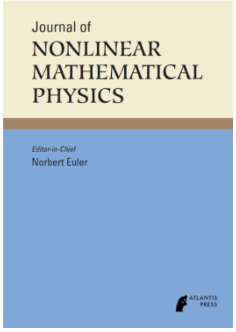

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Marianna Euler, Norbert Euler

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A CLASS OF SEMILINEAR FIFTH-ORDER EVOLUTION EQUATIONS: RECURSION OPERATORS AND MULTIPOTENTIALISATIONS

MARIANNA EULER* and NORBERT EULER†

Department of Engineering Sciences and Mathematics
Luleå University of Technology
SE-971 87 Luleå, Sweden
**marianna@ltu.se*
†*norbert@ltu.se*

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We apply a list of criteria which leads to a class of fifth-order symmetry-integrable evolution equations. The recursion operators for this class are given explicitly. Multipotentialisations are then applied to the equations in this class in order to extend this class of integrable equations.

Keywords: Nonlinear fifth-order evolution equations; recursion operators; Lie–Bäcklund symmetries; integrability; multipotentialisations.

1. Introduction

Autonomous symmetry-integrable evolution equations in $(1+1)$ dimensions are equations of the form

$$u_t = F(x, u, u_x, u_{xx}, \dots) \tag{1.1}$$

which admit an infinite number of local commuting Lie–Bäcklund symmetries [3, 11],

$$Z = \eta(x, u, u_x, u_{xx}, \dots)\frac{\partial}{\partial u}. \tag{1.2}$$

Recursion operators, $R[u]$, can be defined which generate these infinite sets of Lie–Bäcklund symmetries [15]. In most cases these recursion operators for autonomous evolution equations are integro-differential operators of the form

$$R[u] = \sum_{j=0}^{p} G_j D_x^j + \sum_{i=1} I_i(u_x, u_t) D_x^{-1} \circ \Lambda_i, \tag{1.3}$$

where G_j are functions of x, u and x-derivatives of u, D_x is the total derivative operator, D_x^{-1} the integral operator and Λ_i are integrating factors for the evolution equation.

By applying the above mentioned criteria, we classified several classes of second- and third-order evolution equations with respect to its recursion operators of the form (1.3). These results have been published in the papers [4–6, 16].

It is of some interest to potentialise and, moreover, multipotentialise integrable evolution equationsas this often extends the class of integrable equations. In particular, multipotentialisations of symmetry-integrable evolution equations may lead to equations with different types of recursion operators. In some cases, even nonlocal recursion operators are necessary in ordert to generate the locaol Lie–Bäcklund symmetries (an example is given in [16]). Multipotentialisations of symmetry-integrable evolution equations was studied and applied for several classes of the equations and published in [8–10].

In the current paper we report a class of fifth-order symmetry-integrable evolution equations and apply multipotentialisations to each equation to extend this class. The classification criteria are given in Sec. 2, where the resulting evolution equations (Eqs. I–VII) are also listed in Proposition 1. In Sec. 3 we give the recursion operators for Eqs. I–VIII. In Sec. 4 we report the multipotentialisations of these equations and display their connections in Figs. 1–5.

2. The Classification Criteria

Classification criteria:

We classify all semi-linear fifth-order evolution equations of the form

$$u_t = u_{5x} + F(u, u_x, \dots, u_{4x}) \tag{2.1}$$

which admit the following criteria:

- **Criterion 1:** *Equation* (2.1) *admits at least two integrating factors*, Λ, *of order zero*, $\Lambda(u)$, *of order two*, $\Lambda(u, u_x, u_{xx})$, *of order four*, $\Lambda(u, u_x, u_{xx}, u_{xxx}, u_{4x})$, *or of order six*, $\Lambda(u, u_x, \dots, u_{6x})$.
- **Criterion 2:** *Equation* (2.1) *admits an integro-differential recursion operator of order six*:

$$R[u] = \sum_{j=0}^{6} G_j D_x^j + (b_1 u_x + b_0) D_x^{-1} \circ \Lambda_1 + (b_2 u_t + b_3) D_x^{-1} \circ \Lambda_2, \tag{2.2}$$

  where $G_j = G_j(u, u_x, u_{xx}, \dots)$ *with* $G_6 \neq 0$, b_k *are real constants and* Λ_i *are integrating factors of the equation.*
- **Criterion 3:** *Equation* (2.1) *does not belong to a hierarchy of evolution equations*

$$u_t = R[u]^n\, u_x, \quad n \in \mathcal{N} \tag{2.3}$$

  where $R[u]$ *is a first- or second-order integro-differential recursion operator.*
- **Criterion 4:** *Equation* (2.1) *admits local Lie–Bäcklund symmetries of lowest order seven, that can be obtained by acting the recursion operator from Criterion 2 on* u_x, *i.e.* $R[u]u_x$.

By applying the above criteria we obtained the following list of eight equations:

Proposition 2.1. *The following list of eight equations are the only fifth-order evolution equations in the class,*

$$u_t = u_{5x} + F(u, u_x, u_{xx}, \dots, u_{5x}), \tag{2.4}$$

which satisfy all four criteria listed above:

Equation I:

$$u_t = u_{5x} + \alpha u u_{xxx} + \alpha u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x + \beta \left(u_{xxx} + \frac{2\alpha}{5} u u_x \right) + \left(\frac{\beta^2}{5} + c_1 \right) u_x. \tag{2.5}$$

Equation I is known as the Sawada–Kotera equation for $\alpha = 5,\ \beta = 0$ [18] *and the Caudrey–Dodd–Gibbon–Sawada–Kotera equation for* $\alpha = 30, \beta = 0$ [2]. *For* $\alpha = 5$ *and* $\beta = 0$, *Equation I has been listed as Eq.* (4.2.2) *in* [14]. *A recursion operator for arbitrary* α *and* $\beta = 0$ *is given in* [13].

Equation II:

$$u_t = u_{5x} + \alpha u u_{xxx} + \frac{5\alpha}{2} u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x + \beta \left(u_{xxx} + \frac{2\alpha}{5} u u_x \right) + c_1 u_x. \tag{2.6}$$

Equation II is known as the Kaup-Kupershmidt equation for $\alpha = 10, \beta = 0$ [12]. *For* $\alpha = 5$ *and* $\beta = 0$ *it has been listed as Eq.* (4.2.3) *in* [14]. *A recursion operator for all* α *and* $\beta = 0$ *is given in* [13].

Equation III:

$$u_t = u_{5x} + \alpha u_x u_{xxx} + \frac{\alpha^2}{15} u_x^3 + \beta \left(u_{xxx} + \frac{\alpha}{5} u_x^2 \right) + c_1 u_x. \tag{2.7}$$

Equation III is known as the potential Sawada–Kotera equation for $\beta = 0$. *For* $\alpha = 5$ *and* $\beta = 0$ *it has been listed as Eq.* (4.2.4) *in* [14]. *For* $\alpha = 5$ *and* $\beta = 0$ *a recursion operator is given in* [1].

Equation IV:

$$u_t = u_{5x} + \alpha u_x u_{xxx} + \frac{3\alpha}{4} u_{xx}^2 + \frac{\alpha^2}{15} u_x^3 + \beta \left(u_{xxx} + \frac{\alpha}{5} u_x^2 \right) + c_1 u_x. \tag{2.8}$$

Equation IV is known as the potential Kaup–Kupershmidt equation for $\alpha = 5$ *and* $\beta = 0$. *For* $\alpha = 5$ *and* $\beta = 0$ *it has been listed as Eq.* (4.2.5) *in* [14]. *For* $\alpha = 5$ *and* $\beta = 0$ *a recursion operator is given in* [1].

Equation V:

$$\begin{aligned} u_t = u_{5x} &- \frac{\alpha^2}{5} u^2 u_{xxx} + \alpha u_x u_{xxx} - \frac{4\alpha^2}{5} u u_x u_{xx} + \alpha u_{xx}^2 - \frac{\alpha^2}{5} u_x^3 + \frac{\alpha^4}{125} u^4 u_x \\ &+ \beta \left(2u_x u_{xx} + u u_{xxx} - \frac{2\alpha^2}{25} u^3 u_x \right) + \beta^2 \left(\frac{3}{10} u^2 u_x - \frac{5}{4\alpha^2} u_{xxx} \right) - \frac{\beta^3}{2\alpha^2} u u_x + c_1 u_x. \end{aligned} \tag{2.9}$$

Equation V is known as the Kupershmidt equation for $\alpha = 5$ *and* $\beta = 0$ *and it has been listed as Eq.* (4.2.6) *in* [14]. *A recursion operator for* $\alpha = 5$ *and* $\beta = 0$ *is given in* [1, 13].

Equation VI:

$$\begin{aligned} u_t &= u_{5x} + \alpha u_{xx}u_{xxx} - \frac{\alpha^2}{5}u_x^2u_{xxx} - \frac{\alpha^2}{5}u_xu_{xx}^2 + \frac{\alpha^4}{624}u_x^5 \\ &\quad + \beta\left(u_xu_{xxx} - \frac{\alpha^2}{50}u_x^4 + \frac{1}{2}u_{xx}^2\right) + \beta^2\left(-\frac{5}{4\alpha^2}u_{xxx} + \frac{1}{10}u_x^3\right) - \frac{\beta^3}{4\alpha^2}u_x^2 + c_1u_x. \end{aligned} \tag{2.10}$$

Equation VI is known as the potential Kupershmidt equation. For $\alpha = 5, \beta = 0$ it has been listed as Eq. (4.2.7) *in* [14]. *For $\alpha = 5$ and $\beta = 0$ a recursion operator is given in* [1]. *As far as we know, the recursion operator for the full Eq.* (2.10) *has not be reported so far in the literature.*

Equation VII:

$$\begin{aligned} u_t &= u_{5x} + \left(5u_{xx} - 5u_x^2 + \beta e^{2u} + \alpha e^{-4u}\right)u_{xxx} - 5u_xu_{xx}^2 + 3\left(\beta e^{2u} - 4\alpha e^{-4u}\right)u_xu_{xx} \\ &\quad + u_x^5 + 18\alpha e^{-4u}u_x^3 + \frac{1}{5}\left(\beta e^{2u} + \alpha e^{-4u}\right)^2u_x + c_1u_x. \end{aligned} \tag{2.11}$$

Equation VII, with $\alpha = -5\lambda_2^2$ and $\beta = 5\lambda_1$, has been listed as Eq. (4.2.8) *in* [14]. *As far as we know, the recursion operator for* (2.11) *has not be reported so far in the literature.*

Equation VIII:

$$\begin{aligned} u_t &= u_{5x} + \left(5u_{xx} - 5u_x^2 + \alpha e^{-u} + \beta e^{2u}\right)u_{xxx} - 5u_xu_{xx}^2 + 3\beta e^{2u}u_xu_{xx} \\ &\quad + u_x^5 + \frac{1}{5}\left(\alpha e^{-u} + \beta e^{2u}\right)^2u_x + c_1u_x. \end{aligned} \tag{2.12}$$

Equation VIII, with $\alpha = 5\lambda_2$ and $\beta = -5\lambda_1^2$, has been listed as Eq. (4.2.9) *in* [14]. *As far as we know, the recursion operator for* (2.12) *has not be reported so far in the literature.*

In Sec. 2 we report the recursion operators for the equations in Proposition 1 and in Sec. 3 we multipotentialise each equation and, in doing so, we extend the list of equations to it multipotential forms. The resulting equations are given in Proposition 3 and the relations between the equations are shown in Figs. 1–5.

3. The Recursion Operators for Equations I–VIII

Equation I admits a recursion operator of the form (2.2) with

$$\begin{aligned} &G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5}u + \frac{6\beta}{5}, \quad G_3 = \frac{9\alpha}{5}u_x \\ &G_2 = \frac{11\alpha}{5}u_{xx} + \frac{9\alpha^2}{25}u^2 + \frac{18\alpha\beta}{25}u + \frac{9\beta^2}{25} \\ &G_1 = 2\alpha u_{xxx} + \frac{21\alpha^2}{25}uu_x + \frac{21\alpha\beta}{25}u_x \end{aligned}$$

$$G_0 = \alpha u_{4x} + \frac{16\alpha^2}{25} u u_{xx} + \frac{16\alpha\beta}{25} u_{xx} + \frac{6\alpha^2}{25} u_x^2 + \frac{4\alpha^3}{125} u^3 + \frac{12\alpha^2\beta}{125} u^2 + \frac{12\alpha\beta^2}{125} u + k_0$$

$$\Lambda_1 = \frac{2\alpha^2}{25} u_{xx} + \frac{\alpha^3}{125} u^2 + \frac{2\alpha^2\beta}{125} u + \frac{\alpha}{6} c_1 + \frac{\alpha\beta^2}{125}, \quad \Lambda_2 = \frac{\alpha}{5}$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 0,$$

where k_0 is an arbitrary constant.

Equation II admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u + \frac{6\beta}{5}, \quad G_3 = \frac{18\alpha}{5} u_x$$

$$G_2 = \frac{18\alpha\beta}{25} u + \frac{9\alpha^2}{25} u^2 + \frac{49\alpha}{10} u_{xx} + \frac{9\beta^2}{25}$$

$$G_1 = \frac{7\alpha}{2} u_{xxx} + \frac{6\alpha^2}{5} u u_x + \frac{6\alpha\beta}{5} u_x$$

$$G_0 = \frac{13\alpha}{10} u_{4x} + \frac{41\alpha\beta}{50} u_{xx} + \frac{41\alpha^2}{50} u u_{xx} + \frac{69\alpha^2}{100} u_x^2 + \frac{4\alpha^3}{125} u^3 + \frac{12\alpha^2\beta}{125} u^2 + \frac{12\alpha\beta^2}{125} u + k_0$$

$$\Lambda_1 = \frac{\alpha^2}{50} u_{xx} + \frac{\alpha^3}{125} u^2 + \frac{2\alpha^2\beta}{125} u + \frac{\alpha}{125}(6\beta^2 - 25c_1), \quad \Lambda_2 = \frac{\alpha}{5}$$

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 0.$$

Equation III admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u_x + \frac{6\beta}{5}$$

$$G_3 = \frac{3\alpha}{5} u_{xx}, \quad G_2 = \frac{8\alpha}{5} u_{xxx} + \frac{9\alpha^2}{25} u_x^2 + \frac{18\alpha\beta}{25} u_x + \frac{9\beta^2}{25}$$

$$G_1 = \frac{2\alpha}{5} u_{4x} + \frac{3\alpha^2}{25} u_x u_{xx} + \frac{3\alpha\beta}{25} u_{xx}$$

$$G_0 = \frac{3\alpha}{5} u_{5x} + \frac{13\alpha^2}{25} u_x u_{xxx} + \frac{13\alpha\beta}{25} u_{xxx} + \frac{3\alpha^2}{25} u_{xx}^2 + \frac{4\alpha^3}{125} u_x^3 + \frac{12\alpha^2\beta}{125} u_x^2 + \frac{12\alpha\beta^2}{125} u_x + k_0$$

$$\Lambda_1 = u_{4x} + \frac{\alpha}{5} u_x u_{xx} + \frac{\beta}{5} u_{xx}$$

$$\Lambda_2 = u_{6x} + \frac{3\alpha}{5} u_x u_{4x} + \frac{6\alpha}{5} u_{xx} u_{xxx} + \frac{2\alpha^2}{25} u_x^2 u_{xx} + \frac{\alpha\beta}{25} u_x u_{xx} - \frac{\beta^2}{25} u_{xx}$$

$$b_0 = -\frac{8\alpha\beta}{25}, \quad b_1 = -\frac{2\alpha^2}{25}, \quad b_2 = 0, \quad b_3 = -\frac{2\alpha}{5}.$$

Equation IV admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5} u_x + \frac{6\beta}{5}, \quad G_3 = \frac{12\alpha}{5} u_{xx}$$

$$G_2 = \frac{5\alpha}{2} u_{xxx} + \frac{9\alpha^2}{25} u_x^2 + \frac{18\alpha\beta}{25} u_x + \frac{9\beta^2}{25}, \quad G_1 = \alpha u_{4x} + \frac{12\alpha^2}{25} u_x u_{xx} + \frac{12\alpha\beta}{25} u_{xx}$$

$$G_0 = \frac{3\alpha}{10}u_{5x} + \frac{17\alpha^2}{50}u_x u_{xxx} + \frac{17\alpha\beta}{50}u_{xxx} + \frac{21\alpha^2}{100}u_{xx}^2 + \frac{4\alpha^3}{125}u_x^3 + \frac{12\alpha^2\beta}{125}u_x^2$$
$$+ \frac{12\alpha\beta^2}{125}u_x + \frac{6\beta c_1}{25} + k_0$$
$$b_0 = k_1, \quad b_1 = -\frac{\alpha^2}{50}, \quad b_2 = 0, \quad b_3 = -\frac{\alpha}{10}$$
$$\Lambda_1 = u_{4x} + \frac{4\alpha}{5}u_{xx}u_x + \frac{4\beta}{5}u_{xx}$$
$$\Lambda_2 = u_{6x} + \frac{6\alpha}{5}u_{4x}u_x + \frac{7\beta}{5}u_{4x} + \frac{10k_1}{\alpha}u_{4x} + \frac{12\alpha}{5}u_{xxx}u_{xx} + \frac{8\alpha^2}{25}u_{xx}u_x^2 + 8k_1 u_{xx}u_x$$
$$+ \frac{4\alpha\beta}{5}u_{xx}u_x + \frac{12\beta^2}{25}u_{xx} + \frac{8\beta k_1}{\alpha}u_{xx}.$$

Equation V admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5}u_x - \frac{6\alpha^2}{25}u^2 + \frac{6\beta}{5}u - \frac{3\beta^2}{2\alpha^2}$$
$$G_3 = 3\alpha u_{xx} - \frac{6\alpha^2}{5}uu_x + 3\beta u_x$$
$$G_2 = \frac{14\alpha}{5}u_{xxx} - \frac{8\alpha^2}{5}uu_{xx} + 4\beta u_{xx} - \frac{31\alpha^2}{25}u_x^2 - \frac{6\alpha^3}{125}u^2u_x + \frac{6\alpha\beta}{25}uu_x - \frac{3\beta^2}{10\alpha}u_x$$
$$+ \frac{9\alpha^4}{625}u^4 - \frac{18\alpha^2\beta}{125}u^3 + \frac{27\beta^2}{50}u^2 - \frac{9\beta^3}{10\alpha^2}u + \frac{9\beta^4}{16\alpha^4}$$
$$G_1 = \frac{6\alpha}{5}u_{4x} - \frac{6\alpha^2}{5}uu_{xxx} + 3\beta u_{xxx} - \frac{63\alpha^2}{25}u_x u_{xx} - \frac{9\alpha^3}{125}u^2u_{xx} + \frac{9\alpha\beta}{25}uu_{xx}$$
$$- \frac{9\beta^2}{20\alpha}u_{xx} - \frac{18\alpha^3}{125}uu_x^2 + \frac{9\alpha\beta}{25}u_x^2 + \frac{54\alpha^4}{625}u^3u_x - \frac{81\alpha^2\beta}{125}u^2u_x + \frac{81\beta^2}{50}uu_x - \frac{27\beta^3}{20\alpha^2}u_x$$
$$G_0 = \frac{\alpha}{5}u_{5x} - \frac{12\alpha^2}{25}uu_{4x} + \frac{6\beta}{5}u_{4x} - \frac{23\alpha^2}{25}u_xu_{xxx} - \frac{3\alpha^3}{125}u^2u_{xxx} + \frac{3\alpha\beta}{25}uu_{xxx} - \frac{3\beta^2}{20\alpha}u_{xxx}$$
$$- \frac{3\alpha^2}{5}u_{xx}^2 - \frac{38\alpha^3}{125}uu_xu_{xx} + \frac{19\alpha\beta}{25}u_xu_{xx} + \frac{38\alpha^4}{625}u^3u_{xx} - \frac{57\alpha^2\beta}{125}u^2u_{xx}$$
$$+ \frac{57\beta^2}{50}uu_{xx} - \frac{19\beta^3}{20\alpha^2}u_{xx} - \frac{6\alpha^3}{125}u_x^3 + \frac{74\alpha^4}{625}u^2u_x^2 - \frac{74\alpha^2\beta}{125}uu_x^2 + \frac{37\beta^2}{50}u_x^2$$
$$- \frac{4\alpha^6}{15625}\alpha^6u^6 + \frac{12\alpha^4\beta}{3125}u^5 - \frac{3\alpha^2\beta^2}{125}u^4 + \frac{2\beta^3}{25}u^3 - \frac{3\beta^4}{20\alpha^2}u^2 + \frac{3\beta^5}{20\alpha^4}u + k_0$$
$$\Lambda_1 = -\frac{2\alpha^2}{25}u_{4x} - \frac{2\alpha^3}{25}u_xu_{xx} + \frac{2\alpha^4}{125}u^2u_{xx} - \frac{2\alpha^2\beta}{25}uu_{xx} + \frac{\beta^2}{10}u_{xx} + \frac{2\alpha^4}{125}uu_x^2 - \frac{\alpha^2\beta}{25}u_x^2$$
$$- \frac{2\alpha^6}{15625}u^5 + \frac{\alpha^4\beta}{625}u^4 - \frac{\alpha^2\beta^2}{125}u^3 + \frac{\beta^3}{50}u^2 - \frac{\beta^4}{20\alpha^2}u + \frac{2\alpha^2c_1}{25}u + \frac{3\beta^5}{40\alpha^4} - \frac{\beta c_1}{5}$$
$$\Lambda_2 = -\frac{2\alpha^2}{25}u + \frac{\beta}{5}$$
$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 1, \quad b_3 = 0.$$

Equation VI admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = \frac{6\alpha}{5}u_{xx} - \frac{6\alpha^2}{25}u_x^2 + \frac{6\beta}{5}u_x - \frac{3\beta^2}{2\alpha^2}$$

$$G_3 = \frac{9\alpha}{5}u_{xxx} - \frac{18\alpha^2}{25}u_x u_{xx} + \frac{9\beta}{5}u_{xx}$$

$$G_2 = \alpha u_{4x} - \frac{22\alpha^2}{25}u_x u_{xxx} + \frac{11\beta}{5}u_{xxx} - \frac{13\alpha^2}{25}u_{xx}^2 - \frac{6\alpha^3}{125}u_x^2 u_{xx} + \frac{6\alpha\beta}{25}u_x u_{xx} - \frac{3\beta^2}{10\alpha}u_{xx}$$
$$+ \frac{9\alpha^4}{625}u_x^4 - \frac{18\alpha^2\beta}{125}u_x^3 + \frac{27\beta^2}{50}u_x^2 - \frac{9\beta^3}{10\alpha^2}u_x + \frac{9\beta^4}{16\alpha^4}$$

$$G_1 = \frac{\alpha}{5}u_{5x} - \frac{8\alpha^2}{25}u_x u_{4x} + \frac{4\beta}{5}u_{4x} - \frac{3\alpha^2}{5}u_{xx}u_{xxx} - \frac{3\alpha^3}{125}u_x^2 u_{xxx} + \frac{3\alpha\beta}{25}u_x u_{xxx} - \frac{3\beta^2}{20\alpha}u_{xxx}$$
$$- \frac{6\alpha^3}{125}u_x u_{xx}^2 + \frac{3\alpha\beta}{25}u_{xx}^2 + \frac{18\alpha^4}{625}u_x^3 u_{xx} - \frac{27\alpha^2\beta}{125}u_x^2 u_{xx} + \frac{27\beta^2}{50}u_x u_{xx} - \frac{9\beta^3}{20\alpha^2}u_{xx}$$

$$G_0 = -\frac{4\alpha^4}{25}u_x u_{5x} + \frac{2\beta}{5}u_{5x} - \frac{4\alpha^3}{25}u_x u_{xx} u_{xxx} + \frac{2\alpha\beta}{5}u_{xx}u_{xxx} + \frac{4\alpha^4}{125}u_x^3 u_{xxx} - \frac{6\alpha^2\beta}{25}u_x^2 u_{xxx}$$
$$+ \frac{3\beta^2}{5}u_x u_{xxx} - \frac{\beta^3}{2\alpha^2}u_{xxx} + \frac{4\alpha^4}{125}u_x^2 u_{xx}^2 - \frac{4\alpha^2\beta}{25}u_x u_{xx}^2 + \frac{\beta^2}{5}u_{xx}^2 - \frac{4\alpha^6}{15625}u_x^6 + \frac{12\alpha^4\beta}{3125}u_x^5$$
$$- \frac{3\alpha^2\beta^2}{125}u_x^4 + \frac{2\beta^3}{25}u_x^3 - \frac{3\beta^4}{20\alpha^2}u_x^2 + \frac{3\beta^5}{20\alpha^4}u_x + k_0$$

$$\Lambda_1 = u_{6x} + \alpha u_{xx}u_{4x} - \frac{\alpha^2}{5}u_x^2 u_{4x} + \beta u_x u_{4x} - \frac{5\beta^2}{4\alpha^2}u_{4x} + \alpha u_{xxx}^2 - \frac{4\alpha^2}{5}u_x u_{xx} u_{xxx}$$
$$+ 2\beta u_{xx}u_{xxx} - \frac{\alpha^2}{5}u_{xx}^3 + \frac{\alpha^4}{125}u_x^4 u_{xx} - \frac{2\alpha^2\beta}{25}u_x^3 u_{xx} + \frac{3\beta^2}{10}u_x^2 u_{xx} - \frac{\beta^3}{2\alpha^2}u_x u_{xx}$$
$$+ \frac{5\beta^4}{8\alpha^4}u_{xx} - c_1 u_{xx}$$

$$\Lambda_2 = u_{xx}, \quad b_0 = -\frac{\beta}{5}, \quad b_1 = \frac{2\alpha^2}{25}, \quad b_2 = \frac{2\alpha^2}{25}, \quad b_3 = -\frac{\beta c_1}{5} + \frac{\beta^5}{\alpha^4}.$$

Equation VII admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = 6u_{xx} - 6u_x^2 + \frac{6\alpha}{5}e^{-4u} + \frac{6\beta}{5}e^{2u}$$

$$G_3 = 9u_{xxx} - 18u_x u_{xx} - \frac{84\alpha}{5}u_x e^{-4u} + \frac{24\beta}{5}e^{2u}u_x$$

$$G_2 = 5u_{4x} - 22u_x u_{xxx} - 13u_{xx}^2 + \left(\frac{37\beta}{5}e^{2u} - 22\alpha e^{-4u}\right)u_{xx} - 6u_x^2 u_{xx} + 9u_x^4$$
$$+ \left(\frac{494\alpha}{5}e^{-4u} + 4\beta e^{2u}\right)u_x^2 + \frac{9}{25}\left(\alpha e^{-4u} + \beta e^{2u}\right)^2$$

$$G_1 = u_{5x} - 8u_x u_{4x} - 15u_{xx}u_{xxx} - 3u_x^2 u_{xxx} + \left(\frac{28\beta}{5}e^{2u} - \frac{77\alpha}{5}e^{-4u}\right) u_{xxx} - 6u_x u_{xx}^2$$
$$+ 18u_x^3 u_{xx} + \left(186\alpha e^{-4u} + \frac{84\beta}{5}e^{2u}\right) u_x u_{xx} + \left(\frac{4\beta}{5}e^{2u} - 268\alpha e^{-4u}\right) u_x^3$$
$$+ \left(\frac{48\beta^2}{25}e^{4u} - \frac{84\alpha\beta}{25}e^{-2u} - \frac{132\alpha^2}{25}e^{-8u}\right) u_x$$
$$G_0 = -4u_x u_{5x} + \left(\frac{11\beta}{5}e^{2u} - \frac{28\alpha}{5}e^{-4u}\right) u_{4x} + 20u_x^3 u_{xxx} - 20u_x u_{xx} u_{xxx}$$
$$+ \left(60\alpha e^{-4u} + \frac{24\beta}{5}e^{2u}\right) u_x u_{xxx} + \left(\frac{204\alpha}{5}e^{-4u} + \frac{39\beta}{5}e^{2u}\right) u_{xx}^2 + 20u_x^2 u_{xx}^2$$
$$- \left(344\alpha e^{-4u} + \frac{\beta}{5}e^{2u}\right) u_x^2 u_{xx} + \left(\frac{7\beta^2}{5}e^{4u} - \frac{88\alpha^2}{25}e^{-8u} - \frac{53\alpha\beta}{25}e^{-2u}\right) u_{xx}$$
$$- 4u_x^6 + \left(\frac{988\alpha}{5}e^{-4u} - \frac{2\beta}{5}e^{2u}\right) u_x^4 + \left(\frac{11\beta^2}{5}e^{4u} + \frac{14\alpha\beta}{25}e^{-2u} + \frac{116\alpha^2}{5}e^{-8u}\right) u_x^2$$
$$+ \frac{4}{125}\left(\alpha^3 e^{-12u} + \beta^3 e^{6u} + 3\alpha^2\beta e^{-6u} + 3\alpha\beta^2\right) + k_0$$
$$\Lambda_1 = u_{6x} + 5u_{xx}u_{4x} - 5u_x^2 u_{4x} + \left(\frac{9\alpha}{5}e^{-4u} + \frac{6\beta}{5}e^{2u}\right) u_{4x} + 5u_{xxx}^2 - 20u_x u_{xx} u_{xxx}$$
$$+ \left(\frac{24\beta}{5}e^{2u} - \frac{72\alpha}{5}e^{-4x}\right) u_x u_{xxx} - 5u_{xx}^3 + \left(\frac{18\beta}{5}e^{2u} - \frac{54\alpha}{5}e^{-4u}\right) u_{xx}^2$$
$$+ 5u_x^4 u_{xx} + \left(\frac{26\beta}{5}e^{2u} + \frac{506\alpha}{5}e^{-4u}\right) u_x^2 u_{xx} + \left(\frac{2\beta^2}{5}e^{4u} + \alpha^2 e^{-8u} + \frac{26\alpha\beta}{25}e^{-2u}\right) u_{xx}$$
$$- c_1 u_{xx} + \left(\frac{\beta}{5}e^{2u} - \frac{362\alpha}{5}e^{-4u}\right) u_x^4 + \left(\frac{4\beta^2}{5}e^{4u} - 4\alpha^2 e^{-8u} - \frac{26\alpha\beta}{25}e^{-2u}\right) u_x^2$$
$$- \frac{2\alpha^3}{125}e^{-12u} + \frac{\beta^3}{125}e^{6u} - \frac{3\alpha^2\beta}{125}e^{-6u}$$
$$\Lambda_2 = 2u_{xx} - \frac{4\alpha}{5}e^{-4u} + \frac{2\beta}{5}e^{2u}, \quad b_0 = 0, \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 0.$$

Equation VIII admits a recursion operator of the form (2.2) with

$$G_6 = 1, \quad G_5 = 0, \quad G_4 = 6u_{xx} - 6u_x^2 + \frac{6\beta}{5}e^{2u} + \frac{6\alpha}{5}e^{-u}$$
$$G_3 = 9u_{xxx} - 18u_x u_{xx} + \left(\frac{24\beta}{5}e^{2u} - \frac{3\alpha}{5}e^{-u}\right) u_x$$
$$G_2 = 5u_{4x} - 22u_x u_{xxx} - 13u_{xx}^2 - 6u_x^2 u_{xx} + \left(\frac{4\alpha}{5}e^{-u} + \frac{37\beta}{5}e^{2u}\right) u_{xx}$$
$$+ 9u_x^4 + \left(4\beta e^{2u} - \frac{4\alpha}{5}e^{-u}\right) u_x^2 + \frac{9}{25}\left(\alpha e^{-u} + \beta e^{2u}\right)^2$$

$$
\begin{aligned}
G_1 &= u_{5x} - 8u_x u_{4x} - 15u_{xx}u_{xxx} - 3u_x^2 u_{xxx} + \left(\frac{28\beta}{5}e^{2u} - \frac{2\alpha}{5}e^{-u}\right)u_{xxx} - 6u_x u_{xx}^2 \\
&\quad + 18u_x^3 u_{xx} + \left(\frac{84\beta}{5}e^{2u} - \frac{6\alpha}{5}e^{-u}\right)u_x u_{xx} + \left(\frac{4\beta}{5}e^{2u} - \frac{2\alpha}{5}e^{-u}\right)u_x^3 \\
&\quad + \left(\frac{48\beta^2}{25}e^{4u} + \frac{33\alpha\beta}{25}e^{u} - \frac{3\alpha^2}{5}e^{-2u}\right)u_x \\
G_0 &= -4u_x u_{5x} + \left(\frac{11\beta}{5}e^{2u} - \frac{4\alpha}{5}e^{-u}\right)u_{4x} - 20u_x u_{xx}u_{xxx} + 20u_x^3 u_{xxx} \\
&\quad + \left(\frac{24\beta}{5}e^{2u} - \frac{9\alpha}{5}e^{-u}\right)u_x u_{xxx} + 20u_x^2 u_{xx}^2 + \frac{39\beta}{5}e^{2u}u_{xx}^2 \\
&\quad + \left(\frac{8\alpha}{5}e^{-u} - \frac{\beta}{5}e^{2u}\right)u_x^2 u_{xx} + \left(\frac{7\beta^2}{5}e^{4u} + \frac{22\alpha\beta}{25}e^{u} - \frac{13\alpha^2}{25}e^{-2u}\right)u_{xx} - 4u_x^6 \\
&\quad - \frac{2}{5}\left(\alpha e^{-u} + \beta e^{2u}\right)u_x^4 + \left(\frac{11\beta^2}{5}e^{4u} - \frac{2\alpha^2}{25}e^{-2u} - \frac{\alpha\beta}{25}e^{u}\right)u_x^2 \\
&\quad + \frac{4}{125}\left(\alpha e^{-u} + \beta e^{2u}\right)^3 + k_0 \\
\Lambda_1 &= u_{6x} + 5u_{xx}u_{4x} - 5u_x^2 u_{4x} + \frac{6\beta}{5}e^{2u}u_{4x} + \frac{3\alpha}{5}e^{-u}u_{4x} + 5u_{xxx}^2 - 20u_x u_{xx}u_{xxx} \\
&\quad + \left(\frac{24\beta}{5}e^{2u} - \frac{6\alpha}{5}e^{-u}\right)u_x u_{xxx} - 5u_{xx}^3 + \left(\frac{18\beta}{5}e^{2u} - \frac{9\alpha}{10}e^{-u}\right)u_{xx}^2 \\
&\quad + \left(\alpha e^{-u} + \frac{26\beta}{5}e^{2u}\right)u_x^2 u_{xx} - c_1 u_{xx} + 5u_x^4 u_{xx} + \left(\frac{4\alpha^2}{25}e^{-2u} + \frac{2\beta^2}{5}e^{4u}\right. \\
&\quad \left. + \frac{\alpha\beta}{5}e^{u}\right)u_{xx} + \left(\frac{\beta}{5}e^{2u} - \frac{\alpha}{10}e^{-u}\right)u_x^4 + \left(\frac{4\beta^2}{5}e^{4u} + \frac{\alpha\beta}{10}e^{u} - \frac{4\alpha^2}{25}e^{-2u}\right)u_x^2 \\
&\quad + \frac{3\alpha\beta^2}{250}e^{3u} - \frac{\alpha^3}{250}e^{-3u} + \frac{\beta^3}{125}e^{6u} + \frac{\alpha c_1}{10}e^{-u} - \frac{\beta c_1}{5}e^{2u} \\
\Lambda_2 &= 2u_{xx} - \frac{\alpha}{5}e^{-u} + \frac{2\beta}{5}e^{2u}, \quad b_0 = 0, \quad b_1 = 2, \quad b_2 = 1, \quad b_3 = 0.
\end{aligned}
$$

4. Multipotentialisations of Equations I–VIII

We now multipotentialise Eqs. I–VIII. This process identifies some more symmetry-integrable equations that are outside of the class I–VIII, in that the recursion operators for those equations require integrating factors, Λ, of order eight.

To establish the potentialisations of Eqs. I–VIII, we first calculate the integrating factors, Λ, for each equation and then use the relation

$$\Lambda = \hat{E}_u \Phi^t \tag{4.1}$$

Figure 1: Equation V

$$U_t = U_{5x} - \frac{\alpha^2}{5}U^2U_{xxx} + \alpha U_xU_{xxx} - \frac{4\alpha^2}{5}UU_xU_{xx} + \alpha U_{xx}^2 - \frac{\alpha^2}{5}U_x^3 + \frac{\alpha^4}{125}U^4U_x + \beta\left(2U_xU_{xx} + UU_{xxx} - \frac{2\alpha^2}{25}U^3U_x\right) + \beta^2\left(\frac{3}{10}U^2U_x - \frac{5}{4\alpha^2}U_{xxx}\right) - \frac{\beta^3}{2\alpha^2}UU_x + c_1U_x$$

$\tilde{u}_x = \Phi_1^t$ (Equation V → $\tilde{u}$ equation)

$$\tilde{u}_t = \tilde{u}_{5x} - \frac{5}{2}\frac{\tilde{u}_{xx}\tilde{u}_{4x}}{\tilde{u}_x} - \frac{5}{4}\frac{\tilde{u}_{xxx}^2}{\tilde{u}_x} + \frac{5\tilde{u}_{xx}^2\tilde{u}_{xxx}}{\tilde{u}_x^2} - \frac{35}{16}\frac{\tilde{u}_{xx}^4}{\tilde{u}_x^3} + \frac{\alpha}{2}\frac{\tilde{u}_{xx}\tilde{u}_{xxx}}{\tilde{u}_x^{1/2}} - \frac{\alpha^2}{5}\tilde{u}_x\tilde{u}_{xxx} - \frac{\alpha}{3}\frac{\tilde{u}_{xx}^3}{u_x^{3/2}} + \frac{\alpha^2}{20}\tilde{u}_{xx}^2 + \frac{\alpha^4}{375}\tilde{u}_x^3 + \beta\tilde{u}_x^{1/2}\tilde{u}_{xxx} - \frac{\beta}{2}\frac{\tilde{u}_{xx}^2}{\tilde{u}_x^{1/2}} - \frac{4\alpha^2\beta}{125}\tilde{u}_x^{5/2} + \frac{3\beta^2}{20}\tilde{u}_x^2 - \frac{5\beta^2}{4\alpha^2}\tilde{u}_{xxx} + \frac{15\beta^2}{16\alpha^2}\frac{\tilde{u}_{xx}^2}{\tilde{u}_x} - \frac{\beta^3}{3\alpha^2}\tilde{u}_x^{3/2} + c_1\tilde{u}_x$$

$u_x = \Phi_2^t$ (Equation V → Equation VI)

Equation VI:

$$u_t = u_{5x} + \alpha u_{xx}u_{xxx} - \frac{\alpha^2}{5}u_x^2u_{xxx} - \frac{\alpha^2}{5}u_xu_{xx}^2 + \frac{\alpha^4}{625}u_x^5 + \beta\left(u_xu_{xxx} - \frac{\alpha^2}{50}u_x^4 + \frac{1}{2}u_{xx}^2\right) + \beta^2\left(-\frac{5}{4\alpha^2}u_{xxx} + \frac{1}{10}u_x^3\right) - \frac{\beta^3}{4\alpha^2}u_x^2 + c_1u_x$$

$\tilde{u}_x = \Phi_3^t$ (Equation VI → $\tilde{u}$ equation)

$v_x = \Phi_4^t$ (Equation VI → $\tilde{v}$ equation)

$$\tilde{v}_t = \tilde{v}_{5x} - \frac{5\tilde{v}_{xx}\tilde{v}_{4x}}{\tilde{v}_x} - \frac{15}{4}\frac{\tilde{v}_{xxx}^2}{\tilde{v}_x} + \frac{65}{4}\frac{\tilde{v}_{xx}^2\tilde{v}_{xxx}}{\tilde{v}_x^2} - \frac{135}{16}\frac{\tilde{v}_{xx}^4}{\tilde{v}_x^3} - \frac{\beta^4}{16\alpha^4}\tilde{v}_x$$

$v_x = \Phi_5^t$ (Equation VI → v equation)

$$v_t = v_{5x} - \frac{5v_{xx}^2v_{4x}}{v_x} + \frac{5v_{xx}^2v_{xxx}}{v_x^2} - \frac{1}{16}\frac{\beta^4}{\alpha^4}v_x$$

$\tilde{v}_x = \Phi_6^t$ (v equation → $\tilde{v}$ equation)

$\tilde{w}_x = \Phi_7^t$ (v equation → w equation)

$$w_t = w_{5x} - \frac{5w_{xx}w_{4x}}{w_x} - \frac{15}{4}\frac{w_{xxx}^2}{w_x} + \frac{65}{4}\frac{w_{xx}^2w_{xxx}}{w_x^2} - \frac{135}{16}\frac{w_{xx}^4}{w_x^3} - \frac{\beta^4}{16\alpha^4}w_x + \frac{5\gamma}{6}\frac{w_{xx}}{w_x} - \frac{35\gamma}{24}\frac{w_{xx}^2}{w_x^2} - \frac{5\gamma^2}{36}\frac{1}{w_x}$$

$\tilde{w}_x = \Phi_8^t$ (w equation → $\tilde{w}$ equation)

$$\tilde{w}_t = \tilde{w}_{5x} - \frac{5\tilde{w}_{xx}\tilde{w}_{4x}}{\tilde{w}_x} + \frac{5\tilde{w}_{xx}^2\tilde{w}_{xxx}}{\tilde{w}_x^2} + \frac{15\gamma}{32}\tilde{w}_x^2\tilde{w}_{xxx} - \frac{15\gamma}{32}\tilde{w}_{xx}^2\tilde{w}_x + \left(\frac{3\gamma}{32}\right)^2\tilde{w}_x^5 - \frac{\beta^4}{16\alpha^4}\tilde{w}_x, \qquad \gamma < 0$$

$\tilde{v}_x = \Phi_{9\pm}^t$ ($\tilde{w}$ equation → $\tilde{v}$ equation)

Fig. 1. Equations V and VI