35051

### 1. Introduction

Some of the authors [6] have applied the classical Lie Method [2, 10] not just to a partial differential equation (PDE) in  $2 + 1$  dimensions, but also to the Lax pair associated with this PDE [9]. This requires consideration not only of the independent variables and the fields involved in the PDE, but also the eigenfunctions of the Lax pair. The bonus is that once we have identified the symmetries, we can proceed to the corresponding  $1 + 1$  reductions [12], which provide us with the reduced  $1 + 1$  equations as well as their associated spectral problems. The spectral parameter of the reduced  $1 + 1$  Lax pair is introduced in a very natural way.

At this point, it is interesting to recall the Ablowitz–Ramani–Segur conjecture [1], which establishes that a PDE is integrable in the Painlevé sense [11] if all its reductions pass the

Painlevé test [13]. This means that solutions of a PDE can be achieved by solving its reductions to ordinary differential equations (ODE). Nevertheless, it is well known [5] that it is often more difficult to find the linear problem associated with a  $1+1$  PDE than with a  $2+1$  PDE. In this sense, our approach is the opposite of that of Ablowitz–Ramani–Segur. We start with a  $2+1$  spectral problem and we obtain several nontrivial  $1+1$  Lax pairs through classical Lie reductions of the former.

The problem we are considering here concerns the following  $2+1$  system of PDEs:

$$0 = u_t + u_{xxx} + u_{xy} - 6u\omega u_x + 2m_y u \quad (1.1)$$

$$0 = \omega_t + \omega_{xxx} - \omega_{xy} - 6u\omega\omega_x - 2m_y\omega \quad (1.2)$$

$$0 = m_x + u\omega. \quad (1.3)$$

This system is the real version of the PDE proposed by Maccari in [8]. This equation is a particular member of a class of integrable equations found by Calogero and Degasperis in [3] and constitutes a  $2+1$  generalization of the equation proposed by Hirota in [7]. In reference [4] one of us proved that the system has the Painlevé property and the singular manifold method [13] was used to derive the following nonisospectral two-component Lax pair associated with the system (1.1)–(1.3).

$$0 = \psi_x + \frac{\lambda}{2}\psi + u\chi \quad (1.4)$$

$$0 = \chi_x - \frac{\lambda}{2}\chi + \omega\psi \quad (1.5)$$

$$0 = \psi_t - \lambda\psi_y + \left[ m_y - \omega u_x + u\omega_x + \lambda u\omega - \frac{1}{2}\lambda_y - \frac{1}{2}\lambda^3 \right] \psi \\ + [2\omega u^2 - u_y - u_{xx} + \lambda u_x - \lambda^2 u] \chi \quad (1.6)$$

$$0 = \chi_t - \lambda\chi_y + \left[ -m_y + \omega u_x - u\omega_x - \lambda u\omega - \frac{1}{2}\lambda_y + \frac{1}{2}\lambda^3 \right] \chi \\ + [2\omega^2 u + \omega_y - \omega_{xx} - \lambda\omega_x - \lambda^2\omega] \psi. \quad (1.7)$$

The system is particularly interesting because the compatibility conditions  $\psi_{xt} = \psi_{tx}$ ,  $\chi_{xt} = \chi_{tx}$  imply that the spectral parameter  $\lambda$  is a function of  $y$  and  $t$  that satisfies:

$$\lambda_t - \lambda\lambda_y = 0, \quad \lambda_x = 0. \quad (1.8)$$

The possible  $1+1$  dimensional reductions of (1.1)–(1.3) are of course interesting, but it is better to study the reductions of the spectral problem (1.4)–(1.7) because, by doing this, we shall know how the spectral parameter appears in the reduction and this is a nontrivial question at all as we shall see.

In Sec. 2, we apply the Classical Lie Method of finding point symmetries to (1.4)–(1.7) system. The different possible reductions corresponding to these symmetries are identified in Sec. 3. Different non trivial  $1+1$  spectral problems are obtained through these reductions. The conclusions are presented in Sec. 4.

## 2. Classical Symmetries

In order to apply the Classical Lie Method [12] to the system of PDEs (1.4)–(1.7) with three independent variables and six fields, we consider the one-parameter Lie group of infinitesimal transformations given by:

$$x' = x + \varepsilon \xi_1(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.1)$$

$$y' = y + \varepsilon \xi_2(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.2)$$

$$t' = t + \varepsilon \xi_3(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2), \quad (2.3)$$

and

$$u' = u + \varepsilon \phi_1(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.4)$$

$$\omega' = \omega + \varepsilon \phi_2(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.5)$$

$$m' = m + \varepsilon \phi_3(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.6)$$

$$\psi' = \psi + \varepsilon \phi_4(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.7)$$

$$\chi' = \chi + \varepsilon \phi_5(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2) \quad (2.8)$$

$$\lambda' = \lambda + \varepsilon \phi_6(x, y, t, u, \omega, m, \psi, \chi, \lambda) + O(\varepsilon^2), \quad (2.9)$$

where  $\varepsilon$  is the group parameter. This transformation must therefore leave the set of solutions of (1.4)–(1.7) invariant. This yields an overdetermined linear system of equations for the infinitesimals  $\xi_1, \xi_2, \xi_3, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5$  and  $\phi_6$ . The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial u} + \phi_2 \frac{\partial}{\partial \omega} + \phi_3 \frac{\partial}{\partial m} + \phi_4 \frac{\partial}{\partial \psi} + \phi_5 \frac{\partial}{\partial \chi} + \phi_6 \frac{\partial}{\partial \lambda}. \quad (2.10)$$

By applying the classical Lie method [12] to the (1.4)–(1.7) system of PDEs, we obtain a system of overdetermined equations whose solutions are (We have used MACSYMA and MAPLE independently to handle the calculations):

$$\xi_1 = k_1 x + M_1 \quad (2.11)$$

$$\xi_2 = 2k_1 y + k_2 \quad (2.12)$$

$$\xi_3 = 3k_1 t + k_3 \quad (2.13)$$

$$\phi_1 = u \left( M_2 + \frac{dM_1}{dt} y - k_1 \right) \quad (2.14)$$

$$\phi_2 = \omega \left( -M_2 - \frac{dM_1}{dt} y - k_1 \right) \quad (2.15)$$

$$\phi_3 = M_3 - \frac{1}{4} \frac{d^2 M_1}{dt^2} y^2 - \frac{1}{2} \frac{dM_2}{dt} y - k_1 m \quad (2.16)$$

$$\phi_4 = \psi \left( \frac{1}{2} \frac{dM_2}{dt} + \frac{1}{2} \frac{dM_1}{dt} y + N \right) \quad (2.17)$$

$$\phi_5 = \chi \left( -\frac{1}{2} \frac{dM_2}{dt} - \frac{1}{2} \frac{dM_1}{dt} y + N \right) \quad (2.18)$$

$$\phi_6 = -k_1 \lambda, \quad (2.19)$$

where  $k_1, k_2$  and  $k_3$  are arbitrary constants;  $M_i = M_i(t)$ , ( $i = 1..3$ ) are arbitrary functions of  $t$ , and  $N = N(y, t)$  is a function of  $y$  and  $t$  that satisfies:

$$N_t - \lambda N_y = 0. \quad (2.19)$$

Having determined the infinitesimals in (2.13)–(2.18), the symmetry variables are found by solving the corresponding characteristic equations

$$\begin{aligned} \frac{dx}{k_1x + M_1} &= \frac{dy}{2k_1y + k_2} = \frac{dt}{3k_1t + k_3} = \frac{-d\lambda}{k_1\lambda} \\ &= \frac{du}{u \left( M_2 + \frac{dM_1}{dt}y - k_1 \right)} = \frac{-d\omega}{\omega \left( M_2 + \frac{dM_1}{dt}y + k_1 \right)} \\ &= \frac{dm}{M_3 - \frac{1}{2} \frac{dM_2}{dt}y - \frac{1}{4} \frac{d^2M_1}{dt^2}y^2 - k_1m} = \\ &= \frac{d\psi}{\psi \left( \frac{1}{2}M_2 + \frac{1}{2} \frac{dM_1}{dt}y + N \right)} = \frac{-d\chi}{\chi \left( \frac{1}{2}M_2 + \frac{1}{2} \frac{dM_1}{dt}y - N \right)}. \end{aligned} \quad (2.20)$$

In the next section we solve (2.20) for the different possibilities.

### 3. Reductions

There are four independent reductions, depending on the values of  $k_1, k_2, k_3$ :

#### 3.1. $k_1 \neq 0$

In this case, there is no restriction in setting  $k_2 = k_3 = 0$  because it only implies a trivial translation in  $y$  and  $t$ .

- By solving (2.20), we have the reduced variables:

$$z_1 = \frac{x}{(3t)^{\frac{1}{3}}} - \frac{1}{k_1} \int \frac{M_1}{(3t)^{\frac{4}{3}}} dt, \quad z_2 = \frac{y}{(3t)^{\frac{2}{3}}}. \quad (3.1)$$

- The reduction of the spectral parameter is:

$$\lambda(y, t) = 2(3t)^{-\frac{1}{3}} \Lambda(z_2), \quad (3.2)$$

where (1.8) yields the following equation for  $\Lambda(z_2)$ :

$$2(z_2 + \Lambda) \frac{d\Lambda}{dz_2} + \Lambda = 0. \quad (3.3)$$

Therefore, **the reduced linear problem is nonisospectral.**

- The reductions for the fields and eigenfunctions are:

$$u(x, y, t) = (3t)^{-\frac{1}{3}} e^{2\Omega[t, z_2]} \alpha(z_1, z_2) \quad (3.4)$$

$$\omega(x, y, t) = (3t)^{-\frac{1}{3}} e^{-2\Omega[t, z_2]} \beta(z_1, z_2) \quad (3.5)$$

$$m(x, y, t) = (3t)^{-\frac{1}{3}} (\gamma(z_1, z_2) + \Delta[t, z_2]) \quad (3.6)$$

$$\psi(x, y, t) = e^{\Omega[t, z_2]} e^{H[t, z_2]} Q(z_1, z_2) \quad (3.7)$$

$$\chi(x, y, t) = e^{-\Omega[t, z_2]} e^{H[t, z_2]} R(z_1, z_2), \quad (3.8)$$

where  $\alpha(z_1, z_2)$ ,  $\beta(z_1, z_2)$ ,  $\gamma(z_1, z_2)$ ,  $Q(z_1, z_2)$ ,  $R(z_1, z_2)$  are the reduced fields and eigenfunctions.

Furthermore  $\Omega[t, z_2]$ ,  $H[t, z_2]$ ,  $\Delta[t, z_2]$  are defined as:

$$\Omega[t, z_2] = \frac{1}{2k_1} \left[ \left( \frac{M_1}{(3t)^{\frac{1}{3}}} + \int \frac{M_1}{(3t)^{\frac{4}{3}}} dt \right) z_2 + \int \frac{M_2}{3t} dt \right] \quad (3.9)$$

$$\begin{aligned} \Delta[t, z_2] = \frac{1}{2k_1} \left[ \frac{M_1}{(3t)^{\frac{1}{3}}} + \int \frac{M_1}{(3t)^{\frac{4}{3}}} dt - \frac{1}{2} \int (3t)^{\frac{2}{3}} \frac{dM_1}{dt} dt \right] z_2^2 \\ + \frac{1}{2k_1} \left[ -z_2 M_2 + 2 \int \frac{M_3}{(3t)^{\frac{2}{3}}} dt \right] \end{aligned} \quad (3.10)$$

$$H[t, z_2] = \frac{1}{k_1} \int \frac{\hat{N}[t, z_2]}{3t} dt, \quad (3.11)$$

where

$$\hat{N}[t, z_2] = N[t, y(t, z_2)].$$

Note that with this definition of  $\hat{N}[t, z_2]$ , equation (2.19) yields:

$$\frac{\partial \hat{N}}{\partial z_2} = \frac{1}{2} \left( \frac{3t}{z_2 + \Lambda(z_2)} \right) \frac{\partial \hat{N}}{\partial t}.$$

• Substitution of the reductions in the 2 + 1 spectral problem (1.4)–(1.7) gives us the following 1 + 1 Lax pair:

$$Q_{z_1} + \Lambda Q + \alpha R = 0 \quad (3.12)$$

$$R_{z_1} - \Lambda R + \beta Q = 0 \quad (3.13)$$

$$\begin{aligned} 2(z_2 + \Lambda) Q_{z_2} = Q (\gamma_{z_2} + \alpha \beta_{z_1} - \beta \alpha_{z_1} + 2\Lambda \alpha \beta - 4\Lambda^3 - \Lambda_{z_2} + z_1 \Lambda) \\ + R (-\alpha_{z_2} - \alpha_{z_1 z_1} + 2\beta \alpha^2 - 4\Lambda^2 \alpha + 2\Lambda \alpha_{z_1} + z_1 \alpha) \end{aligned} \quad (3.14)$$

$$\begin{aligned} 2(z_2 + \Lambda) R_{z_2} = R (-\gamma_{z_2} - \alpha \beta_{z_1} + \beta \alpha_{z_1} - 2\Lambda \alpha \beta + 4\Lambda^3 - \Lambda_{z_2} - z_1 \Lambda) \\ + Q (\beta_{z_2} - \beta_{z_1 z_1} + 2\alpha \beta^2 - 4\Lambda^2 \beta - 2\Lambda \beta_{z_1} + z_1 \beta) \end{aligned} \quad (3.15)$$

• It is trivial to check that the compatibility condition of Eqs. (3.12)–(3.15) yields the system of **nonautonomous** equations:

$$(\gamma_{z_1} + \alpha \beta)_{z_2} = 0 \quad (3.16)$$

$$\alpha_{z_1 z_1 z_1} + \alpha_{z_1 z_2} - 6\alpha \beta \alpha_{z_1} + 2\alpha \gamma_{z_2} - 2z_2 \alpha_{z_2} - z_1 \alpha_{z_1} - \alpha = 0 \quad (3.17)$$

$$\beta_{z_1 z_1 z_1} - \beta_{z_1 z_2} - 6\alpha \beta \beta_{z_1} - 2\beta \gamma_{z_2} - 2z_2 \beta_{z_2} - z_1 \beta_{z_1} - \beta = 0, \quad (3.18)$$

It is not difficult to check that (3.16)–(3.18) has the Painlevé property. This property and the existence of the nonlinear associated problem (3.12)–(3.15) prove that the system is integrable although the spectral parameter is not a constant because it should satisfy the **nonisospectral** condition (3.3).

**3.2.  $k_1 = 0$ ,  $k_2 \neq 0$ ,  $k_3 \neq 0$** 

- The reduced variables provided by integration of (2.20) are:

$$z_1 = \frac{k_2}{6k_3}x - \frac{k_2}{18k_3^2} \int M_1 dt, \quad z_2 = \left( \frac{k_2}{6k_3} \right)^2 y - 2 \left( \frac{k_2}{6k_3} \right)^3 t. \quad (3.19)$$

- The spectral parameter is reduced as:

$$\lambda(y, t) = \frac{k_2}{3k_3} \Lambda_0, \quad (3.20)$$

where  $\Lambda_0$  is a constant and therefore, **the reduced linear problem is isospectral.**

- The reductions for the fields are:

$$u(x, y, t) = \frac{k_2}{6k_3} e^{2\Omega[t, z_2]} \alpha(z_1, z_2) \quad (3.21)$$

$$\omega(x, y, t) = \frac{k_2}{6k_3} e^{-2\Omega[t, z_2]} \beta(z_1, z_2) \quad (3.22)$$

$$m(x, y, t) = \frac{k_2}{6k_3} \gamma(z_1, z_2) + \frac{1}{3k_3} \Delta[t, z_2] \quad (3.23)$$

$$\psi(x, y, t) = e^{\Omega[t, z_2]} e^{H[t, z_2]} Q(z_1, z_2) \quad (3.24)$$

$$\chi(x, y, t) = e^{-\Omega[t, z_2]} e^{H[t, z_2]} R(z_1, z_2), \quad (3.25)$$

where the functions  $\Omega[t, z_2]$ ,  $H[t, z_2]$  and  $\Delta[t, z_2]$  are:

$$\Omega[t, z_2] = \frac{1}{6k_3} \left( \left[ \left( \frac{6k_3}{k_2} \right)^2 M_1 \right] z_2 + \frac{k_2}{3k_3} \int t \frac{dM_1}{dt} dt + \int M_2 dt \right) \quad (3.26)$$

$$\begin{aligned} \Delta[t, z_2] = & \left[ -\frac{1}{4} \left( \frac{6k_3}{k_2} \right)^4 \frac{dM_1}{dt} \right] z_2^2 + \left[ -\frac{1}{2} \left( \frac{6k_3}{k_2} \right)^2 M_2 - \left( \frac{6k_3}{k_2} \right) \left( t \frac{dM_1}{dt} - M_1 \right) \right] z_2 \\ & + \left[ \int M_3 dt - \left( \frac{k_2}{6k_3} \right) \int t \frac{dM_2}{dt} - \left( \frac{k_2}{6k_3} \right)^2 \int t^2 \frac{d^2 M_1}{dt^2} dt \right] \end{aligned} \quad (3.27)$$

$$H[t, z_2] = \frac{1}{3k_3} \int \hat{N}[t, z_2] dt, \quad (3.28)$$

where

$$\hat{N}[t, z_2] = N[t, y(t, z_2)]$$

Therefore, (2.19) yields

$$\frac{\partial \hat{N}}{\partial z_2} = \left( \frac{6k_3}{k_2} \right)^2 \left( \frac{1}{1 + \Lambda_0} \right) \frac{\partial \hat{N}}{\partial t}.$$

- The reduction of the Lax pair is:

$$0 = Q_{z_1} + \Lambda_0 Q + \alpha R \quad (3.29)$$

$$0 = R_{z_1} - \Lambda_0 R + \beta Q \quad (3.30)$$

$$2(1 + \Lambda_0) Q_{z_2} = Q(\gamma_{z_2} + \alpha \beta_{z_1} - \beta \alpha_{z_1} + 2\Lambda_0 \alpha \beta - 4\Lambda_0^3) + R(-\alpha_{z_2} - \alpha_{z_1 z_1} + 2\beta \alpha^2 - 4\Lambda_0^2 \alpha + 2\Lambda_0 \alpha_{z_1}) \quad (3.31)$$

$$2(1 + \Lambda_0) R_{z_2} = R(-\gamma_{z_2} - \alpha \beta_{z_1} + \beta \alpha_{z_1} - 2\Lambda_0 \alpha \beta + 4\Lambda_0^3) + Q(\beta_{z_2} - \beta_{z_1 z_1} + 2\alpha \beta^2 - 4\Lambda_0^2 \beta - 2\Lambda_0 \beta_{z_1}). \quad (3.32)$$

- The compatibility condition of this Lax pair gives us the system:

$$(\gamma_{z_1} + \alpha \beta)_{z_2} = 0 \quad (3.33)$$

$$\alpha_{z_1 z_1 z_1} + \alpha_{z_1 z_2} - 6\alpha \beta \alpha_{z_1} + 2\alpha \gamma_{z_2} - 2\alpha_{z_2} = 0 \quad (3.34)$$

$$\beta_{z_1 z_1 z_1} - \beta_{z_1 z_2} - 6\alpha \beta \beta_{z_1} - 2\beta \gamma_{z_2} - 2\beta_{z_2} = 0. \quad (3.35)$$

### 3.3. $k_1 = 0$ , $k_2 \neq 0$ , $k_3 = 0$

- In this case, (2.20) indicates that  $t$  is one of the reduced variables. The other reduced variable is  $z_1$  defined by:

$$z_1 = x - \frac{M_1}{k_2} y. \quad (3.36)$$

- For the spectral parameter we have the reduction:

$$\lambda(y, t) = 2\Lambda_0, \quad (3.37)$$

where (3.37) implies that  $\Lambda_0$  is a constant, and therefore the reduced problem is isospectral.

- The reduction for the fields is:

$$u(x, y, t) = e^{2\Omega(y, t)} \alpha(z_1, t) \quad (3.38)$$

$$\omega(x, y, t) = e^{-2\Omega(y, t)} \beta(z_1, t) \quad (3.39)$$

$$m(x, y, t) = \gamma(z_1, t) + \frac{1}{k_2} \Delta(y, t) \quad (3.40)$$

$$\psi(x, y, t) = e^{\Omega(y, t)} e^{H(y, t)} Q(z_1, t) \quad (3.41)$$

$$\chi(x, y, t) = e^{-\Omega(y, t)} e^{H(y, t)} R(z_1, t), \quad (3.42)$$

where  $\Omega(y, t)$ ,  $H(y, t)$  and  $\Delta(y, t)$  are:

$$\Omega(y, t) = \frac{1}{2k_2} \left( M_2 y + \frac{1}{2} \frac{dM_1}{dt} y^2 \right) \quad (3.43)$$

$$\Delta(y, t) = \frac{1}{k_2} \left( M_3 y - \frac{1}{4} \frac{dM_2}{dt} y^2 - \frac{1}{12} \frac{d^2 M_1}{dt^2} y^3 \right) \quad (3.44)$$

$$H(y, t) = \frac{1}{k_2} \int N(y, t) dy. \quad (3.45)$$



- The reduced spectral problem is:

$$0 = Q_{z_1} + \lambda_0 Q + \alpha R \quad (3.46)$$

$$0 = R_{z_1} - \lambda_0 R + \beta Q \quad (3.47)$$

$$Q_t = Q \left( \beta \alpha_{z_1} - \alpha \beta_{z_1} - 2\Lambda_0 \alpha \beta + 4\Lambda_0^3 + \frac{1}{k_2} [-M_3 + \Lambda_0 M_2 + 2\Lambda_0^2 M_1 + M_1 \gamma_{z_1}] \right) \\ + R \left( \alpha_{z_1 z_1} - 2\beta \alpha^2 + 4\Lambda_0^2 \alpha - 2\Lambda_0 \alpha_{z_1} + \frac{1}{k_2} [2\Lambda_0 M_1 \alpha + M_2 \alpha - M_1 \alpha_{z_1}] \right) \quad (3.48)$$

$$R_t = R \left( -\beta \alpha_{z_1} + \alpha \beta_{z_1} + 2\Lambda_0 \alpha \beta - 4\Lambda_0^3 + \frac{1}{k_2} [M_3 - \Lambda_0 M_2 - 2\Lambda_0^2 M_1 - M_1 \gamma_{z_1}] \right) \\ + Q \left( \beta_{z_1 z_1} - 2\alpha \beta^2 + 4\Lambda_0^2 \beta + 2\Lambda_0 \beta_{z_1} + \frac{1}{k_2} [2\Lambda_0 M_1 \beta + M_2 \beta + M_1 \beta_{z_1}] \right), \quad (3.49)$$

where  $\Lambda_0$  is the spectral parameter.

- The compatibility condition of (3.46)–(3.49) is the following system of PDEs:

$$(\gamma_{z_1} + \alpha \beta)_{z_1} = 0 \quad (3.50)$$

$$\alpha_{z_1 z_1 z_1} + \alpha_t - 6\alpha \beta \alpha_{z_1} + \frac{M_2 \alpha_{z_1} + 2M_3 \alpha - M_1 (\alpha_{z_1 z_1} + 2\alpha \gamma_{z_1})}{k_2} = 0 \quad (3.51)$$

$$\beta_{z_1 z_1 z_1} + \beta_t - 6\alpha \beta \beta_{z_1} + \frac{M_2 \beta_{z_1} - 2M_3 \beta + M_1 (\beta_{z_1 z_1} + 2\beta \gamma_{z_1})}{k_2} = 0, \quad (3.52)$$

which are the reduction of (1.1)–(1.3). This case includes for  $M_2 = M_3 = 0$  the Hirota equation of reference [7].

### 3.4. $k_1 = 0$ , $k_2 = 0$ , $k_3 \neq 0$ .

- $y$  is now one of the reduced variables. The other reduced variable is  $z_1$ , defined by:

$$z_1 = x - \frac{\int M_1(t) dt}{3k_3}. \quad (3.53)$$

- The spectral parameter is reduced as follows:

$$\Lambda(z_1, y) = 2\Lambda_0, \quad (3.54)$$

with  $\lambda_0$  an arbitrary constant.

- The reduction for the fields is:

$$u(x, y, t) = e^{2\Omega(y, t)} \alpha(z_1, y) \quad (3.55)$$

$$\omega(x, y, t) = e^{-2\Omega(y, t)} \beta(z_1, y) \quad (3.56)$$

$$m(x, y, t) = \gamma(z_1, y) + \Delta(y, t) \quad (3.57)$$

$$\psi(x, y, t) = e^{\Omega(y, t)} e^{H(y, t)} Q(z_1, y) \quad (3.58)$$

$$\chi(x, y, t) = e^{-\Omega(y, t)} e^{H(y, t)} R(z_1, y), \quad (3.59)$$

where  $\Omega(y, t)$ ,  $H(y, t)$  and  $\Delta(y, t)$  are:

$$\Omega(y, t) = \frac{1}{6k_3} \left( M_1 y + \int M_2(t) dt \right) \quad (3.60)$$

$$\Delta(y, t) = \frac{1}{3k_3} \left( \int M_3(t) dt - \frac{M_2}{2} y - \frac{1}{4} \frac{dM_1}{dt} y^2 \right) \quad (3.61)$$

$$H(y, t) = \frac{1}{3k_3} \int N(y, t) dt. \quad (3.62)$$

- The reduction of the Lax pair yields:

$$0 = Q_{z_1} + \Lambda_0 Q + \alpha R \quad (3.63)$$

$$0 = R_{z_1} - \Lambda_0 R + \beta Q \quad (3.64)$$

$$\begin{aligned} 2\Lambda_0 Q_y = Q & (\gamma_y + \alpha \beta_{z_1} - \beta \alpha_{z_1} + 2\Lambda_0 \alpha \beta - 4\Lambda_0^3) \\ & + R (-\alpha_{z_1 z_1} - \alpha_y + 2\alpha^2 \beta + 2\Lambda_0 \alpha_{z_1} - 4\Lambda_0^2 \alpha) \end{aligned} \quad (3.65)$$

$$\begin{aligned} 2\Lambda_0 R_y = R & (-\gamma_y - \alpha \beta_{z_1} + \beta \alpha_{z_1} - 2\Lambda_0 \alpha \beta + 4\Lambda_0^3) \\ & + Q (-\beta_{z_1 z_1} + \beta_y + 2\alpha \beta^2 - 2\Lambda_0 \beta_{z_1} - 4\Lambda_0^2 \beta), \end{aligned} \quad (3.66)$$

whose compatibility condition of (3.63)–(3.66) is the system of equations:

$$(\gamma_{z_1} + \alpha \beta)_{z_2} = 0 \quad (3.67)$$

$$\alpha_{z_1 z_1 z_1} + \alpha_{z_1 y} - 6\alpha \beta \alpha_{z_1} + 2\alpha \gamma_y = 0 \quad (3.68)$$

$$\beta_{z_1 z_1 z_1} - \beta_{z_1 y} - 6\alpha \beta \beta_{z_1} - 2\beta \gamma_y = 0. \quad (3.69)$$

#### 4. Conclusions

A spectral problem in  $2 + 1$  dimensions is presented. The compatibility conditions of this Lax pair yields a  $2 + 1$  system that was introduced in [8]. An important fact is that the spectral parameter is nonisospectral.

If we wish to know the  $1 + 1$  reductions of the spectral problem, it is specially important to establish how the spectral parameter reduces. One possibility is to identify the classical Lie symmetries of the Lax pair instead of those of the system of PDEs. We have identified these symmetries **by considering the spectral parameter as an additional field**. This means that we obtain symmetries that are symmetries of the fields  $m, u, \omega$ , the eigenfunctions  $\psi, \chi$  and the spectral parameter  $\lambda$ . The symmetries that we have obtained include three arbitrary constants and several arbitrary functions.

We attempt to go from  $2 + 1$  to  $1 + 1$  dimensions by using the reductions arising from the former Classical Lie symmetries. Four possible reductions arise from the Classical Symmetries. They yield highly nontrivial systems of nonautonomous PDEs in  $1 + 1$  dimensions as well as their associated spectral problems, with spectral parameters that are obtained from reductions of the function  $\lambda(y, t)$ . Particularly interesting is case 3.1, where the spectral parameter is not a constant even in the  $1 + 1$  reduction.

Obviously, classical Lie symmetries are not the only symmetries that can be identified. Other symmetries such as nonclassical or potential symmetries can be studied in the future.

Nevertheless the purpose of this paper is not an exhaustive study of the symmetries of (1.1)–(1.3) but prove that the study of the symmetries of the Lax pair and the spectral parameter provides much more interesting information than the symmetries of the PDE's system. This information is specially relevant when we obtain the similarity reductions that provides the reduced spectral problem that yields to the reduced system and the reduction of the spectral parameter.

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