Journal of Nonlinear Mathematical Physics

ISSN (Online): 1776-0852

ISSN (Print): 1402-9251

Journal Home Page: <https://www.atlantis-press.com/journals/jnmp>

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To cite this article: Marianna Euler, Norbert Euler (2011) A Class of Semilinear Fifth-Order Evolution Equations: Recursion Operators and Multipotentialisations, Journal of Nonlinear Mathematical Physics 18:Supplement 1, 61–75, DOI: <https://doi.org/10.1142/S1402925111001283>

To link to this article: <https://doi.org/10.1142/S1402925111001283>

Published online: 04 January 2021

A CLASS OF SEMILINEAR FIFTH-ORDER EVOLUTION EQUATIONS: RECURSION OPERATORS AND MULTIPOTENTIALISATIONS

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Received 1 October 2010

Accepted 3 December 2010

We apply a list of criteria which leads to a class of fifth-order symmetry-integrable evolution equations. The recursion operators for this class are given explicitly. Multipotentialisations are then applied to the equations in this class in order to extend this class of integrable equations.

Keywords: Nonlinear fifth-order evolution equations; recursion operators; Lie–Bäcklund symmetries; integrability; multipotentialisations.

1. Introduction

Autonomous symmetry-integrable evolution equations in $(1 + 1)$ dimensions are equations of the form

$$u_t = F(x, u, u_x, u_{xx}, \dots) \quad (1.1)$$

which admit an infinite number of local commuting Lie–Bäcklund symmetries [3, 11],

$$Z = \eta(x, u, u_x, u_{xx}, \dots) \frac{\partial}{\partial u}. \quad (1.2)$$

Recursion operators, $R[u]$, can be defined which generate these infinite sets of Lie–Bäcklund symmetries [15]. In most cases these recursion operators for autonomous evolution equations are integro-differential operators of the form

$$R[u] = \sum_{j=0}^p G_j D_x^j + \sum_{i=1} I_i(u_x, u_t) D_x^{-1} \circ \Lambda_i, \quad (1.3)$$

where G_j are functions of x , u and x -derivatives of u , D_x is the total derivative operator, D_x^{-1} the integral operator and Λ_i are integrating factors for the evolution equation.

By applying the above mentioned criteria, we classified several classes of second- and third-order evolution equations with respect to its recursion operators of the form (1.3). These results have been published in the papers [4–6, 16].

It is of some interest to potentialise and, moreover, multipotentialise integrable evolution equations as this often extends the class of integrable equations. In particular, multipotentialisations of symmetry-integrable evolution equations may lead to equations with different types of recursion operators. In some cases, even nonlocal recursion operators are necessary in order to generate the local Lie–Bäcklund symmetries (an example is given in [16]). Multipotentialisations of symmetry-integrable evolution equations was studied and applied for several classes of the equations and published in [8–10].

In the current paper we report a class of fifth-order symmetry-integrable evolution equations and apply multipotentialisations to each equation to extend this class. The classification criteria are given in Sec. 2, where the resulting evolution equations (Eqs. I–VII) are also listed in Proposition 1. In Sec. 3 we give the recursion operators for Eqs. I–VIII. In Sec. 4 we report the multipotentialisations of these equations and display their connections in Figs. 1–5.

2. The Classification Criteria

Classification criteria:

We classify all semi-linear fifth-order evolution equations of the form

$$u_t = u_{5x} + F(u, u_x, \dots, u_{4x}) \quad (2.1)$$

which admit the following criteria:

- **Criterion 1:** Equation (2.1) admits at least two integrating factors, Λ , of order zero, $\Lambda(u)$, of order two, $\Lambda(u, u_x, u_{xx})$, of order four, $\Lambda(u, u_x, u_{xx}, u_{xxx}, u_{4x})$, or of order six, $\Lambda(u, u_x, \dots, u_{6x})$.
- **Criterion 2:** Equation (2.1) admits an integro-differential recursion operator of order six:

$$R[u] = \sum_{j=0}^6 G_j D_x^j + (b_1 u_x + b_0) D_x^{-1} \circ \Lambda_1 + (b_2 u_t + b_3) D_x^{-1} \circ \Lambda_2, \quad (2.2)$$

where $G_j = G_j(u, u_x, u_{xx}, \dots)$ with $G_6 \neq 0$, b_k are real constants and Λ_i are integrating factors of the equation.

- **Criterion 3:** Equation (2.1) does not belong to a hierarchy of evolution equations

$$u_t = R[u]^n u_x, \quad n \in \mathcal{N} \quad (2.3)$$

where $R[u]$ is a first- or second-order integro-differential recursion operator.

- **Criterion 4:** Equation (2.1) admits local Lie–Bäcklund symmetries of lowest order seven, that can be obtained by acting the recursion operator from Criterion 2 on u_x , i.e. $R[u]u_x$.

By applying the above criteria we obtained the following list of eight equations:

Proposition 2.1. *The following list of eight equations are the only fifth-order evolution equations in the class,*

$$u_t = u_{5x} + F(u, u_x, u_{xx}, \dots, u_{5x}), \quad (2.4)$$

which satisfy all four criteria listed above:

Equation I:

$$u_t = u_{5x} + \alpha u u_{xxx} + \alpha u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x + \beta \left(u_{xxx} + \frac{2\alpha}{5} u u_x \right) + \left(\frac{\beta^2}{5} + c_1 \right) u_x. \quad (2.5)$$

Equation I is known as the Sawada–Kotera equation for $\alpha = 5$, $\beta = 0$ [18] and the Caudrey–Dodd–Gibbon–Sawada–Kotera equation for $\alpha = 30$, $\beta = 0$ [2]. For $\alpha = 5$ and $\beta = 0$, Equation I has been listed as Eq. (4.2.2) in [14]. A recursion operator for arbitrary α and $\beta = 0$ is given in [13].

Equation II:

$$u_t = u_{5x} + \alpha u u_{xxx} + \frac{5\alpha}{2} u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x + \beta \left(u_{xxx} + \frac{2\alpha}{5} u u_x \right) + c_1 u_x. \quad (2.6)$$

Equation II is known as the Kaup–Kupershmidt equation for $\alpha = 10$, $\beta = 0$ [12]. For $\alpha = 5$ and $\beta = 0$ it has been listed as Eq. (4.2.3) in [14]. A recursion operator for all α and $\beta = 0$ is given in [13].

Equation III:

$$u_t = u_{5x} + \alpha u_x u_{xxx} + \frac{\alpha^2}{15} u_x^3 + \beta \left(u_{xxx} + \frac{\alpha}{5} u_x^2 \right) + c_1 u_x. \quad (2.7)$$

Equation III is known as the potential Sawada–Kotera equation for $\beta = 0$. For $\alpha = 5$ and $\beta = 0$ it has been listed as Eq. (4.2.4) in [14]. For $\alpha = 5$ and $\beta = 0$ a recursion operator is given in [1].

Equation IV:

$$u_t = u_{5x} + \alpha u_x u_{xxx} + \frac{3\alpha}{4} u_{xx}^2 + \frac{\alpha^2}{15} u_x^3 + \beta \left(u_{xxx} + \frac{\alpha}{5} u_x^2 \right) + c_1 u_x. \quad (2.8)$$

Equation IV is known as the potential Kaup–Kupershmidt equation for $\alpha = 5$ and $\beta = 0$. For $\alpha = 5$ and $\beta = 0$ it has been listed as Eq. (4.2.5) in [14]. For $\alpha = 5$ and $\beta = 0$ a recursion operator is given in [1].

Equation V:

$$\begin{aligned} u_t = & u_{5x} - \frac{\alpha^2}{5} u^2 u_{xxx} + \alpha u_x u_{xxx} - \frac{4\alpha^2}{5} u u_x u_{xx} + \alpha u_{xx}^2 - \frac{\alpha^2}{5} u_x^3 + \frac{\alpha^4}{125} u^4 u_x \\ & + \beta \left(2u_x u_{xx} + u u_{xxx} - \frac{2\alpha^2}{25} u^3 u_x \right) + \beta^2 \left(\frac{3}{10} u^2 u_x - \frac{5}{4\alpha^2} u_{xxx} \right) - \frac{\beta^3}{2\alpha^2} u u_x + c_1 u_x. \end{aligned} \quad (2.9)$$

Equation V is known as the Kupershmidt equation for $\alpha = 5$ and $\beta = 0$ and it has been listed as Eq. (4.2.6) in [14]. A recursion operator for $\alpha = 5$ and $\beta = 0$ is given in [1, 13].

Equation VI:

$$\begin{aligned}
u_t = & u_{5x} + \alpha u_{xx} u_{xxx} - \frac{\alpha^2}{5} u_x^2 u_{xxx} - \frac{\alpha^2}{5} u_x u_{xx}^2 + \frac{\alpha^4}{624} u_x^5 \\
& + \beta \left(u_x u_{xxx} - \frac{\alpha^2}{50} u_x^4 + \frac{1}{2} u_{xx}^2 \right) + \beta^2 \left(-\frac{5}{4\alpha^2} u_{xxx} + \frac{1}{10} u_x^3 \right) - \frac{\beta^3}{4\alpha^2} u_x^2 + c_1 u_x.
\end{aligned} \tag{2.10}$$

Equation VI is known as the potential Kupershmidt equation. For $\alpha = 5, \beta = 0$ it has been listed as Eq. (4.2.7) in [14]. For $\alpha = 5$ and $\beta = 0$ a recursion operator is given in [1]. As far as we know, the recursion operator for the full Eq. (2.10) has not been reported so far in the literature.

Equation VII:

$$\begin{aligned}
u_t = & u_{5x} + (5u_{xx} - 5u_x^2 + \beta e^{2u} + \alpha e^{-4u}) u_{xxx} - 5u_x u_{xx}^2 + 3(\beta e^{2u} - 4\alpha e^{-4u}) u_x u_{xx} \\
& + u_x^5 + 18\alpha e^{-4u} u_x^3 + \frac{1}{5} (\beta e^{2u} + \alpha e^{-4u})^2 u_x + c_1 u_x.
\end{aligned} \tag{2.11}$$

Equation VII, with $\alpha = -5\lambda_2^2$ and $\beta = 5\lambda_1$, has been listed as Eq. (4.2.8) in [14]. As far as we know, the recursion operator for (2.11) has not been reported so far in the literature.

Equation VIII:

$$\begin{aligned}
u_t = & u_{5x} + (5u_{xx} - 5u_x^2 + \alpha e^{-u} + \beta e^{2u}) u_{xxx} - 5u_x u_{xx}^2 + 3\beta e^{2u} u_x u_{xx} \\
& + u_x^5 + \frac{1}{5} (\alpha e^{-u} + \beta e^{2u})^2 u_x + c_1 u_x.
\end{aligned} \tag{2.12}$$

Equation VIII, with $\alpha = 5\lambda_2$ and $\beta = -5\lambda_1^2$, has been listed as Eq. (4.2.9) in [14]. As far as we know, the recursion operator for (2.12) has not been reported so far in the literature.

In Sec. 2 we report the recursion operators for the equations in Proposition 1 and in Sec. 3 we multipotentialise each equation and, in doing so, we extend the list of equations to its multipotential forms. The resulting equations are given in Proposition 3 and the relations between the equations are shown in Figs. 1–5.

3. The Recursion Operators for Equations I–VIII

Equation I admits a recursion operator of the form (2.2) with

$$\begin{aligned}
G_6 = & 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u + \frac{6\beta}{5}, \quad G_3 = \frac{9\alpha}{5} u_x \\
G_2 = & \frac{11\alpha}{5} u_{xx} + \frac{9\alpha^2}{25} u^2 + \frac{18\alpha\beta}{25} u + \frac{9\beta^2}{25} \\
G_1 = & 2\alpha u_{xxx} + \frac{21\alpha^2}{25} u u_x + \frac{21\alpha\beta}{25} u_x
\end{aligned}$$

$$\begin{aligned}
G_0 &= \alpha u_{4x} + \frac{16\alpha^2}{25} u u_{xx} + \frac{16\alpha\beta}{25} u_{xx} + \frac{6\alpha^2}{25} u_x^2 + \frac{4\alpha^3}{125} u^3 + \frac{12\alpha^2\beta}{125} u^2 + \frac{12\alpha\beta^2}{125} u + k_0 \\
\Lambda_1 &= \frac{2\alpha^2}{25} u_{xx} + \frac{\alpha^3}{125} u^2 + \frac{2\alpha^2\beta}{125} u + \frac{\alpha}{6} c_1 + \frac{\alpha\beta^2}{125}, \quad \Lambda_2 = \frac{\alpha}{5} \\
b_0 &= 0, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 0,
\end{aligned}$$

where k_0 is an arbitrary constant.

Equation II admits a recursion operator of the form (2.2) with

$$\begin{aligned}
G_6 &= 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u + \frac{6\beta}{5}, \quad G_3 = \frac{18\alpha}{5} u_x \\
G_2 &= \frac{18\alpha\beta}{25} u + \frac{9\alpha^2}{25} u^2 + \frac{49\alpha}{10} u_{xx} + \frac{9\beta^2}{25} \\
G_1 &= \frac{7\alpha}{2} u_{xxx} + \frac{6\alpha^2}{5} u u_x + \frac{6\alpha\beta}{5} u_x \\
G_0 &= \frac{13\alpha}{10} u_{4x} + \frac{41\alpha\beta}{50} u_{xx} + \frac{41\alpha^2}{50} u u_{xx} + \frac{69\alpha^2}{100} u_x^2 + \frac{4\alpha^3}{125} u^3 + \frac{12\alpha^2\beta}{125} u^2 + \frac{12\alpha\beta^2}{125} u + k_0 \\
\Lambda_1 &= \frac{\alpha^2}{50} u_{xx} + \frac{\alpha^3}{125} u^2 + \frac{2\alpha^2\beta}{125} u + \frac{\alpha}{125} (6\beta^2 - 25c_1), \quad \Lambda_2 = \frac{\alpha}{5} \\
b_0 &= 0, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 0.
\end{aligned}$$

Equation III admits a recursion operator of the form (2.2) with

$$\begin{aligned}
G_6 &= 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u_x + \frac{6\beta}{5} \\
G_3 &= \frac{3\alpha}{5} u_{xx}, \quad G_2 = \frac{8\alpha}{5} u_{xxx} + \frac{9\alpha^2}{25} u_x^2 + \frac{18\alpha\beta}{25} u_x + \frac{9\beta^2}{25} \\
G_1 &= \frac{2\alpha}{5} u_{4x} + \frac{3\alpha^2}{25} u_x u_{xx} + \frac{3\alpha\beta}{25} u_{xx} \\
G_0 &= \frac{3\alpha}{5} u_{5x} + \frac{13\alpha^2}{25} u_x u_{xxx} + \frac{13\alpha\beta}{25} u_{xxx} + \frac{3\alpha^2}{25} u_{xx}^2 + \frac{4\alpha^3}{125} u_x^3 + \frac{12\alpha^2\beta}{125} u_x^2 + \frac{12\alpha\beta^2}{125} u_x + k_0 \\
\Lambda_1 &= u_{4x} + \frac{\alpha}{5} u_x u_{xx} + \frac{\beta}{5} u_{xx} \\
\Lambda_2 &= u_{6x} + \frac{3\alpha}{5} u_x u_{4x} + \frac{6\alpha}{5} u_{xx} u_{xxx} + \frac{2\alpha^2}{25} u_x^2 u_{xx} + \frac{\alpha\beta}{25} u_x u_{xx} - \frac{\beta^2}{25} u_{xx} \\
b_0 &= -\frac{8\alpha\beta}{25}, \quad b_1 = -\frac{2\alpha^2}{25}, \quad b_2 = 0, \quad b_3 = -\frac{2\alpha}{5}.
\end{aligned}$$

Equation IV admits a recursion operator of the form (2.2) with

$$\begin{aligned}
G_6 &= 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u_x + \frac{6\beta}{5}, \quad G_3 = \frac{12\alpha}{5} u_{xx} \\
G_2 &= \frac{5\alpha}{2} u_{xxx} + \frac{9\alpha^2}{25} u_x^2 + \frac{18\alpha\beta}{25} u_x + \frac{9\beta^2}{25}, \quad G_1 = \alpha u_{4x} + \frac{12\alpha^2}{25} u_x u_{xx} + \frac{12\alpha\beta}{25} u_{xx}
\end{aligned}$$

$$G_0 = \frac{3\alpha}{10}u_{5x} + \frac{17\alpha^2}{50}u_x u_{xxx} + \frac{17\alpha\beta}{50}u_{xxx} + \frac{21\alpha^2}{100}u_{xx}^2 + \frac{4\alpha^3}{125}u_x^3 + \frac{12\alpha^2\beta}{125}u_x^2 \\ + \frac{12\alpha\beta^2}{125}u_x + \frac{6\beta c_1}{25} + k_0$$

$$b_0 = k_1, \quad b_1 = -\frac{\alpha^2}{50}, \quad b_2 = 0, \quad b_3 = -\frac{\alpha}{10}$$

$$\Lambda_1 = u_{4x} + \frac{4\alpha}{5}u_{xx}u_x + \frac{4\beta}{5}u_{xx}$$

$$\Lambda_2 = u_{6x} + \frac{6\alpha}{5}u_{4x}u_x + \frac{7\beta}{5}u_{4x} + \frac{10k_1}{\alpha}u_{4x} + \frac{12\alpha}{5}u_{xxx}u_{xx} + \frac{8\alpha^2}{25}u_{xx}u_x^2 + 8k_1u_{xx}u_x \\ + \frac{4\alpha\beta}{5}u_{xx}u_x + \frac{12\beta^2}{25}u_{xx} + \frac{8\beta k_1}{\alpha}u_{xx}.$$

Equation V admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5}u_x - \frac{6\alpha^2}{25}u^2 + \frac{6\beta}{5}u - \frac{3\beta^2}{2\alpha^2}$$

$$G_3 = 3\alpha u_{xx} - \frac{6\alpha^2}{5}uu_x + 3\beta u_x$$

$$G_2 = \frac{14\alpha}{5}u_{xxx} - \frac{8\alpha^2}{5}uu_{xx} + 4\beta u_{xx} - \frac{31\alpha^2}{25}u_x^2 - \frac{6\alpha^3}{125}u^2u_x + \frac{6\alpha\beta}{25}uu_x - \frac{3\beta^2}{10\alpha}u_x \\ + \frac{9\alpha^4}{625}u^4 - \frac{18\alpha^2\beta}{125}u^3 + \frac{27\beta^2}{50}u^2 - \frac{9\beta^3}{10\alpha^2}u + \frac{9\beta^4}{16\alpha^4}$$

$$G_1 = \frac{6\alpha}{5}u_{4x} - \frac{6\alpha^2}{5}uu_{xxx} + 3\beta u_{xxx} - \frac{63\alpha^2}{25}u_x u_{xx} - \frac{9\alpha^3}{125}u^2u_{xx} + \frac{9\alpha\beta}{25}uu_{xx} \\ - \frac{9\beta^2}{20\alpha}u_{xx} - \frac{18\alpha^3}{125}uu_x^2 + \frac{9\alpha\beta}{25}u_x^2 + \frac{54\alpha^4}{625}u^3u_x - \frac{81\alpha^2\beta}{125}u^2u_x + \frac{81\beta^2}{50}uu_x - \frac{27\beta^3}{20\alpha^2}u_x$$

$$G_0 = \frac{\alpha}{5}u_{5x} - \frac{12\alpha^2}{25}uu_{4x} + \frac{6\beta}{5}u_{4x} - \frac{23\alpha^2}{25}u_x u_{xxx} - \frac{3\alpha^3}{125}u^2u_{xxx} + \frac{3\alpha\beta}{25}uu_{xxx} - \frac{3\beta^2}{20\alpha}u_{xxx} \\ - \frac{3\alpha^2}{5}u_{xx}^2 - \frac{38\alpha^3}{125}uu_x u_{xx} + \frac{19\alpha\beta}{25}u_x u_{xx} + \frac{38\alpha^4}{625}u^3u_{xx} - \frac{57\alpha^2\beta}{125}u^2u_{xx} \\ + \frac{57\beta^2}{50}uu_{xx} - \frac{19\beta^3}{20\alpha^2}u_{xx} - \frac{6\alpha^3}{125}u_x^3 + \frac{74\alpha^4}{625}u^2u_x^2 - \frac{74\alpha^2\beta}{125}uu_x^2 + \frac{37\beta^2}{50}u_x^2 \\ - \frac{4\alpha^6}{15625}\alpha^6u^6 + \frac{12\alpha^4\beta}{3125}u^5 - \frac{3\alpha^2\beta^2}{125}u^4 + \frac{2\beta^3}{25}u^3 - \frac{3\beta^4}{20\alpha^2}u^2 + \frac{3\beta^5}{20\alpha^4}u + k_0$$

$$\Lambda_1 = -\frac{2\alpha^2}{25}u_{4x} - \frac{2\alpha^3}{25}u_x u_{xx} + \frac{2\alpha^4}{125}u^2u_{xx} - \frac{2\alpha^2\beta}{25}uu_{xx} + \frac{\beta^2}{10}u_{xx} + \frac{2\alpha^4}{125}uu_x^2 - \frac{\alpha^2\beta}{25}u_x^2 \\ - \frac{2\alpha^6}{15625}u^5 + \frac{\alpha^4\beta}{625}u^4 - \frac{\alpha^2\beta^2}{125}u^3 + \frac{\beta^3}{50}u^2 - \frac{\beta^4}{20\alpha^2}u + \frac{2\alpha^2c_1}{25}u + \frac{3\beta^5}{40\alpha^4} - \frac{\beta c_1}{5}$$

$$\Lambda_2 = -\frac{2\alpha^2}{25}u + \frac{\beta}{5}$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 0.$$

Equation VI admits a recursion operator of the form (2.2) with

$$\begin{aligned}
 G_6 &= 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5}u_{xx} - \frac{6\alpha^2}{25}u_x^2 + \frac{6\beta}{5}u_x - \frac{3\beta^2}{2\alpha^2} \\
 G_3 &= \frac{9\alpha}{5}u_{xxx} - \frac{18\alpha^2}{25}u_x u_{xx} + \frac{9\beta}{5}u_{xx} \\
 G_2 &= \alpha u_{4x} - \frac{22\alpha^2}{25}u_x u_{xxx} + \frac{11\beta}{5}u_{xxx} - \frac{13\alpha^2}{25}u_{xx}^2 - \frac{6\alpha^3}{125}u_x^2 u_{xx} + \frac{6\alpha\beta}{25}u_x u_{xx} - \frac{3\beta^2}{10\alpha}u_{xx} \\
 &\quad + \frac{9\alpha^4}{625}u_x^4 - \frac{18\alpha^2\beta}{125}u_x^3 + \frac{27\beta^2}{50}u_x^2 - \frac{9\beta^3}{10\alpha^2}u_x + \frac{9\beta^4}{16\alpha^4} \\
 G_1 &= \frac{\alpha}{5}u_{5x} - \frac{8\alpha^2}{25}u_x u_{4x} + \frac{4\beta}{5}u_{4x} - \frac{3\alpha^2}{5}u_{xx} u_{xxx} - \frac{3\alpha^3}{125}u_x^2 u_{xxx} + \frac{3\alpha\beta}{25}u_x u_{xxx} - \frac{3\beta^2}{20\alpha}u_{xxx} \\
 &\quad - \frac{6\alpha^3}{125}u_x u_{xx}^2 + \frac{3\alpha\beta}{25}u_{xx}^2 + \frac{18\alpha^4}{625}u_x^3 u_{xx} - \frac{27\alpha^2\beta}{125}u_x^2 u_{xx} + \frac{27\beta^2}{50}u_x u_{xx} - \frac{9\beta^3}{20\alpha^2}u_{xx} \\
 G_0 &= -\frac{4\alpha^4}{25}u_x u_{5x} + \frac{2\beta}{5}u_{5x} - \frac{4\alpha^3}{25}u_x u_{xx} u_{xxx} + \frac{2\alpha\beta}{5}u_{xx} u_{xxx} + \frac{4\alpha^4}{125}u_x^3 u_{xxx} - \frac{6\alpha^2\beta}{25}u_x^2 u_{xxx} \\
 &\quad + \frac{3\beta^2}{5}u_x u_{xxx} - \frac{\beta^3}{2\alpha^2}u_{xxx} + \frac{4\alpha^4}{125}u_x^2 u_{xx}^2 - \frac{4\alpha^2\beta}{25}u_x u_{xx}^2 + \frac{\beta^2}{5}u_{xx}^2 - \frac{4\alpha^6}{15625}u_x^6 + \frac{12\alpha^4\beta}{3125}u_x^5 \\
 &\quad - \frac{3\alpha^2\beta^2}{125}u_x^4 + \frac{2\beta^3}{25}u_x^3 - \frac{3\beta^4}{20\alpha^2}u_x^2 + \frac{3\beta^5}{20\alpha^4}u_x + k_0 \\
 \Lambda_1 &= u_{6x} + \alpha u_{xx} u_{4x} - \frac{\alpha^2}{5}u_x^2 u_{4x} + \beta u_x u_{4x} - \frac{5\beta^2}{4\alpha^2}u_{4x} + \alpha u_{xxx}^2 - \frac{4\alpha^2}{5}u_x u_{xx} u_{xxx} \\
 &\quad + 2\beta u_{xx} u_{xxx} - \frac{\alpha^2}{5}u_{xx}^3 + \frac{\alpha^4}{125}u_x^4 u_{xx} - \frac{2\alpha^2\beta}{25}u_x^3 u_{xx} + \frac{3\beta^2}{10}u_x^2 u_{xx} - \frac{\beta^3}{2\alpha^2}u_x u_{xx} \\
 &\quad + \frac{5\beta^4}{8\alpha^4}u_{xx} - c_1 u_{xx} \\
 \Lambda_2 &= u_{xx}, \quad b_0 = -\frac{\beta}{5}, \quad b_1 = \frac{2\alpha^2}{25}, \quad b_2 = \frac{2\alpha^2}{25}, \quad b_3 = -\frac{\beta c_1}{5} + \frac{\beta^5}{\alpha^4}.
 \end{aligned}$$

Equation VII admits a recursion operator of the form (2.2) with

$$\begin{aligned}
 G_6 &= 1, \quad G_5 = 0, \quad G_4 = 6u_{xx} - 6u_x^2 + \frac{6\alpha}{5}e^{-4u} + \frac{6\beta}{5}e^{2u} \\
 G_3 &= 9u_{xxx} - 18u_x u_{xx} - \frac{84\alpha}{5}u_x e^{-4u} + \frac{24\beta}{5}e^{2u} u_x \\
 G_2 &= 5u_{4x} - 22u_x u_{xxx} - 13u_{xx}^2 + \left(\frac{37\beta}{5}e^{2u} - 22\alpha e^{-4u} \right) u_{xx} - 6u_x^2 u_{xx} + 9u_x^4 \\
 &\quad + \left(\frac{494\alpha}{5}e^{-4u} + 4\beta e^{2u} \right) u_x^2 + \frac{9}{25}(\alpha e^{-4u} + \beta e^{2u})^2
 \end{aligned}$$

$$\begin{aligned}
G_1 = & u_{5x} - 8u_x u_{4x} - 15u_{xx} u_{xxx} - 3u_x^2 u_{xxx} + \left(\frac{28\beta}{5} e^{2u} - \frac{77\alpha}{5} e^{-4u} \right) u_{xxx} - 6u_x u_{xx}^2 \\
& + 18u_x^3 u_{xx} + \left(186\alpha e^{-4u} + \frac{84\beta}{5} e^{2u} \right) u_x u_{xx} + \left(\frac{4\beta}{5} e^{2u} - 268\alpha e^{-4u} \right) u_x^3 \\
& + \left(\frac{48\beta^2}{25} e^{4u} - \frac{84\alpha\beta}{25} e^{-2u} - \frac{132\alpha^2}{25} e^{-8u} \right) u_x
\end{aligned}$$

$$\begin{aligned}
G_0 = & -4u_x u_{5x} + \left(\frac{11\beta}{5} e^{2u} - \frac{28\alpha}{5} e^{-4u} \right) u_{4x} + 20u_x^3 u_{xxx} - 20u_x u_{xx} u_{xxx} \\
& + \left(60\alpha e^{-4u} + \frac{24\beta}{5} e^{2u} \right) u_x u_{xxx} + \left(\frac{204\alpha}{5} e^{-4u} + \frac{39\beta}{5} e^{2u} \right) u_{xx}^2 + 20u_x^2 u_{xx}^2 \\
& - \left(344\alpha e^{-4u} + \frac{\beta}{5} e^{2u} \right) u_x^2 u_{xx} + \left(\frac{7\beta^2}{5} e^{4u} - \frac{88\alpha^2}{25} e^{-8u} - \frac{53\alpha\beta}{25} e^{-2u} \right) u_{xx} \\
& - 4u_x^6 + \left(\frac{988\alpha}{5} e^{-4u} - \frac{2\beta}{5} e^{2u} \right) u_x^4 + \left(\frac{11\beta^2}{5} e^{4u} + \frac{14\alpha\beta}{25} e^{-2u} + \frac{116\alpha^2}{5} e^{-8u} \right) u_x^2 \\
& + \frac{4}{125} (\alpha^3 e^{-12u} + \beta^3 e^{6u} + 3\alpha^2 \beta e^{-6u} + 3\alpha \beta^2) + k_0
\end{aligned}$$

$$\begin{aligned}
\Lambda_1 = & u_{6x} + 5u_{xx} u_{4x} - 5u_x^2 u_{4x} + \left(\frac{9\alpha}{5} e^{-4u} + \frac{6\beta}{5} e^{2u} \right) u_{4x} + 5u_{xxx}^2 - 20u_x u_{xx} u_{xxx} \\
& + \left(\frac{24\beta}{5} e^{2u} - \frac{72\alpha}{5} e^{-4u} \right) u_x u_{xxx} - 5u_{xx}^3 + \left(\frac{18\beta}{5} e^{2u} - \frac{54\alpha}{5} e^{-4u} \right) u_{xx}^2 \\
& + 5u_x^4 u_{xx} + \left(\frac{26\beta}{5} e^{2u} + \frac{506\alpha}{5} e^{-4u} \right) u_x^2 u_{xx} + \left(\frac{2\beta^2}{5} e^{4u} + \alpha^2 e^{-8u} + \frac{26\alpha\beta}{25} e^{-2u} \right) u_{xx} \\
& - c_1 u_{xx} + \left(\frac{\beta}{5} e^{2u} - \frac{362\alpha}{5} e^{-4u} \right) u_x^4 + \left(\frac{4\beta^2}{5} e^{4u} - 4\alpha^2 e^{-8u} - \frac{26\alpha\beta}{25} e^{-2u} \right) u_x^2 \\
& - \frac{2\alpha^3}{125} e^{-12u} + \frac{\beta^3}{125} e^{6u} - \frac{3\alpha^2\beta}{125} e^{-6u}
\end{aligned}$$

$$\Lambda_2 = 2u_{xx} - \frac{4\alpha}{5} e^{-4u} + \frac{2\beta}{5} e^{2u}, \quad b_0 = 0, \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 0.$$

Equation VIII admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = 6u_{xx} - 6u_x^2 + \frac{6\beta}{5} e^{2u} + \frac{6\alpha}{5} e^{-u}$$

$$G_3 = 9u_{xxx} - 18u_x u_{xx} + \left(\frac{24\beta}{5} e^{2u} - \frac{3\alpha}{5} e^{-u} \right) u_x$$

$$\begin{aligned}
G_2 = & 5u_{4x} - 22u_x u_{xxx} - 13u_{xx}^2 - 6u_x^2 u_{xx} + \left(\frac{4\alpha}{5} e^{-u} + \frac{37\beta}{5} e^{2u} \right) u_{xx} \\
& + 9u_x^4 + \left(4\beta e^{2u} - \frac{4\alpha}{5} e^{-u} \right) u_x^2 + \frac{9}{25} (\alpha e^{-u} + \beta e^{2u})^2
\end{aligned}$$

$$\begin{aligned}
 G_1 &= u_{5x} - 8u_x u_{4x} - 15u_{xx} u_{xxx} - 3u_x^2 u_{xxx} + \left(\frac{28\beta}{5} e^{2u} - \frac{2\alpha}{5} e^{-u} \right) u_{xxx} - 6u_x u_{xx}^2 \\
 &\quad + 18u_x^3 u_{xx} + \left(\frac{84\beta}{5} e^{2u} - \frac{6\alpha}{5} e^{-u} \right) u_x u_{xx} + \left(\frac{4\beta}{5} e^{2u} - \frac{2\alpha}{5} e^{-u} \right) u_x^3 \\
 &\quad + \left(\frac{48\beta^2}{25} e^{4u} + \frac{33\alpha\beta}{25} e^u - \frac{3\alpha^2}{5} e^{-2u} \right) u_x \\
 G_0 &= -4u_x u_{5x} + \left(\frac{11\beta}{5} e^{2u} - \frac{4\alpha}{5} e^{-u} \right) u_{4x} - 20u_x u_{xx} u_{xxx} + 20u_x^3 u_{xxx} \\
 &\quad + \left(\frac{24\beta}{5} e^{2u} - \frac{9\alpha}{5} e^{-u} \right) u_x u_{xxx} + 20u_x^2 u_{xx}^2 + \frac{39\beta}{5} e^{2u} u_{xx}^2 \\
 &\quad + \left(\frac{8\alpha}{5} e^{-u} - \frac{\beta}{5} e^{2u} \right) u_x^2 u_{xx} + \left(\frac{7\beta^2}{5} e^{4u} + \frac{22\alpha\beta}{25} e^u - \frac{13\alpha^2}{25} e^{-2u} \right) u_{xx} - 4u_x^6 \\
 &\quad - \frac{2}{5} (\alpha e^{-u} + \beta e^{2u}) u_x^4 + \left(\frac{11\beta^2}{5} e^{4u} - \frac{2\alpha^2}{25} e^{-2u} - \frac{\alpha\beta}{25} e^u \right) u_x^2 \\
 &\quad + \frac{4}{125} (\alpha e^{-u} + \beta e^{2u})^3 + k_0 \\
 \Lambda_1 &= u_{6x} + 5u_{xx} u_{4x} - 5u_x^2 u_{4x} + \frac{6\beta}{5} e^{2u} u_{4x} + \frac{3\alpha}{5} e^{-u} u_{4x} + 5u_{xxx}^2 - 20u_x u_{xx} u_{xxx} \\
 &\quad + \left(\frac{24\beta}{5} e^{2u} - \frac{6\alpha}{5} e^{-u} \right) u_x u_{xxx} - 5u_{xx}^3 + \left(\frac{18\beta}{5} e^{2u} - \frac{9\alpha}{10} e^{-u} \right) u_{xx}^2 \\
 &\quad + \left(\alpha e^{-u} + \frac{26\beta}{5} e^{2u} \right) u_x^2 u_{xx} - c_1 u_{xx} + 5u_x^4 u_{xx} + \left(\frac{4\alpha^2}{25} e^{-2u} + \frac{2\beta^2}{5} e^{4u} \right. \\
 &\quad \left. + \frac{\alpha\beta}{5} e^u \right) u_{xx} + \left(\frac{\beta}{5} e^{2u} - \frac{\alpha}{10} e^{-u} \right) u_x^4 + \left(\frac{4\beta^2}{5} e^{4u} + \frac{\alpha\beta}{10} e^u - \frac{4\alpha^2}{25} e^{-2u} \right) u_x^2 \\
 &\quad + \frac{3\alpha\beta^2}{250} e^{3u} - \frac{\alpha^3}{250} e^{-3u} + \frac{\beta^3}{125} e^{6u} + \frac{\alpha c_1}{10} e^{-u} - \frac{\beta c_1}{5} e^{2u} \\
 \Lambda_2 &= 2u_{xx} - \frac{\alpha}{5} e^{-u} + \frac{2\beta}{5} e^{2u}, \quad b_0 = 0, \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 0.
 \end{aligned}$$

4. Multipotentialisations of Equations I–VIII

We now multipotentialise Eqs. I–VIII. This process identifies some more symmetry-integrable equations that are outside of the class I–VIII, in that the recursion operators for those equations require integrating factors, Λ , of order eight.

To establish the potentialisations of Eqs. I–VIII, we first calculate the integrating factors, Λ , for each equation and then use the relation

$$\Lambda = \hat{E}_u \Phi^t \quad (4.1)$$

Fig. 1. Equations V and VI

Equation VII with $\alpha \neq 0$ and $\beta \neq 0$

$$U_t = U_{5x} + (5U_{xx} - 5U_x^2 + \beta e^{2U} + \alpha e^{-4U}) U_{xxx} - 5U_x U_{xx}^2$$

$$+ 3(\beta e^{2U} - 4\alpha e^{-4U}) U_x U_{xx} + U_x^5 + 18\alpha e^{-4U} U_x^3$$

$$+ \frac{1}{5} (\beta e^{2U} + \alpha e^{-4U})^2 U_x + c_1 U_x$$

$\downarrow u_x = -e^{-2U}/2$

$$u_t = u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} - \frac{15}{4} \frac{u_{xxx}^2}{u_x} + \frac{65}{4} \frac{u_{xx}^2 u_{xxx}}{u_x^2} - \frac{135}{16} \frac{u_{xx}^4}{u_x^3}$$

$$- \frac{5\beta}{2} \frac{u_{xxx}}{u_x} + \frac{35\beta}{8} \frac{u_{xx}^2}{u_x^2} - 20\alpha^2 u_x^2 u_{xxx} - 10\alpha^2 u_x u_{xx}^2 - \frac{5\beta^2}{4} \frac{1}{u_x}$$

$$+ 10\alpha^2 \beta u_x^2 + 16\alpha^4 u_x^5 + c_1 u_x$$

 Fig. 2. Equation VII with $\alpha \neq 0$ and $\beta \neq 0$

Equation VII with $\alpha \neq 0$ and $\beta = 0$

$$U_t = U_{5x} + (5U_{xx} - 5U_x^2 + \alpha e^{-4U}) U_{xxx} - 5U_x U_{xx}^2 + U_x^5 + c_1 U_x$$

$$- 12\alpha e^{-4U} U_x U_{xx} + 18\alpha e^{-4U} U_x^3 + \frac{1}{5} \alpha^2 e^{-8U} U_x$$

$\downarrow u_x = e^{-2U}$

$$u_t = u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} - \frac{15}{4} \frac{u_{xxx}^2}{u_x} + \frac{65}{4} \frac{u_{xx}^2 u_{xxx}}{u_x^2} - \frac{135}{16} \frac{u_{xx}^4}{u_x^3}$$

$$+ \alpha u_x^2 u_{xxx} + \frac{1}{2} \alpha u_x u_{xx}^2 + \frac{\alpha^2}{25} u_x^5 + c_1 u_x$$

$\downarrow v_x = (u_x)^{-1/2} q(u) \text{ where } q''(u) + \alpha q(u)/5 = 0$

$$v_t = v_{5x} + \frac{5v_{xx}^2 v_{xxx}}{v_x^2} - \frac{5v_{xx} v_{4x}}{v_x} + c_1 v_x$$

see Figure for Eq. V

 Fig. 3. Equation VII with $\alpha \neq 0$ and $\beta = 0$

Fig. 4. Equation VII with $\alpha = 0$ and $\beta \neq 0$

Equation VIII with $\alpha \neq 0$ and $\beta \neq 0$

$$U_t = U_{5x} + (5U_{xx} - 5U_x^2 + \beta e^{2U} + \alpha e^{-U}) U_{xxx} - 5U_x U_{xx}^2$$

$$+ 3\beta e^{2U} U_x U_{xx} + U_x^5 + \frac{1}{5} (\beta e^{2U} + \alpha e^{-U})^2 U_x + c_1 U_x$$

$\downarrow u_x = e^U$

$$u_t = u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} + \frac{5u_{xx}^2 u_{xxx}}{u_x^2} + \beta u_x^2 u_{xxx} - \beta u_x u_{xx}^2 + \frac{\alpha u_{xxx}}{u_x}$$

$$- \frac{\alpha u_{xx}^2}{u_x^2} + \frac{\beta^2}{25} u_x^5 + \frac{\alpha\beta}{5} u_x^2 - \frac{\alpha^2}{5} \frac{1}{u_x} + c_1 u_x$$

 Fig. 5. Equation VIII with $\alpha \neq 0$ and $\beta \neq 0$

to calculate Φ^t , for the conserved form of the equation $u_t = F$, i.e.

$$D_t \Phi^t + D_x \Phi^x \big|_{u_t=F} = 0. \quad (4.2)$$

Here $\hat{E}_u$ is the Euler operator, defined by

$$\hat{E}_u = \frac{\partial}{\partial u} - D_t \circ \frac{\partial}{\partial u_t} - D_x \circ \frac{\partial}{\partial u_x} + D_x^2 \circ \frac{\partial}{\partial u_{xx}} - \dots. \quad (4.3)$$

The potential variable, v , is then given by the relation

$$v_x = \Phi^t(x, u, u_x, \dots) \quad (4.4a)$$

$$v_t = -\Phi^x(x, u, u_x, \dots). \quad (4.4b)$$

The following proposition is useful for establishing the different orders of Φ^t :

Proposition 4.1. [7] *An evolution equation of order n which admits an integrating factor of order $2p$ ($p \in \mathcal{N}$), $\Lambda = \Lambda(x, u, u_x, \dots, u_{(2p)x})$, may admit a conserved density, Φ^t , of minimum order p and maximum order $2p + 2$, i.e. $\Phi^t = \Phi^t(x, u, u_x, \dots, u_{kx})$, where $k = p, p + 1, \dots, 2p + 2$.*

For Eq. I the most general potentialisation is $u_x \mapsto u$, which leads to Eq. III. Also for Eq. II the only potentialisation is $u_x \mapsto u$ and results in Eq. IV. No further multipotentialisations are possible for Eq. I and Eq. II (nor for Eq. III and Eq. IV). The remaining equations, Eqs. V–VIII, have more interesting multipotentialisations and lead to an extension of the class of symmetry-integrable equations, listed in the following

Proposition 4.2. *A multipotentialisation of Eqs. I–VIII leads to the following semilinear evolution equations, all of which admit Lie–Bäcklund symmetries of order seven but not recursion operators with maximal six order integrating factors (below α and β are arbitrary constants):*

$$\begin{aligned}
u_t = & u_{5x} - \frac{5}{2} \frac{u_{xx}u_{4x}}{u_x} - \frac{5}{4} \frac{u_{xxx}^2}{u_x} + \frac{5u_{xx}^2u_{xxx}}{u_x} - \frac{35}{16} \frac{u_{xx}^4}{u_x^3} + \frac{\alpha}{2} \frac{u_{xx}u_{xxx}}{u_x^{1/2}} - \frac{\alpha^2}{5} u_x u_{xxx} \\
& - \frac{\alpha}{3} \frac{u_{xx}^3}{u_x^{3/2}} + \frac{\alpha^2}{20} u_{xx}^2 + \frac{\alpha^4}{375} u_x^3 + \beta u_x^{1/2} u_{xxx} - \frac{\beta}{2} \frac{u_{xx}^2}{u_x^{1/2}} - \frac{4\alpha^2\beta}{125} u_x^{5/2} + \frac{3\beta^2}{20} u_x^2 \\
& - \frac{5\beta^2}{4\alpha^2} u_{xxx} + \frac{15\beta^2}{16\alpha^2} \frac{u_{xx}^2}{u_x} - \frac{\beta^3}{3\alpha^2} u_x^{3/2} + c_1 u_x
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
u_t = & u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} + \frac{5u_x^2u_{xxx}}{u_x^2} + \beta u_x^2 u_{xxx} - \beta u_x u_{xx}^2 + \frac{\alpha u_{xxx}}{u_x} \\
& - \frac{\alpha u_{xx}^2}{u_x^2} + \frac{\beta^2}{25} u_x^5 + \frac{\alpha\beta}{5} u_x^2 - \frac{\alpha^2}{5} \frac{1}{u_x} + c_1 u_x
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
u_t = & u_{5x} - \frac{5u_{xx}u_{4x}}{u_x} - \frac{15}{4} \frac{u_{xxx}^2}{u_x} + \frac{65}{4} \frac{u_{xx}^2u_{xxx}}{u_x^2} - \frac{135}{16} \frac{u_{xx}^4}{u_x^3} \\
& - \frac{5\beta}{2} \frac{u_{xxx}}{u_x} + \frac{35\beta}{8} \frac{u_{xx}^2}{u_x^2} - 20\alpha^2 u_x^2 u_{xxx} - 10\alpha^2 u_x u_{xx}^2 - \frac{5\beta^2}{4} \frac{1}{u_x} \\
& + 10\alpha^2 \beta u_x^2 + 16\alpha^4 u_x^5 + c_1 u_x.
\end{aligned} \tag{4.7}$$

Some remarks on Proposition 3 are in order:

Regarding (4.6): Equation (4.6) is equivalent to Eq. (4.2.11) listed in [14] for $\lambda_1 = \alpha/5$ and $\lambda_2 = -\beta/5$. Equation (4.7) is listed as Eq. (4.2.12) in [14] for $\lambda_1 = -\beta/2$ and $\lambda_2 = -4\alpha^2$. Regarding (4.5): If $\beta = 0$ and $\alpha = 5$, then (4.5) is equivalent to Eq. (4.2.10) with $\lambda = 0$ listed in [14]. However if $\lambda \neq 0$ then Eq. (4.2.10) in [14] is not equivalent to (4.5) for any choice of β and α . If $\beta \neq 0$, then (4.5) is not listed in [14] and, as far as we know, also not listed elsewhere in the literature. Moreover, (4.5) admits a Lie–Bäcklund symmetry of order seven, whereas Eq. (4.2.10) with $\lambda \neq 0$ in [14] does not admit a Lie–Bäcklund symmetry of order seven.

The resulting multipotentialisation of Eqs. V–VII are shown in Figs. 1–5 below. Referring to Fig. 1, we list the following:

$$\begin{aligned}
\Phi_1^t &= U^2, \quad \Phi_2^t = U, \quad \Phi_3^t = u_x^2, \quad \Phi_4^t = -\frac{5}{2\alpha} e^{\beta x/\alpha} e^{-2\alpha u/5} \\
\Phi_5^t &= \frac{5}{\alpha} e^{-\beta x/(2\alpha)} e^{\alpha u/5}, \quad \Phi_6^t = \frac{v^n}{v_x^2} \quad (n = 0 \text{ or } n = 4) \\
\Phi_7^t &= -\frac{\gamma}{6} \frac{v^2}{v_x^2}, \quad \Phi_8^t = -\frac{4}{3} \frac{1}{\sqrt{w_x}}, \quad \Phi_{9\pm}^t = e^{\pm\sqrt{-6\gamma}\tilde{w}/4}, \quad \gamma < 0.
\end{aligned}$$

We remark that, under $\beta \rightarrow 5\beta$, Eq. VIII is equivalent to Eq. VII with $\alpha = 0$.

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